Algebraic Topology
(Mathematics 734, Prof. Rosenberg)
Final Examination Solutions
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2. (a) By the Euler-Poincaré principle, $\chi(S^n) = \chi(X^G) + \chi(X, X^G)$. But $\chi(X, X^G)$ is the alternating sum of the number of relative cells in each dimension. Since G permutes these cells freely, the number of them in each dimension is divisible by p, and hence $\chi(X, X^G)$ is a multiple of p. The result follows.

(b) Free action means there is no fixed set, i.e., $X^G = \emptyset$. Then if n is even, we have $2 = \chi(S^n) \equiv 0 \mod p$. If p is odd, this is impossible.

3. **Method 1 (cell decomposition).** Choose the standard CW decompositions of D^2 with a 0-cell, a 1-cell, and a 2-cell, and of S^1 with a 0-cell and a 1-cell. Taking products gives a CW decomposition of X with 6 cells. Now collapse the subcomplex T^2 to a point. This removes two 1-cells and a 2-cell, and leaves a CW decomposition of $X/\partial X$ with a 3-cell, a 2-cell, and a 0-cell. The one cellular boundary map that must be computed goes from 3-chains to 2-chains. The 3-cell in X is (2-cell)×(1-cell), and its cellular boundary is (1-cell)×(1-cell) which is collapsed to the 0-cell in $X/\partial X$, and thus the cellular boundary map can be seen to be 0. So $H_j(X/\partial X) \cong \mathbb{Z}$ for $j = 0, 2, 3$, 0 otherwise. By the universal coefficient theorem, one also has $H^j(X/\partial X) \cong \mathbb{Z}$ for $j = 0, 2, 3$, 0 otherwise.

Method 2 (long exact sequence). Since ∂X is nicely embedded in X (i.e., is a deformation retract of a collar neighborhood), $\widetilde{H}_\bullet(X/\partial X) \cong H_\bullet(X, \partial X)$. The latter can be computed from the long exact sequence of the pair $(X, \partial X)$. First of all, X is homotopy equivalent to S^1, so $H_j(X) = 0$ for $j \geq 2$. And in dimension 1, the map $H_1(\partial X) \to H_1(X)$ is clearly equivalent to the map $H_1(S^1 \times S^1) \to H_1(S^1)$ induced by projection onto the second factor, which is split surjective. So the exact sequence is

$$\begin{aligned} 0 \to H_3(X, \partial X) \to H_2(\partial X) = \mathbb{Z} \to 0 \\ \to H_2(X, \partial X) \to H_1(\partial X) = \mathbb{Z}^2 \twoheadrightarrow H_1(X) = \mathbb{Z} \to H_1(X, \partial X) \\ \to H_0(\partial X) = \mathbb{Z} \xrightarrow{\cong} H_0(X) = \mathbb{Z}. \end{aligned}$$

From this we see $H_j(X, \partial X) \cong \mathbb{Z}$ for $j = 2, 3$, 0 otherwise. Hence $H_j(X/\partial X) \cong \mathbb{Z}$ for $j = 0, 2, 3$, 0 otherwise. By the universal coefficient theorem, one also has $H^j(X/\partial X) \cong \mathbb{Z}$ for $j = 0, 2, 3$, 0 otherwise.

Method 3 (Poincaré duality). By Poincaré duality for manifolds with boundary, $H^j(X, \partial X) \cong H_{3-j}(X)$. The latter is non-zero only for $3 - j = 0$ or 1. So $H^j(X, \partial X) \cong \mathbb{Z}$ for $j = 2$ or 3, 0 otherwise. By the UCT, we similarly have $H_j(X, \partial X) \cong \mathbb{Z}$ for $j = 2$ or 3, 0 otherwise. The rest of the solution is as in Method 2.

4. By the UCT, $H^n(M) \cong \mathrm{Hom}(H_n(M), \mathbb{Z}) \cong \mathbb{Z}$, and this identification is natural, so f^* on $H^n(M)$ is just the adjoint of f_* on $H_n(M)$. Hence the degree can also be described as the integer m such that f^*: $H^n(M) \to H^n(M)$ is multiplication by m.

