DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
AUGUST, 2002

Probability (Ph. D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (**NOT YOUR NAME**) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

---

1. A r.v. Y with pdf $g(x)$ is called stochatically bigger than a r.v. X with pdf $f(x)$ if $P(Y > u) \geq P(X > u)$ for all $u \in R$. Prove that if the likelihood ratio $g(x)/f(x)$ is monotone increasing, then Y is stochastically bigger than X.

2. The lifetimes of n computer systems are assumed to be independent and exponentially distributed with mean 1.
(a) Calculate the pdf of L, the life time of the system that survives the longest.
(b) Show that the times between failures are independent.

3. Let $X_1, X_2, \ldots$ be positive. Prove that if X_n converges to X in probability, and $E(X_n)$ converges to $E(X)$, then X_n converges to X in L^1, as n tends to ∞.

4. Pick $(2n+1)$ numbers at random in $(0, 1)$, i.e., the numbers are independent and uniformly distributed. Let V_{n+1} be the $(n+1)$th largest number, and $Y_n = \sqrt{2n}(2V_{n+1} - 1)$.
(a) Calculate the pdf of V_{n+1}.
(b) Prove that Y_n converges in distribution, and find the limit distribution.

5. Consider a population in which each member acts independently, and gives birth at exponential rate 1. That is, each member gives birth to one new member after an exponential waiting time with mean 1, independently of all other members. This birth mechanism applies to each new member, and all exponential waiting times are independent. Suppose that no one ever dies. Let X_t represent the population size at time t, and $X_0 = 1$. Assume that the age of the member at time 0 is 0. Prove that the expected sum of the ages at time t is $e^t - 1$.

6. Consider successive flips of a fair coin. Use optional stopping theorems of martingales to compute the expected number of flips until the following sequences appear.
(a) H
(b) HT