TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

AUGUST 8, 2008
SOLUTIONS

Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear which theorem you are using and why its use is justified. In problems with multiple parts, be sure to go on to the rest of the problem even if there is some part you cannot do. In working on any part, you may assume the answer to any previous part, even if you have not proved it.

Problem 1.

Let (M, d) be a metric space.

(a) Show that the topology on M induced by the metric is Hausdorff.
Solution. Let $x \neq y$ in M. If $\varepsilon = d(x, y)/2$, then the open ε-balls $B_\varepsilon(x)$ and $B_\varepsilon(y)$ around x and y do not intersect. □

(b) Show that $d\colon M \times M \longrightarrow \mathbb{R}$ is continuous with respect to the product topology on $M \times M$.
Solution. It suffices to show that if $a < b$ in $\mathbb{R}$, then $d^{-1}(a, b)$ is open in $M \times M$. Let $(x, y) \in d^{-1}(a, b)$, so that $a < d(x, y) < b$. Let $\varepsilon = \frac{1}{2}\min(b - d(x, y), d(x, y) - a)$. If $(x', y') \in B_\varepsilon(x) \times B_\varepsilon(y)$, we have by the triangle inequality

$$d(x', y') \leq d(x, y) + d(x, x') + d(y, y'), \quad d(x, y) \leq d(x', y') + d(x, x') + d(y, y'),$$

so that $|d(x, y) - d(x', y')| \leq d(x, x') + d(y, y') < 2\varepsilon$, so $a < d(x', y') < b$ and $B_\varepsilon(x) \times B_\varepsilon(y) \subseteq d^{-1}(a, b)$. □

(c) Find an example for which M is a smooth manifold, but $d\colon M \times M \longrightarrow \mathbb{R}$ is *not* smooth.
Solution. Simply take $M = \mathbb{R}$ with $d(x, y) = |x - y|$. This is not smooth along the diagonal in $\mathbb{R} \times \mathbb{R}$. □

Problem 2.

Let X and Y, be manifolds, and let U and Z be submanifolds of Y.

(a) Assume that $f\colon X \to Y$ is a smooth map transversal to Z in Y, so that $W = f^{-1}(Z)$ is a submanifold of X. Prove that $T_x(W)$ is the preimage of $T_{f(x)}(Z)$ under the linear map $df_x\colon T_x(X) \to T_{f(x)}(Y)$.

Solution. First of all, observe that if $g\colon Y \to V$ is a smooth map with regular value $v \in V$, then the tangent space to the submanifold $Z = g^{-1}(v)$ at any point $y \in Z$ is the kernel of the derivative $dg_y\colon T_y(Y) \to T_v(V)$. In fact, since g is constant on Z, we have that dg_y is zero on $T_y(Z)$, so that $T_y(Z) \subset \ker(dg_y)$. But v is a regular value, hence dg_y is surjective, which implies that

$$\dim \ker(dg_y) = \dim T_v(V) - \dim T_y(Y) = \dim V - \dim Y = \dim Z.$$

Now let $x \in W = f^{-1}(Z)$ be such that $f(x) = y$ and let $k = \operatorname{codim}_Y(Z)$. Then in a neighborhood of x we can write $W = (g \circ f)^{-1}(0)$, where 0 is a regular value of a map g defined on a neighborhood of y and with values in $\mathbb{R}^k$. Because f is transversal to Z we have that 0 is a regular value of $(g \circ f)$ and, by the above observation,

$$T_x(W) = \ker d(g \circ f)_x = \ker(dg_y \circ df_x) = (df_x)^{-1}(\ker dg_y) = (df_x)^{-1}(T_y(Z). \quad \square$$

(b) Assume that U is transversal to Z. Show that for $y \in U \cap Z$, $T_y(U \cap Z) = T_y(U) \cap T_y(Z)$.
Solution. Saying that U and Z are transversal submanifolds is equivalent to saying that the inclusion map $i\colon U \to Y$ is transversal to the submanifold $Z \subset Y$. Let $W = i^{-1}(Z) = U \cap Z \subset U$ and let $y \in U \cap Z$. Then, by (i), we have that $T_y(U \cap Z) = (di_y)^{-1}(T_y(Z)) = T_y(U) \cap T_y(Z)$, since $di_y\colon T_y(U) \to T_{i(y)}(Y) = T_y(Y)$. $\square$

Problem 3.