(a) If $M = \mathbb{CP}^2$, then $n = 4$ and $H^4(M)$ is generated by α^2, where α is a generator of $H^2(M)$. For any map $f\colon M \to M$, f^* is a ring homomorphism, and thus $f^*(\alpha^2) = (f^*\alpha)^2$. Since $H^2(M) \cong \mathbb{Z}$, $f^*\alpha = k\alpha$ for some k, and then $f^*(\alpha^2) = k^2\alpha^2$, so $\deg f = k^2$ is a perfect square.

(b) If M is smooth, then the de Rham theorem says that $H^n_{\mathrm{deR}}(M) \cong H^n(M;\mathbb{R}) \cong H^n(M;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$, and the isomorphisms are natural. Hence we may also compute $\deg f$ from the factor by which f^* acts on $H^n_{\mathrm{deR}}(M)$. Now every n-form on M is closed (since $\dim M = n$), and so such a form ω represents a class in $H^n_{\mathrm{deR}}(M)$. So we have $f^*(\omega) \equiv (\deg f)\omega$ modulo exact forms. By Stokes' Theorem, exact forms integrate to 0. Hence $\int_M f^*(\omega) = (\deg f)\int_M \omega$.

5. (a) Since X has no cells in adjacent dimensions, the cellular boundary maps must vanish, and $H_j(X) \cong \mathbb{Z}$ for $j = 0, 2, 4$, and is 0 otherwise. Similarly (via the cellular cochain complex), $H^{(}X) \cong \mathbb{Z}$ for $j = 0, 2, 4$, and is 0 otherwise.

(b) Since β generates $H^4(X)$, $\alpha \cup \alpha$ must be a multiple of β.

(c) Each point in D^4 can be written as $\lambda \cdot x$ with $0 \le \lambda \le 1$ and $x \in S^3$, and this decomposition ("polar coordinates") is unique unless $\lambda = 0$, in which case all choices for x give the same point $0 \in \mathbb{R}^4$. Thus we can define a map $\widetilde{g}\colon D^4 \to D^4$ by $\widetilde{g}(\lambda \cdot x) = \lambda \cdot g(x)$, and this map sends 0 to 0 and restricts to g on S^3. We claim that $\mathrm{id} \coprod \widetilde{g}$ descends to a well defined map $X \to \mathbb{CP}^2$. It suffices to check that the identifications are preserved.

If $x \in S^3$, then in X, x is identified in X to $f(x)$, which maps under id_{S^2} to $f(x) = h \circ g(x)$. On the other hand, $\widetilde{g}(x) = g(x)$ is identified in $\mathbb{CP}^2$ to $h(g(x)) = f(x)$. So this shows we get a well defined map $\varphi\colon X \to \mathbb{CP}^2$.

Finally, let α' be the usual generator of $H^2(\mathbb{CP}^2)$ (coming from the standard orientation on $\mathbb{CP}^1 = S^2 \subset \mathbb{CP}^2$). Since φ was defined using the identity map on S^2 (which is the 2-skeleton of both X and $\mathbb{CP}^2$), $\varphi^*(\alpha') = \alpha$. And since φ was defined using $\widetilde{g}$ on the 4-cell, $\varphi^*\colon H^4(\mathbb{CP}^2) \to H^4(X)$ can be identified with the map induced by $\widetilde{g}$ on $H^4(D^4, S^3) \cong \mathbb{Z}$. But we have a commutative diagram

$$\begin{array}{ccc} H^3(S^3) & \xrightarrow{\ \partial\ } & H^4(D^4, S^3) \\ {\scriptstyle g^*}\downarrow & & {\scriptstyle \widetilde{g}^*}\downarrow \\ H^3(S^3) & \xrightarrow{\ \partial\ } & H^4(D^4, S^3). \end{array}$$

Since g has degree k, that means φ^* sends the generator $(\alpha')^2$ of $H^4(\mathbb{CP}^2)$ to k times the corresponding generator β of $H^4(X)$. But then

$$\alpha^2 = (\varphi^*(\alpha'))^2 = \varphi^*({\alpha'}^2) = k\beta$$

and $m = k$.