MATH 734 Ph.D. QUALIFYING EXAM

August, 1989

1. Let M be a compact, connected, n-dimensional manifold without boundary.
   (a) Show that if M is orientable then $H_{n-1}(M;\mathbb{Z})$ is torsion free.
   (b) Show that if M is non-orientable then the group of 2-torsion elements of $H_{n-1}(M;\mathbb{Z})$ is cyclic of order 2. (The group of 2-torsion elements is $\{x \in H_{n-1}(M;\mathbb{Z}) \mid 2x = 0\}$.)

2. Define a singular 1-cochain f on $\mathbb{R}$ as follows: The value of f on a 1-simplex will be the oriented length of the line segment joining the vertices of the simplex intersected with a compact interval in $\mathbb{R}$. More precisely, let $\Delta_1 = \{(t_0, t_1) \in \mathbb{R}^2 \mid t_0 + t_1 = 1,\ t_i \geq 0\}$ be the standard 1-simplex; let $e_0 = (0,1)$ and $e_1 = (1,0)$. Let I be the closed interval $[-1,1]$ in $\mathbb{R}$. If $\sigma : \Delta_1 \to \mathbb{R}$ is a singular 1-simplex then define $\pi(\sigma) = 1$ if $\sigma(e_0) \leq \sigma(e_1)$; define $\pi(\sigma) = -1$ otherwise. Finally, define $f(\sigma) = \pi(\sigma) \cdot \text{length}([\pi(\sigma)\sigma(e_0), \pi(\sigma)\sigma(e_1)] \cap I)$.
   a) Show that f is a cocycle. Show that f defines a non-trivial element of $H_c^1(\mathbb{R};\mathbb{R})$, the first cohomology group of $\mathbb{R}$ with compact supports and coefficients in $\mathbb{R}$.
   b) Let $p : \mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong \mathbb{T}^1$ be the covering map. Show that f is not the image by $S^1(p) : S^1(\mathbb{T}^1) \to S^1(\mathbb{R})$ of a cochain on $\mathbb{T}^1$. ($S^1(\cdot)$ is the singular 1-cochain functor.)

3. Denote by $\mathbb{C}P^n$ the complex projective space of complex lines in $\mathbb{C}^{n+1}$.
   a) Prove that $\mathbb{C}P^n$ is a spherical complex (or CW complex) with one cell in each even dimension less than or equal to $2n$.
   b) Show that if $p : \mathbb{C}P^2 \to X$ is a covering, then p is a homeomorphism. (Suggestion: Consider the Euler characteristic.)