Probability (Ph. D. Version)

Instructions to the Student

1. Answer all six questions. Each will be graded from 0 to 10.
2. Use a different booklet for each question. Write the problem number and your code number (NOT YOUR NAME) on the outside cover.
3. Keep scratch work on separate pages in the same booklet.
4. If you use a “well known” theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

1. Let $X_1, X_2, \ldots$ be an infinite sequence of independent identically distributed random variables having double exponential distribution with density
   \[ f(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < +\infty. \]
   Prove that with probability one,
   \[ \limsup_{n \to \infty} \frac{|X_n|}{\ln n} = 1. \]

2. Let $\{X_n, n = 1, 2, \ldots\}$ be a random process with constant mean $E(X_n) = \mu$ and covariance $R(n-m) = \text{cov}(X_n, X_m)$ depending only on $n-m$.
   Prove that if $R(n) \to 0$, $n \to \infty$, then $(1/n) \sum_1^n X_i \to \mu$ in probability as $n \to \infty$. 


3. Let $X_1, X_2, \ldots$ be a sequence of independent random variables, $X_n$ having a uniform distribution on $[-n, n]$, $n = 1, 2, \ldots$. Prove that the Central Limit Theorem (CLT) holds: for $S_n = X_1 + \ldots + X_n$, $(S_n - E(S_n)) / \sqrt{\text{var}(S_n)}$ converges in distribution to $Z \sim N(0, 1)$ as $n \to \infty$.

4. Let $\xi_1, \xi_2, \xi_3$ be random variables with finite second moments. Prove that the relations

$$E(\xi_1 | \xi_2) = \xi_2, \ E(\xi_2 | \xi_3) = \xi_3, \ E(\xi_3 | \xi_1) = \xi_1$$

imply that with probability one $\xi_1 = \xi_2 = \xi_3$.

5. Let $X$ be a random variable with symmetric distribution. Prove the following properties of its characteristic function $f(t) = E(e^{itX})$:

(i) $1 + f(2t) \geq 2[f(t)]^2$,
(ii) if $E|X| < \infty$, then $f(t)$ satisfies the Lipschitz condition, i.e.,

$$|f(t + \Delta) - f(t)| \leq C|\Delta|$$

for some $C > 0$ and all $t, \Delta$.

6. For a sequence of random variables $X_1, X_2, \ldots$ set $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ (the $\sigma$-algebra generated by $X_1, \ldots, X_n$).

(i) Prove that if $(X_n, \mathcal{F}_n)$, $n = 1, 2, \ldots$ is a martingale with $E|X_n|$ finite for all $n$, the sequence of variances $\sigma_n^2 = \text{var}(X_n), n = 1, 2, \ldots$ is nondecreasing. You must consider the cases of both finite and infinite variances.

(ii) What are necessary and sufficient conditions for $\sigma_N^2 = \sigma_{N+1}^2 = \sigma_{N+2}^2 = \ldots$ for some $N$?