734 PH.D. QUALIFYING EXAMINATION

1. Let $\Delta_1 = \{(t_0, t_1)|t_0 + t_1 = 1, t_i \geq 0\}$ be the standard 1-simplex and let $e_0 = (0,1)$, $e_1 = (1,0)$. Define a 1-cochain $f \in \mathrm{C}^1(\mathbf{R}, \mathbf{Z})$ by

$$f(\sigma) = \left\{ \begin{array}{rl} 1 & \sigma(e_0) \leq 0 < \sigma(e_1) \\ -1 & \sigma(e_1) \leq 0 < \sigma(e_0) \\ 0 & \text{otherwise} \end{array} \right\}$$

for any singular 1-simplex $\sigma : \Delta_1 \to \mathbf{R}$.

a) Show f is a cocyle.

b) Write f as a coboundary, i.e. find a 0-cochain $g \in \mathrm{C}^1(\mathbf{R}, \mathbf{Z})$ such that $f = \delta g$.

c) Show that there is no 0-cochain h with compact support such that $f = \delta h$.

2. Using Mayer-Vietoris type arguments, answer the following two otherwise unrelated questions. i) Let X^n be a closed, connected manifold of dimension $n \geq 1$. Let $B \subset X^n$ be the homeomorphic image of the closed unit ball in R^n and B° its interior. Let $X' = X - B^\circ$. Note that X' is an n–manifold with boundary equal to the $n-1$ sphere, S^{n-1}. Now let M and N be n–manifolds as above and let $M\#N$ be the disjoint union of M' and N' with the boundary $n-1$ spheres identified.

Show that if M and N are orientable, so is $M\#N$.

ii) An open cover $\{U_\alpha\}$ of a topological space X is called a good cover if all non-empty finite intersections $U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$ are contractible. (In particular, each U_α is contractible.)

Show that if M is a closed manifold of dimension $n > 0$, then the cardinality of a good cover of M is at least $n+2$.

3. Let n be an odd integer, S^n the n-sphere, P^n the real projective plane and $p : S^n \to P^n$ the natural projection. Suppose we are given continuous maps $g : S^n \to S^n$ and $f : P^n \to P^n$ such that $pg = fp$, i.e., the following diagram commutes.

$$\begin{array}{ccc} S^n & \xrightarrow{g} & S^n \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p} \\ P^n & \xrightarrow{f} & P^n \end{array}$$

a) Show degree(g) = degree(f).

b) Prove there exists a map $f' : P^n \to S^n$ such that $pf' = f$ if and only if degree(f) is even. (The cases $n = 1$ and $n > 1$ are separate. You may find cohomology with mod 2 coefficients useful).

c) Prove that if $h : S^n \to S^n$ commutes with the antipodal map of S^n then degree(h) is odd.