(a) Compute the fundamental group of the space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times 1$ in one torus with the corresponding circle $S^1 \times 1$ in the other torus.
Solution. Let the fundamental group with basepoint $(1, 1)$ of the first torus be written as the group $\langle a, b \mid ab = ba\rangle$, and the second as $\langle c, d \mid cd = dc\rangle$. Let a in the first and c in the second represent the loop $S^1 \times 1$. Then by the Van Kampen Theorem, in the fundamental group of the identification space, a and c are identified, so the group is $\langle a, b, d \mid ab = ba, ad = da\rangle$, the product of an infinite cyclic group (with generator a) and a free group on two generators (b and d). $\square$

(b) Let $X \subset \mathbb{R}^m$ be the union of convex open sets $X_1, \cdots, X_n$ such that $X_i \cap X_j \cap X_k \neq \emptyset$ for all $i, j, k = 1, \ldots, n$. Show that X is connected and simply-connected.
Solution. We prove this by induction on n. For $n = 1, 2$, or 3, X is star-shaped about any point in the intersection of all the spaces, hence is contractible, hence connected and simply-connected. Let $n \geq 4$, and assume the result is true for $n - 1$. Assume that $X \subset \mathbb{R}^m$ is the union of convex open sets $X_1, \cdots, X_n$, and that $X_i \cap X_j \cap X_k = \emptyset$ for all $i, j, k = 1, \ldots, n$. Let X' be the union of $X_1, \cdots, X_{n-1}$. Then $X = X' \cup X_n$. Each space is connected and their intersection is non-empty, so X is connected. Also, $X_n \cap X' = \cup_{i<n}(X_n \cap X_i)$ is connected, because each $X_n \cap X_i$ is connected and and the intersection of any two is of the form $X_i \cap X_j \cap X_n$, which is non-empty. Now

we can apply Van Kampen to the union of simply connected spaces with connected intersection. Thus X is simply-connected. □

Problem 4.

Let Top be the category of pairs of topological spaces and continuous maps (as usual, we identify a single space X with the pair $(X, \emptyset)$) and let ChCompl be the category of chain complexes $C_\bullet$ of abelian groups (with $C_n = 0$ for $n < 0$) and chain maps. Let $F\colon \mathsf{Top} \rightsquigarrow \mathsf{ChCompl}$ be a functor and define a "homology theory" H^F by $H_n^F(X) = H_n(F(X))$, $H_n^F(X, A) = H_n(F(X, A))$. Assume that for each $(X, A) \in \mathsf{Top}$, one has a natural short exact sequence

$$0 \to F_\bullet(A) \to F_\bullet(X) \to F_\bullet(X, A) \to 0.$$

Also assume that if X is contractible, then

$$H_n^F(X) \cong \begin{cases} \mathbb{Z} \quad \text{(with a natural choice of generator)}, & n = 0, \\ 0, & n > 0. \end{cases}$$

(a) Suppose $x, y \in X$ lie the same path component of X. Show that the images of $H_0^F(x)$ and of $H_0^F(y)$ in $H_0^F(X)$ must be equal.
Solution. Let γ be a path in X from x_0 to x_1. This can be viewed as a map $\gamma\colon [0, 1] \to X$ with $\gamma(0) = x_0$, $\gamma(1) = x_1$. Since $[0, 1]$ is contractible, the inclusions $\{0\} \hookrightarrow [0, 1]$ and $\{1\} \hookrightarrow [0, 1]$ induce isomorphisms on H^F. So look at the commuting diagram

$$\begin{array}{ccc} \{0\} & \longrightarrow & \{x_0\} \\ \downarrow & & \downarrow \\ [0,1] & \xrightarrow{\gamma} & X \\ \uparrow & & \uparrow \\ \{1\} & \longrightarrow & \{x_1\} \end{array}$$

and the induced diagram in H^F homology

$$\begin{array}{ccc} H_0^F(\{0\}) & \xrightarrow{\cong} & H_0^F(\{x_0\}) \\ \cong\downarrow & & \downarrow \\ H_0^F([0,1]) & \xrightarrow{\gamma_*} & H_0^F(X) \\ \cong\uparrow & & \uparrow \\ H_0^F(\{1\}) & \xrightarrow{\cong} & H_0^F(\{x_1\}). \end{array}$$

This shows that the images of $H_0^F(\{x_0\})$ and $H_0^F(\{x_1\})$ in $H_0^F(X)$ are equal. □

