TOPOLOGY/GEOMETRY QUALIFYING
EXAMINATION

UNIVERSITY OF MARYLAND

Unless otherwise stated, you may appeal to a “well known theorem” in your solution to a problem. If you do, it is your responsibility to clarify exactly which theorem you are using and to justify its use. In any part of a problem with multiple parts, you may assume the answer to any previous part, even if you have not proved it.

NOTE: On this exam not all the problems are equally weighted. Problem 5 is worth 20 points and problems 1-4 are each worth 10.

1. Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. A map \(\pi : X \to Y\) is called a submetry if for every \(x \in X\), and any \(r > 0\),
\[
\pi(D(x, r)) = D(\pi(x), r)
\]
where \(D(x, r)\) denotes the closed \(r\)-ball about \(x\).
(a) Show that \(\pi\) is surjective if \(X\) is nonempty.
(b) Show that \(\pi\) is continuous.
(c) Show that \(\pi\) is open. [A map \(f : A \to B\) is open if and only if for every open subset \(U \subset A\), the image \(f(U)\) is open in \(B\).]
(d) Suppose that \(y_1, y_2 \in Y\). Suppose that \(x_1 \in X\) satisfies \(\pi(x_1) = y_1\). Show that there exists \(x_2 \in X\) such that \(\pi(x_2) = y_2\) and \(d_X(x_1, x_2) = d_Y(y_1, y_2)\).

2. Let \(F : \mathbb{R}^4 \to \mathbb{R}\) be the quadratic function
\[
F(x, y, z, t) = 4x^2 + 3y^2 + 3z^2 + t^2.
\]
Let \(f : S^3 \to \mathbb{R}\) be the restriction of \(F\) to the unit sphere \(S^3 \subset \mathbb{R}^4\).
(a) Let \(\mathbb{RP}^3\) be real projective space and let \(\pi : S^3 \to \mathbb{RP}^3\) be the 2-fold covering map. Give \(\mathbb{RP}^3\) the unique differentiable structure for which \(\pi\) is a local diffeomorphism. Prove that \(f\) descends to a smooth function \(\bar{f}\) on \(\mathbb{RP}^3\); that is, there exists a smooth function \(\bar{f}\) on \(\mathbb{RP}^3\) such that \(\bar{f} \circ \pi = f\).
(b) Find the critical points of \(\bar{f}\).

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(3) The picture on the following page illustrates the map $p : X \longrightarrow Y$ of adjunction spaces $X$, $Y$ which we describe precisely as follows. For $n = 1, 2, 3, 4, 5$ let $C_n$ denote the circle $\{(e^{i\theta}, n) \mid \theta \in \mathbb{R}\}$. Choose basepoints

\begin{align*}
a_1 &= (1, 2) \in C_2 \\
b_1 &= (-1, 2) \in C_2 \\
c_1 &= (1, 1) \in C_1
\end{align*}

and

\begin{align*}
a_2 &= (1, 3) \in C_3 \\
b_2 &= (e^{2\pi i/3}, 3) \in C_3 \\
c_2 &= (e^{-2\pi i/3}, 3) \in C_3.
\end{align*}

Let $X$ denote the identification space of $C_1 \coprod C_2 \coprod C_3$ under the equivalence relation defined by:

\begin{align*}
a_1 &\sim a_2, \\
b_1 &\sim b_2, \\
c_1 &\sim c_2.
\end{align*}

Let $a, b, c \in X$ be the corresponding images in $X$. Let $Y$ denote the identification space of $C_4 \coprod C_5$ under the equivalence relation defined by $(1, 4) \sim (1, 5)$ and let $y \in Y$ be the common image of these points in $Y$.

