TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

AUGUST 8, 2008
MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND

Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear which theorem you are using and why its use is justified. In problems with multiple parts, be sure to go on to the rest of the problem even if there is some part you cannot do. In working on any part, you may assume the answer to any previous part, even if you have not proved it.

Problem 1.

Let (M, d) be a metric space.

(a) Show that the topology on M induced by the metric is Hausdorff.
(b) Show that $d\colon M \times M \longrightarrow \mathbb{R}$ is continuous with respect to the product topology on $M \times M$.
(c) Find an example for which M is a smooth manifold, but $d\colon M \times M \longrightarrow \mathbb{R}$ is *not* smooth.

Problem 2.

Let X and Y, be manifolds, and let U and Z be submanifolds of Y.

(a) Assume that $f\colon X \to Y$ is a smooth map transversal to Z in Y, so that $W = f^{-1}(Z)$ is a submanifold of X. Prove that $T_x(W)$ is the preimage of $T_{f(x)}(Z)$ under the linear map $df_x\colon T_x(X) \to T_{f(x)}(Y)$.
(b) Assume that U is transversal to Z. Show that for $y \in U \cap Z$, $T_y(U \cap Z) = T_y(U) \cap T_y(Z)$.

Problem 3.

(a) Compute the fundamental group of the space obtained from the disjoint union of two spaces, each homeomorphic to the torus $S^1 \times S^1$, by identifying a circle $S^1 \times 1$ in one torus with the corresponding circle $S^1 \times 1$ in the other torus.

(b) Let $X \subset \mathbb{R}^m$ be the union of convex open sets $X_1, \cdots, X_n$ such that $X_i \cap X_j \cap X_k \neq \emptyset$ for all $i, j, k = 1, \ldots, n$. Show that X is connected and simply-connected.

Problem 4.

Let $\mathsf{TopPair}$ be the category of pairs of topological spaces and continuous maps (as usual, we identify a single space X with the pair $(X, \emptyset)$) and let $\mathsf{ChCompl}$ be the category of chain complexes $C_\bullet$ of abelian groups (with $C_n = 0$ for $n < 0$) and chain maps. Let $F\colon \mathsf{TopPair} \rightsquigarrow \mathsf{ChCompl}$ be a functor and define a "homology theory" H^F by $H_n^F(X) = H_n(F(X))$, $H_n^F(X, A) = H_n(F(X, A))$. Assume that for each $(X, A) \in \mathsf{TopPair}$, one has a natural short exact sequence

$$0 \to F_\bullet(A) \to F_\bullet(X) \to F_\bullet(X, A) \to 0.$$

Also assume that if X is contractible, then

$$H_n^F(X) \cong \begin{cases} \mathbb{Z} & \text{(with a natural choice of generator)}, \quad n = 0, \\ 0, & n > 0. \end{cases}$$

(a) Suppose $x, y \in X$ lie the same path component of X. Show that the images of $H_0^F(x)$ and of $H_0^F(y)$ in $H_0^F(X)$ must be equal.

(b) Let $\mathsf{Sing}\colon \mathsf{TopPair} \rightsquigarrow \mathsf{ChCompl}$ be the singular chain functor. Show that there is a natural transformation $\Phi\colon \mathsf{Sing} \to F$ inducing an isomorphism $H_\bullet \to H_\bullet^F$ on contractible spaces. (Hint: *Naturality* is key; use the method of acyclic models.)

(c) Now assume in addition that the natural map $(D^n, S^{n-1}) \to (S^n, \mathrm{pt})$ (obtained by collapsing S^{n-1} to a point) induces an isomorphism on the relative H_n^F groups for all $n \geq 1$. (This is a weak form of the excision axiom.) Also assume that $F(X \amalg Y) = F(X) \oplus F(Y)$. (Here $\amalg$ denotes the disjoint union of spaces.) Deduce that Φ induces isomorphisms $H_\bullet(S^n) \to H_\bullet^F(S^n)$ for each n. (Hint: Start by proving this for $n = 0$, and proceed by induction on n.)

Problem 5.

Let $n \geq 3$ and suppose X is a CW complex with one 0-cell and all other cells of dimension $\geq n-1$. Suppose

$$H_n(X, \mathbb{Z}) \cong \mathbb{Z}^m \oplus F,$$

where F is a finite abelian group which is the direct sum of k finite cyclic groups.

(a) Show that you can attach $m+k$ $(n+1)$-cells to X, obtaining a new CW complex Y with $H_n(Y, \mathbb{Z}) = 0$ and $H_j(Y, \mathbb{Z}) \cong H_j(X, \mathbb{Z})$ for $j \neq n$, $n+1$. (Hint: The property you need for the attaching maps of the cells has something to do with the Hurewicz map.)

(b) What is $H_{n+1}(Y, \mathbb{Z})$?

(c) Show that if $X = S^n$, Y can be taken to be D^{n+1}.

Problem 6.

Suppose M^n is a compact orientable topological n-manifold with boundary ∂M a rational homology sphere, i.e., with $H_\bullet(\partial M, \mathbb{Q}) \cong H_\bullet(S^{n-1}, \mathbb{Q})$.

(a) Assuming n is odd, use Poincaré duality (with coefficients $\mathbb{Q}$) to show that M has Euler characteristic $\chi(M) = 1$.

(b) Assuming $n \equiv 2 \pmod 4$, show that the Euler characteristic $\chi(M)$ of M is odd. You will need the fact (which you can assume) that if a finite-dimensional vector space over $\mathbb{Q}$ admits a non-degenerate skew-symmetric bilinear form, then the vector space has even dimension.