(b) Let $\mathsf{Sing}\colon \mathsf{Top} \rightsquigarrow \mathsf{ChCompl}$ be the singular chain functor. Show that there is a natural transformation $\Phi\colon \mathsf{Sing} \to F$ inducing an isomorphism $H_\bullet \to H_\bullet^F$ on contractible spaces. (Hint: *Naturality* is key; use the method of acyclic models.)
Solution. It suffices to work with single spaces instead of pairs, since once Φ is given on A and X, that determines it on $\mathsf{Sing}(X, A) = \mathsf{Sing}(X)/\mathsf{Sing}(A)$. Recall that $\mathsf{Sing}_0(X)$ is the free abelian group on the points of X. For $X = \text{pt}$, choose a representative for the canonical generator of $H_0^F(X)$ and use this to define Φ in degree 0 for a point. By naturality, this determines Φ in degree 0 for all X. Use the Acyclic Models Theorem to extend Φ to all dimensions; this is possible since Sing is freely represented by the model spaces Δ^n and $F(\Delta^n)$ is acyclic (except in dimension 0) by assumption. The resulting natural transformation induces isomorphisms $H_\bullet(X) \to H_\bullet^F(X)$ for X contractible, since all the homology is concentrated in degree 0 and this was true in degree 0 for a point. $\square$

(c) Now assume in addition that the natural map $(D^n, S^{n-1}) \to (S^n, \text{pt})$ (obtained by collapsing S^{n-1} to a point) induces an isomorphism on the relative H_n^F groups for all $n \geq 1$. (This is a weak form of the excision axiom.) Also assume that $F(X \amalg Y) = F(X) \oplus F(Y)$. (Here $\amalg$ denotes the disjoint union of spaces.) Deduce that Φ induces isomorphisms $H_\bullet(S^n) \to H_\bullet^F(S^n)$ for each n. (Hint: Start by proving this for $n = 0$, and proceed by induction on n.)
Solution. Since $S^0 = \text{pt} \amalg \text{pt}$ and Φ induces isomorphisms $H_\bullet(\text{pt}) \to H_\bullet^F(\text{pt})$, Φ also induces isomorphisms $H_\bullet(S^0) \to H_\bullet^F(S^0)$ by the additivity axiom. We also know Φ induces isomorphisms $H_\bullet(D^n) \to H_\bullet^F(D^n)$, since D^n is contractible. Now we proceed by induction using the following scheme:

$$\begin{array}{c} \Phi \text{ homology iso for } S^{n-1} \xRightarrow{(1)} \Phi \text{ homology iso for } (D^n, S^{n-1}) \\ \Phi \text{ homology iso for } (D^n, S^{n-1}) \xRightarrow{(2)} \Phi \text{ homology iso for } S^n. \end{array}$$

Implication (1) comes from the Five Lemma applied to the commuting diagram

$$\begin{array}{ccccccccc} H_k(S^{n-1}) & \to & H_k(D^n) & \to & H_k(D^n, S^{n-1}) & \to & H_{k-1}(S^{n-1}) & \to & H_{k-1}(D^n) \\ \downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong \\ H_k^F(S^{n-1}) & \to & H_k^F(D^n) & \to & H_k^F(D^n, S^{n-1}) & \to & H_{k-1}^F(S^{n-1}) & \to & H_{k-1}^F(D^n). \end{array}$$

Implication (2) is similar, using the exact sequences of the pair (S^n, pt) and the excision isomorphisms $H_k(D^n, S^{n-1}) \to H_k(S^n, \text{pt})$ and $H_k^F(D^n, S^{n-1}) \to H_k^F(S^n, \text{pt})$.

Problem 5.

Let $n \geq 3$ and suppose X is a CW complex with one 0-cell and all other cells of dimension $\geq n - 1$. Suppose

$$H_n(X, \mathbb{Z}) \cong \mathbb{Z}^m \oplus F,$$

where F is a finite abelian group which is the direct sum of k finite cyclic groups.

(a) Show that you can attach $m + k$ $(n+1)$-cells to X, obtaining a new CW complex Y with $H_n(Y, \mathbb{Z}) = 0$ and $H_j(Y, \mathbb{Z}) \cong H_j(X, \mathbb{Z})$ for $j \neq n,\ n+1$.

