Algebraic Topology
(Mathematics 734, Prof. Rosenberg)
Mid-Term Examination
Friday, March 19, 2004

Instructions. Answer each question in your exam booklet. The point value of each problem is indicated. The exam is worth a total of 100 points. In problems with multiple parts, whether the parts are related or not, the parts are graded independently of one another. Be sure to go on to subsequent parts even if there is some part you cannot do.

You are allowed to appeal to "standard theorems" proved in class or in the textbook, but if you do so, it's your responsibility to state clearly exactly what you're using and why it applies.

1. (30 points) **Short-Answer Problems.** Give brief definitions or statements (no proofs) for the following terms.
   (a) The singular chain complex $C_\bullet(X)$ of a topological space X.
   (b) The excision property (for an axiomatic homology theory $H_\bullet$).
   (c) The Generalized Jordan Curve Theorem.

2. (20 points) Let M a topological n-manifold (a locally compact Hausdorff space locally homeomorphic to $\mathbb{R}^n$). Use excision to prove that if $x \in M$, $H_j(M, M \smallsetminus \{x\}) \cong \mathbb{Z}$ if $j = n$, 0 otherwise.

3. (50 points) Let X be a CW-complex with exactly 3 cells: a 0-cell, an n-cell, and an m-cell, where $0 < n < m$.
   (a) If $m \geq n+2$, show that $H_j(X)$ is $\cong \mathbb{Z}$ in dimensions 0, n, and m, and 0 otherwise.
   (b) If $m = n + 1$, show that there are three possibilities for the homology: either X is acyclic, or else $H_n(X) \cong H_{n+1}(X) \cong \mathbb{Z}$, or else $H_{n+1}(X) = 0$ and $H_n(X)$ is finite cyclic but non-trivial.
   (c) Give concrete examples to show that all three cases in (b) can occur.
   (d) Show that every homeomorphism $f\colon X \to X$ has a fixed point. (You may assume X is an ENR. Show that $L_{\mathbb{Q}}(f)$ is odd and use the Lefschetz Fixed-Point Theorem.)
   (e) Deduce from (d) that if G is a group (with more than one element), then G cannot act freely on $\mathbb{CP}^2$ or on $\mathbb{RP}^2$.