Algebraic Topology
(Mathematics 734, Prof. Rosenberg)
Final Examination
Sunday, May 16, 2004

Instructions. Answer each question in your exam booklet. The point value of each problem is indicated. The exam is worth a total of 200 points. In problems with multiple parts, whether the parts are related or not, the parts are graded independently of one another. Be sure to go on to subsequent parts even if there is some part you cannot do.

You are allowed to appeal to "standard theorems" proved in class or in the textbook, but if you do so, it's your responsibility to state clearly exactly what you're using and why it applies.

1. (40 points) **Short-Answer Problems.** Give brief definitions or statements (no proofs) for the following terms.
   (a) The Künneth Theorem (for $H_*(X \times Y)$).
   (b) The Mayer-Vietoris sequence (in homology).
   (c) The cap product.
   (d) The Lefschetz Fixed-Point Theorem.

2. (40 points) Suppose a cyclic group $G \cong \mathbb{Z}/p$ of prime order p acts on $X = S^n$. You may assume that there is a finite CW decomposition of X for which the fixed set X^G is a subcomplex and G freely permutes the cells of X outside of X^G.
   (a) Show that the Euler characteristic of X^G is congruent modulo p to $\chi(S^n) = 1 + (-1)^n$.
   (b) Deduce from (a) that if p is odd and n is even, the action of G on $X = S^n$ cannot be free.

3. (40 points) Let $X = D^2 \times S^1$ be the solid torus, which is a 3-manifold with boundary $\partial X = T^2$. Compute the homology and cohomology groups of the quotient space $X/\partial X$ (obtained by collapsing the boundary to a single point). (There are many ways to do this problem, both with and without Poincaré duality.)

4. (40 points) Let M be a compact oriented connected topological n-manifold (without boundary). Recall that the assumptions guarantee that $H_n(M) \cong \mathbb{Z}$. Define the *degree* of a map $f\colon M \to M$ to be the integer m such that $f_*\colon H_n(M) \to H_n(M)$ is multiplication by m. (This generalizes the definition you're familiar with when $M = S^n$.)
   (a) If $M = \mathbb{CP}^2$, show that the degree is always nonnegative and is always a perfect square. (Hint: Use the cup product.)
   (b) Show that if M is smooth and f is smooth, then for any n-form ω on M,

$$\int_M f^*(\omega) = (\deg f) \int_M \omega.$$

   (You may assume the de Rham Theorem.)

5. (40 points) Let X be a CW-complex with exactly 3 cells: a 0-cell, a 2-cell, and a 4-cell, where the 4-cell is attached by means of the map $f = h \circ g \colon \partial D^4 = S^3 \to S^2 = X^{(2)}$, where $g \colon S^3 \to S^3$ is a map of degree $k \geq 1$ and $h \colon S^3 \to S^2$ is the Hopf map, i.e., the attaching map of the 4-cell in $\mathbb{CP}^2$.

(a) Compute the homology and cohomology *groups* of X (with coefficients in $\mathbb{Z}$).

(b) Let α and β be the canonical generators of $H^2(X)$ and of $H^4(X)$ (coming from the usual orientations of the 2-cell and the 4-cell). Show that $\alpha \cup \alpha = m\beta$ for some integer m.

(c) Now compute the multiplication factor m from (b) and show it only depends on the degree k and not on any other invariant of the map g. Here is a suggested method. Recall $X = \left(S^2 \coprod D^4\right)/\!\sim$, where $\sim$ identifies $x \in S^3 \subset D^4$ to $f(x) \in S^2$. Write $\mathbb{CP}^2$ similarly. Show that g extends to a map $\widetilde{g} \colon D^4 \to D^4$ (hint: polar coordinates). Then show that

$$(\mathrm{id} \textstyle\coprod \widetilde{g}) \colon \left(S^2 \coprod D^4\right) \longrightarrow \left(S^2 \coprod D^4\right)$$

defines a map $\varphi \colon X \to \mathbb{CP}^2$. Study φ^* on cohomology.