TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

UNIVERSITY OF MARYLAND

Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem. If you do, it is your responsibility to clarify exactly which theorem you are using and to justify its use. In any part of a problem with multiple parts, you may assume the answer to any previous part, even if you have not proved it.

NOTE: On this exam not all the problems are equally weighted. Problem 5 is worth 20 points and problems 1-4 are each worth 10.

(1) Let (X, d_X) and (Y, d_Y) be metric spaces. A map $\pi : X \longrightarrow Y$ is called a *submetry* if for every $x \in X$, and any $r > 0$,

$$\pi\big(D(x, r)\big) = D(\pi(x), r)$$

where $D(x, r)$ denotes the closed r-ball about x.

(a) Show that π is surjective if X is nonempty.

(b) Show that π is continuous.

(c) Show that π is open. [A map $f : A \longrightarrow B$ is *open* if and only if for every open subset $U \subset A$, the image $f(U)$ is open in B.]

(d) Suppose that $y_1, y_2 \in Y$. Suppose that $x_1 \in X$ satisfies $\pi(x_1) = y_1$. Show that there exists $x_2 \in X$ such that $\pi(x_2) = y_2$ and $d_X(x_1, x_2) = d_Y(y_1, y_2)$.

(2) Let $F : \mathbb{R}^4 \longrightarrow \mathbb{R}$ be the quadratic function

$$F(x, y, z, t) = 4x^2 + 3y^2 + 3z^2 + t^2.$$

Let $f : S^3 \to \mathbb{R}$ be the restriction of F to the unit sphere $S^3 \subset \mathbb{R}^4$.

(a) Let $\mathbb{RP}^3$ be real projective space and let $\pi : S^3 \longrightarrow \mathbb{RP}^3$ be the 2-fold covering map. Give $\mathbb{RP}^3$ the unique differentiable structure for which π is a local diffeomorphism.
Prove that f descends to a smooth function $\bar{f}$ on $\mathbb{RP}^3$; that is, there exists a smooth function $\bar{f}$ on $\mathbb{RP}^3$ such that $\bar{f} \circ \pi = f$.

(b) Find the critical points of $\bar{f}$.

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(3) The picture on the following page illustrates the map $p : X \longrightarrow Y$ of adjunction spaces X, Y which we describe precisely as follows. For $n = 1, 2, 3, 4, 5$ let C_n denote the circle $\{(e^{i\theta}, n) \mid \theta \in \mathbb{R}\}$. Choose basepoints

$$a_1 = (1, 2) \in C_2$$
$$b_1 = (-1, 2) \in C_2$$
$$c_1 = (1, 1) \in C_1$$

and

$$a_2 = (1, 3) \in C_3$$
$$b_2 = (e^{2\pi i/3}, 3) \in C_3$$
$$c_2 = (e^{-2\pi i/3}, 3) \in C_3.$$

Let X denote the identification space of $C_1 \coprod C_2 \coprod C_3$ under the equivalence relation defined by:

$$a_1 \sim a_2,$$
$$b_1 \sim b_2,$$
$$c_1 \sim c_2.$$

Let $a, b, c \in X$ be the corresponding images in X. Let Y denote the identification space of $C_4 \coprod C_5$ under the equivalence relation defined by $(1, 4) \sim (1, 5)$ and let $y \in Y$ be the common image of these points in Y.

There is a continuous map $p : X \longrightarrow Y$ defined as follows:

$$(\zeta, n) \longmapsto \begin{cases} (\zeta, 4) & \text{if } n = 1 \\ (\zeta^2, 4) & \text{if } n = 2 \\ (\zeta^3, 5) & \text{if } n = 3. \end{cases}$$

Informally, p maps the circle C_1 once around C_4 and C_2 twice around C_4. The circle C_4 is attached to C_5 at the point y, and p wraps C_3 three times around C_5. The points a, b, c comprise the inverse image $p^{-1}\{y\}$.

(a) Show that p is a covering space.
(b) Determine k such that X is homotopy equivalent to a wedge of k copies of S^1.
(c) Prove or disprove: *p is a regular covering space.*

(4) Let p, q be relatively prime integers. Consider the following CW-complex: X has one 0-cell x_0, two 1-cells labelled a and b, and two 2-cells labelled c, d. The boundary ∂c is attached to the 1-skeleton

$$X^1 = x_0 \cup a \cup b$$

by the map $a^p b^q$. That is, the attaching map for ∂c wraps p times around the a-circle and then q-times around the b-circle. The boundary ∂d is attached to X^1 by the map $aba^{-1}b^{-1}$, that is the map which wraps ∂d first around a, then around b, then around a in the opposite direction, and finally around b in the opposite direction.

(a) Compute the fundamental group and the integral homology groups of X.

(b) Show X is homotopy equivalent to S^2 with two points identified. [Hint: Think about $(p, q) = (1, 0)$.]

(5) In the following 20-point problem, any part may be used (even if you didn't prove it) in any later part. (χ denotes Euler characteristic. By definition a manifold is *closed* if it is compact and has empty boundary.)

(a) Suppose that M is a closed, connected, orientable odd-dimensional manifold. Show that $\chi(M) = 0$.

(b) Suppose X is a compact, connected, oriented n-manifold with or without boundary. Use Poincaré-Lefschetz duality to show $H_{n-1}(X, \mathbb{Z})$ is free abelian. (You may assume all homology groups are finitely generated abelian groups.)

(c) Let $n \geq 1$ be an integer. Show that there exists a connected, closed, orientable n-dimensional manifold M with $\chi(M) = 0$.

(d) If $M\#N$ denotes the orientable, connected sum of the closed, orientable n-manifolds M and N, show

$$\chi(M\#N) = \chi(M) + \chi(N) - (1 + (-1)^n).$$

(The *connected sum* of $M\#N$ is obtained by gluing together complements $M \setminus D^n_M$ and $N \setminus D^n_N$, where D^n_M and D^n_N are discs in M and N respectively, by an orientation-reversing homeomorphism $\partial D^n_M \approx \partial D^n_N$ of their boundaries.)

(e) Suppose there exists a closed, orientable n-dimensional manifold with $\chi(M)$ an odd integer greater than 1. Show that for any integer l there exists a connected, closed, orientable n-dimensional manifold W with $\chi(W) = l$.
[Hint: Try to find closed orientable manifolds of arbitrarily large even or odd Euler characteristic.]

(f) Suppose n is a positive integer divisible by 4 and m is an integer. Show there exists an closed, orientable n-dimensional manifold of Euler characteristic m.

(g) Suppose M is a closed orientable $2k$-dimensional manifold where k is an integer ≥ 1. Let F be a field of characteristic $\neq 2$. Use the fact that if A is a $m \times m$ skew-symmetric matrix with entries in F having nonzero determinant then m is even to show the following: *Any closed orientable $4n+2$-dimensional manifold has even Euler characteristic.*