DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
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Probability (Ph. D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (**NOT YOUR NAME**) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

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1. Let $X_1, X_2, \ldots$ be i.i.d. r.v.'s with the common probability density function $f(x) = \frac{1}{2}e^{-|x|}, -\infty < x < \infty$. Let the r.v. B_n be given by

$$B_n = \frac{\Sigma_{m=1}^n X_m}{\sqrt{\Sigma_{m=1}^n |X_m|}}.$$

Prove that B_n converges in distribution, and find the limiting distribution as n tends to ∞.

2. Let $X_1, X_2, \ldots$ be i.i.d. uniform on $(0, 1)$, $S_n = X_1 + X_2 + \ldots + X_n$, and $T = \min\{n : S_n > 1\}$.
(a) Calculate $P(T > n)$, and $E(T)$.
(b) Calculate $E(S_T)$.

3. Let F be the interarrival distribution which has a density function (with respect to Lebesgue measure) and a finite mean. Let $H(t)$ be the

probability that there are an even number of renewals in $(0, t]$, given an arrival at time 0, where interarrival times are assumed independent.
(a) Write a renewal equation for $H(t)$, in terms of F.
(b) Use a renewal theorem to find $\lim_{t\to\infty} H(t)$.

4. A spider hunting a fly moves between locations 1 and 2 according to a Markov chain with transition matrix P,

$$P_{11} = P_{22} = 0.7, \qquad P_{12} = P_{21} = 0.3,$$

starting in location 1. The fly, unaware of the spider, starts in location 2 and moves according to a Markov chain with transition matrix Q,

$$Q_{11} = Q_{22} = 0.4, \qquad Q_{12} = Q_{21} = 0.6.$$

The spider catches the fly and the hunt ends at the first time when they occupy the same location. The progress of the hunt, except for knowing the location where it ends, can be described by a three-state Markov chain with a single absorbing state representing the end of the hunt, and the other two states exactly representing the spider and fly at distinct locations.
(a) Define the three states precisely, and obtain the transition matrix for this three-state Markov chain.
(b) Find the average duration of the hunt.
(c) Find the probability that at time n the spider and fly are both at their initial locations with the hunt still in progress.

5. Let X be a positive r.v. with finite mean. Let $Y_{i,j},\ i,\ j \geq 1$ be r.v.'s and

$$X_{m,n} \ = \ E[\, X \mid Y_{i,j},\ i \leq m,\ j \leq n].$$

(a) Prove that $(X_{m,n},\ m, n \geq 1)$ is a uniformly integrable family of r.v.'s.
(b) Explain why the r.v.'s $X_{m,n}$ converge for each fixed n as $m \to \infty$ and describe exactly the limiting r.v.

6. Let $(X_n)_{n\geq 1}$ be a positive martingale. Prove from first principles (*not* by directly citing a theorem) that for each $n \geq 1$ and $a > 0$,

$$P(\max_{k\leq n} X_k \geq a) \leq \frac{E(X_n)}{a}.$$