There is a continuous map $p : X \longrightarrow Y$ defined as follows:

\[
(\zeta, n) \longmapsto \begin{cases}
(\zeta, 4) & \text{if } n = 1 \\
(\zeta^2, 4) & \text{if } n = 2 \\
(\zeta^3, 5) & \text{if } n = 3.
\end{cases}
\]

Informally, $p$ maps the circle $C_1$ once around $C_4$ and $C_2$ twice around $C_4$. The circle $C_4$ is attached to $C_5$ at the point $y$, and $p$ wraps $C_3$ three times around $C_5$. The points $a, b, c$ comprise the inverse image $p^{-1}\{y\}$.

(a) Show that $p$ is a covering space.

(b) Determine $k$ such that $X$ is homotopy equivalent to a wedge of $k$ copies of $S^1$.

(c) Prove or disprove: $p$ is a regular covering space.
(4) Let $p, q$ be relatively prime integers. Consider the following CW-complex: $X$ has one 0-cell $x_0$, two 1-cells labelled $a$ and $b$, and two 2-cells labelled $c, d$. The boundary $\partial c$ is attached to the 1-skeleton

$$X^1 = x_0 \cup a \cup b$$

by the map $a^p b^q$. That is, the attaching map for $\partial c$ wraps $p$ times around the $a$-circle and then $q$-times around the $b$-circle. The boundary $\partial d$ is attached to $X^1$ by the map $aba^{-1}b^{-1}$, that is the map which wraps $\partial d$ first around $a$, then around $b$, then around $a$ in the opposite direction, and finally around $b$ in the opposite direction.

(a) Compute the fundamental group and the integral homology groups of $X$.

(b) Show $X$ is homotopy equivalent to $S^2$ with two points identified. [Hint: Think about $(p, q) = (1, 0)$.]
(5) In the following 20-point problem, any part may be used (even if you didn’t prove it) in any later part. (\(\chi\) denotes Euler characteristic. By definition a manifold is closed if it is compact and has empty boundary.)

(a) Suppose that \(M\) is a closed, connected, orientable odd-dimensional manifold. Show that \(\chi(M) = 0\).

(b) Suppose \(X\) is a compact, connected, oriented \(n\)-manifold with or without boundary. Use Poincaré-Lefschetz duality to show \(H_{n-1}(X, \mathbb{Z})\) is free abelian. (You may assume all homology groups are finitely generated abelian groups.)

(c) Let \(n \geq 1\) be an integer. Show that there exists a connected, closed, orientable \(n\)-dimensional manifold \(M\) with \(\chi(M) = 0\).

(d) If \(M \# N\) denotes the orientable, connected sum of the closed, orientable \(n\)-manifolds \(M\) and \(N\), show
\[
\chi(M \# N) = \chi(M) + \chi(N) - (1 + (-1)^n).
\]
(The connected sum of \(M \# N\) is obtained by gluing together complements \(M \setminus D^m_M\) and \(N \setminus D^m_N\), where \(D^m_M\) and \(D^m_N\) are discs in \(M\) and \(N\) respectively, by an orientation-reversing homeomorphism \(\partial D^m_M \approx \partial D^m_N\) of their boundaries.)

(e) Suppose there exists a closed, orientable \(n\)-dimensional manifold with \(\chi(M)\) an odd integer greater than 1. Show that for any integer \(l\) there exists a connected, closed, orientable \(n\)-dimensional manifold \(W\) with \(\chi(W) = l\).

[Hint: Try to find closed orientable manifolds of arbitrarily large even or odd Euler characteristic.]

(f) Suppose \(n\) is a positive integer divisible by 4 and \(m\) is an integer. Show there exists an closed, orientable \(n\)-dimensional manifold of Euler characteristic \(m\).

(g) Suppose \(M\) is a closed orientable \(2k\)-dimensional manifold where \(k\) is an integer \(\geq 1\). Let \(F\) be a field of characteristic \(\neq 2\). Use the fact that if \(A\) is a \(m \times m\) skew-symmetric matrix with entries in \(F\) having nonzero determinant then \(m\) is even to show the following: Any closed orientable \(4n + 2\)-dimensional manifold has even Euler characteristic.