Solution. We can identify $H_n(X, \mathbb{Z})$ with the homology of the cellular chain complex $C_\bullet(X)$, where $C_n(X)$ is the free abelian group on the n-cells. Choose cycles $c_1, \cdots, c_m, c_{m+1}, \cdots, c_{m+k} \in C_n(X)$ representing generators for $H_n(X, \mathbb{Z})$, so that $c_1, \cdots, c_m$ generate a copy of $\mathbb{Z}^m$ in homology and $c_{m+1}, \cdots, c_{m+k}$ generate F, with each c_j generating a cyclic summand in homology. For each j, we claim we can attach a cell e_j^{n+1} of dimension $n+1$ to X in such a way that in the cellular chain complex of the new space Y obtained by attaching these cells, $\partial e_j^{n+1} = c_j$ (∂ is the cellular boundary map). Assuming that this is the case, we have $C_p(Y) = C_p(X)$ for $p \neq n+1$ and $C_{n+1}(Y) = C_{n+1}(X) \oplus \bigoplus_j \mathbb{Z} e_j^{n+1}$, and the cellular boundary map of Y is the same as for X except that $\partial e_j^{n+1} = c_j$. So $Z_p(Y) = Z_p(X)$ for $p \neq n+1$ and $B_p(Y) = B_p(X)$ for $p \neq n$. Thus the homology of Y can only differ from that of X in dimensions n and $n+1$. In dimension n, $Z_n(Y) = Z_n(X)$, while c_j is now a boundary for each j, so $H_n(Y, \mathbb{Z}) = 0$.

It remains to verify the claim. This comes down to showing that given a cellular cycle c of dimension n in X, we can attach an $(n+1)$-cell having this chain as boundary. The assumption $n \geq 3$ comes into this part of the proof since it implies simple connectivity of X. Since X is simply connected and $(n-2)$-connected, the Hurewicz map $\pi_k(X) \to H_k(X, \mathbb{Z})$ is an isomorphism in dimension $k = n-1$ and surjective in dimension $k = n$. The surjectivity of $\pi_n(X) \to H_n(X, \mathbb{Z})$ means that we choose $f \colon S^n \to X$ representing the homology class of c. If we use this map as an attaching map of an $(n+1)$-cell, this cell will have c as its cellular boundary, which immediately proves the claim.

Alternatively, if one doesn't want to appeal to the Hurewicz theorem, consider the long exact sequence

$$\pi_n(X^n) \to \pi_n(X^n, X^{n-1}) = C_n(X) \xrightarrow{\partial} \pi_{n-1}(X^{n-1}).$$

Think of c as representing a class in $\pi_n(X^n, X^{n-1})$. The map $\pi_n(X^n, X^{n-1}) \xrightarrow{\partial} \pi_{n-1}(X^{n-1})$, followed by the canonical map $\pi_{n-1}(X^{n-1}) \to \pi_{n-1}(X^{n-1}, X^{n-2}) = C_{n-1}(X)$, can be identified with the cellular boundary map, and since c is a homology cycle, ∂c goes to 0 in $\pi_{n-1}(X^{n-1}, X^{n-2})$ and thus comes from a class in $\pi_{n-1}(X^{n-2})$. But X was assumed to have no cells of dimension $1, \cdots, n-2$, so $\partial c = 0$ in $\pi_{n-1}(X^{n-1})$. (We've used the assumption $n \geq 3$ to guarantee that all the homotopy sets are abelian groups.) Thus we can lift c to a map $S^n \to X^n$. This lift is the desired attaching map of the $(n+1)$-cell. □

(b) What is $H_{n+1}(Y, \mathbb{Z})$?

Solution. Look at the exact sequence

$$H_{n+2}(Y, X) \xrightarrow{\partial} H_{n+1}(X) \to H_{n+1}(Y) \to H_{n+1}(Y, X) \xrightarrow{\partial} H_n(X) \to H_n(Y) = 0.$$

Here $H_{n+2}(Y,X)=0$ (since X and Y have the same $(n+2)$-cells), $H_{n+1}(Y,X)$ is the free abelian group on the e_j^{n+1}, and $\partial e_j^{n+1}=[c_j]$ by construction. The $[c_j]$ for $j\le m$ generate a summand of $\mathbb{Z}^m$ inside $H_n(X)$, so the boundary map restricted to $\bigoplus_{j\le m}\mathbb{Z}e_j^{n+1}$ is an isomorphism onto this summand. Hence the exact sequence comes down to

$$0\to H_{n+1}(X)\to H_{n+1}(Y)\to \bigoplus_{j>m}\mathbb{Z}e_j^{n+1}\xrightarrow{\partial} F\to 0.$$

Since F is finite,

$$\ker\left(\bigoplus_{j=m+1}^{m+k}\mathbb{Z}e_j^{n+1}\xrightarrow{\partial} F\right)\cong\mathbb{Z}^k,$$

and so $H_{n+1}(Y,\mathbb{Z})\cong H_n(X,\mathbb{Z})\oplus\mathbb{Z}^k$. □

(c) Show that if $X=S^n$, Y can be taken to be D^{n+1}.
Solution. D^{n+1} can be viewed as the result of attaching an $(n+1)$-cell to S^n, with attaching map the identity map $S^n\to S^n$. □

Problem 6.

