DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
January, 2004

Probability (Ph. D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (**NOT YOUR NAME**) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

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1. X, Y are independent exponentially distributed with $\lambda = 1$. Prove that

$$U = exp\{-2\min(X,Y)\}$$

is uniformly distributed in $[0,1]$.

2. X_n and Y_n are sequences of random variables. Suppose $X_n \to X$ in distribution as $n \to \infty$, and assume that for any finite c,

$$\lim_{n\to\infty} P(Y_n > c) = 1$$

Show that we also have,

$$\lim_{n\to\infty} P(X_n + Y_n > c) = 1$$

3. Suppose that $V_i,\ i = 1, 2, \ldots,$ are iid Unif$[-1, 1]$ random variables.

(a) Show that $\prod_{i=1}^{n} \left(1 + \frac{2}{\sqrt{n}} V_i\right)$ converges in distribution as $n \to \infty$, and find the limiting distribution exactly.

(b) Show that $\prod_{k=1}^{n} \left(1 + k^{-2/3} V_k\right)$ converges almost surely, as $n \to \infty$.

4. You are engaged in an infinite sequence of independent trials conducted under identical conditions. On any given trial the events A, B are mutually exclusive with fixed probabilities $P(A), P(B)$, respectively.

a. What is the probability that A will occur before B?

b. In repeated independent tossings of a pair of fair dice, what is the probability that the sum of 3 will appear before the sum of 7?

Note: The "sum of 3" and "sum of 7" refer to the total number of dots showing up on the faces of the two dice tossed in each trial, and that the 6 faces of each die have 1 through 6 dots on them.

5. Let $X_1, X_2, ...$ be iid Bernoulli random variables and let $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

(a) Show that for $p \neq 1/2$ we have the convergence in distribution

$$\sqrt{n}[Y_n(1 - Y_n) - p(1 - p)] \to \mathrm{N}[0, (1 - 2\mathrm{p})^2\mathrm{p}(1 - \mathrm{p})]$$

(b) Obtain the asymptotic distribution of $Y_n(1 - Y_n) - 1/4$ when $p = 1/2$.

6. Let $\{\epsilon_t\}$, $t = 0, \pm 1, \pm 2, \cdots$, be a sequence of uncorrelated real-valued random variables with mean zero and variance σ_ϵ^2 (i.e. white noise) and define a real-valued weakly stationary stochastic process $\{Z_t\}$ by the stochastic difference equation

$$Z_t = \phi_1 Z_{t-1} + \epsilon_t, \qquad t = 0, \pm 1, \pm 2, \cdots$$

where $|\phi_1| < 1$.

a. Prove that the partial sums

$$\sum_{j=0}^{n} \phi_1^j \epsilon_{t-j}$$

converge to Z_t in mean square as $n \to \infty$.

b. Obtain $E[Z_t]$, $E[Z_t^2]$, and $\mathrm{Cov}[Z_t, Z_{t-k}]$, $k = 0, \pm 1, \pm 2, \cdots$.

c. Describe the behavior of $\{Z_t\}$ in the three cases when ϕ_1 is 0, positive, and negative.