Department of Mathematics
University of Maryland
Written Graduate Qualifying Exam
Topology
August, 2000

Instructions

1. Answer all six questions. Each one will be assigned a grade from 0 to 10. In problems with multiple parts, the parts are graded independently of one another. Be sure to go on to subsequent parts even if there is some part you cannot do. You may assume the answer to any part in subsequent parts of the same problem.

2. Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear exactly which theorem you are using and why its use is justified.

1. Consider the following topological spaces:

a) the subset X of $\mathbb{R}^2$ consisting of all rays $\{(x, x/n) : x > 0\}$, as n runs over the positive integers, with the subspace topology from $\mathbb{R}^2$.

b) the subset Y of $\mathbb{R}^2$ defined by

$$\{(0,0)\} \cup \{(x,y) : -1 < x < 1, y > 0\},$$

with the subspace topology from $\mathbb{R}^2$.

c) the quotient space $Z = W/\sim$ obtained from the subspace

$$W = \{(n,y) : n \in \mathbb{Z}, y \in \mathbb{R}\}$$

of $\mathbb{R}^2$ (with the subspace topology from $\mathbb{R}^2$), where $(n,0) \sim (0,0)$ for all $n \in \mathbb{Z}$ (and there are no other identifications). Note that Z is to be given the quotient topology.

d) the quotient space $Q = \mathbb{R}/\sim$, where $x \sim y$ if $x - y$ is of the form $n + m\sqrt{2}$, $n, m \in \mathbb{Z}$. Note that Q is to be given the quotient topology.

Which of these spaces are locally compact? Which are Hausdorff? Which are metrizable? Give explicit reasons for your answers.

2. Let M^m be a smooth connected m-manifold (without boundary), whose fundamental group is finite of odd order.

a) If $m < n$, show that any continuous map $f: M \to \mathbb{RP}^n$ is null-homotopic (homotopic to a constant map).

b) If M is compact, $m = n$, and n is odd, show that there exists a continuous map $f: M \to \mathbb{RP}^n$ which is *not* null-homotopic. (This is also true if n is even, though you don't have to deal with this case.)

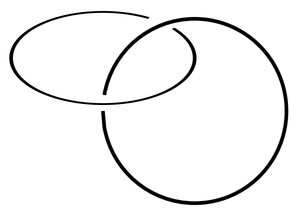
3. Let M^m and N^n be *disjoint* oriented compact connected smooth submanifolds of $\mathbb{R}^{k+1}$, with $\dim M + \dim N = m + n = k$. Define $\lambda\colon M \times N \to S^k$ by

$$\lambda(x, y) = \frac{x - y}{|x - y|}, \quad x \in M,\ y \in N.$$

Let $\operatorname{Lk}(M, N) = \deg \lambda$ (called the *linking number* of M and N in $\mathbb{R}^{k+1}$.

i) If $M = \partial W$, where W is a compact oriented manifold with boundary in $\mathbb{R}^{k+1}$, and $W \cap N = \emptyset$, show that $\operatorname{Lk}(M, N) = 0$.

ii) Compute (up to sign) $\operatorname{Lk}(S^1, S^1)$ for the following link in $\mathbb{R}^3$:



4. Recall that for K^n and L^n, smooth connected n-manifolds without boundary, we can form a new n-manifold, denoted $K\#L$, called the **connected sum** of L and N, by taking smooth embeddings $f\colon \mathbb{R}^n \to K$ and $g\colon \mathbb{R}^n \to L$ and gluing $K \smallsetminus f(0)$ to $L \smallsetminus g(0)$ by identifying $f(tu)$ with $g(t^{-1}u)$ for u in the unit sphere S^{n-1} and $t \in (0, \infty)$. Let $M = \mathbb{RP}^4 \# \mathbb{CP}^2$.

(a) Compute the fundamental group, $\pi_1(M)$ and the homology groups $H_*(M)$ of M. You may use the fact that

$$H_q(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}, & q = 0, \text{ and if } n \text{ is odd also } q = n, \\ \mathbb{Z}_2, & q \text{ odd},\ 1 \le q \le n - 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Is M orientable?

(c) Compute the cohomology **groups** (not the ring structure) of M.

5. Let $f : S^{2n-1} \to S^n$ be defined as follows: Consider S^{2n-1} to be the boundary of $D^{2n} = D^n \times D^n$ where D^n is the n-disk. If $c : D^n \to S^n$ is the map that collapses the boundary to the base-point $*$, then $c \times c : D^{2n} \to S^n \times S^n$ carries S^{2n-1} to the one-point union $S^n \vee S^n$. Then f is the composition of this with the folding map $S^n \vee S^n \to S^n$. Let $X = S^n \cup_f D^{2n}$.

(a) Show that X can also be identified as $S^n \times S^n/\sim$, where $(x, *) \sim (*, x)$.

(b) For $n \ge 2$, calculate the integral cohomology **ring** of X. (Hint: use the map $S^n \times S^n \to X$ from (a).)

(c) Show that X is not homotopy equivalent to a closed manifold for any $n \ge 2$.

6. Recall that $H^*(\mathbb{CP}^n) = \mathbb{Z}[a]/(a^{n+1})$, for $a \in H^2(\mathbb{CP}^n)$.

(a) Determine the cohomology **ring** of $\mathbb{CP}^2 \times \mathbb{CP}^2$.

(b) Show that any homotopy equivalence $f : \mathbb{CP}^2 \to \mathbb{CP}^2$ is orientation preserving.