Suppose M^n is a compact connected orientable topological n-manifold with boundary a rational homology sphere, i.e., with $H_\bullet(\partial M,\mathbb{Q})\cong H_\bullet(S^{n-1},\mathbb{Q})$.

(a) Assuming n is odd, use Poincaré duality (with coefficients $\mathbb{Q}$) to show that M has Euler characteristic $\chi(M)=1$.
Solution. Let $H_\bullet$ denote homology with coefficients in $\mathbb{Q}$. We have the long exact homology sequence

$$\cdots\to H_k(\partial M)\to H_k(M)\to H_k(M,\partial M)\to H_{k-1}(\partial M)\to\cdots$$

as well as the universal coefficient relation $H^k\cong H_k^*$ and Poincaré duality:

$$H^k(M,\partial M)\cong H_{n-k}(M).$$

Certainly $H_n(M)\cong H^0(M,\partial M)=0$ and $H_0(M)\cong\mathbb{Q}$. It helps to separate out the case $n=1$; in this case $M\cong[0,1]$ so the result is obvious. Otherwise, for $1\le k\le n-2$, $H_k(\partial M)=0$, and for $k=0$, the map $H_k(\partial M)\to H_k(M)$ is an isomorphism. So for $1\le k\le n-1$, the exact sequence gives $H_k(M)\cong H_k(M,\partial M)$ and thus

$$\begin{aligned}\beta_k(M)&=\dim H_k(M)=\dim H_k(M,\partial M)\\&=\dim H^k(M,\partial M)\text{ (by UCT)}\\&=\dim H_{n-k}(M)=\beta_{n-k}(M).\end{aligned}$$

Putting everything together,

$$\begin{aligned}\chi(M) &= \beta_0(M) + (-1)^n\beta_n(M) + \sum_{k=1}^{n-1}(-1)^k\beta_k(M) \\ &= 1 + 0 + \sum_{k=1}^{(n-1)/2}\big((-1)^k\beta_k(M) + (-1)^{n-k}\beta_{n-k}(M)\big) \\ &= 1. \quad \square\end{aligned}$$

(b) Assuming $n \equiv 2 \pmod 4$, show that the Euler characteristic $\chi(M)$ of M is odd.
Solution. Everything is the same as before except for the last step. We have

$$\begin{aligned}\chi(M) &= \beta_0(M) + (-1)^n\beta_n(M) + \sum_{k=1}^{n-1}(-1)^k\beta_k(M) \\ &= 1 + 0 + (-1)^{n/2}\beta_{n/2}(M) + \sum_{k=1}^{(n/2)-1}\big((-1)^k\beta_k(M) + (-1)^{n-k}\beta_{n-k}(M)\big) \\ &= 1 - \beta_{n/2}(M) + \sum_{k=1}^{(n/2)-1} 2(-1)^k\beta_k(M).\end{aligned}$$

So we need to show the middle Betti number $\beta_{n/2}(M)$ is even. This follows from the fact that the cup-product pairing $H^{n/2}(M) \otimes H^{n/2}(M, \partial M) \to H^n(M, \partial M) \cong \mathbb{Q}$ is non-degenerate by Poincaré duality, along with the isomorphism $H^{n/2}(M, \partial M) \cong H^{n/2}(M)$ (from the long exact sequence of the pair). So the cup-product pairing $H^{n/2}(M, \partial M) \otimes H^{n/2}(M, \partial M) \to H^n(M, \partial M) \cong \mathbb{Q}$ is non-degenerate, but also skew-symmetric (since $n/2$ is odd). Since any non-degenerate skew-symmetric bilinear form on a finite-dimensional $\mathbb{Q}$-vector space is isomorphic to $\begin{pmatrix} 0_r & 1_r \\ -1_r & 0_r \end{pmatrix}$ for some r, it follows that $\beta_{n/2}(M, \partial M) = \beta_{n/2}(M)$ is even. $\square$