DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
August, 2003

Probability (Ph. D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (**NOT YOUR NAME**) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

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1. Let $X_1, X_2, \ldots$ be an infinite sequence of independent identically distributed random variables having double exponential distribution with density

$$f(x) = (1/2)e^{-|x|},\ -\infty < x < +\infty.$$

Prove that with probability one,

$$\limsup_{n\to\infty} \frac{|X_n|}{\ln n} = 1.$$

2. Let $\{X_n,\, n = 1, 2, \ldots\}$ be a random process with constant mean $E(X_n) = \mu$ and covariance $R(n-m) = \mathrm{cov}(X_n, X_m)$ depending only on $n - m$.
Prove that if $R(n) \to 0$, $n \to \infty$, then $(1/n)\sum_1^n X_i \to \mu$ in probability as $n \to \infty$.

3. Let $X_1, X_2, \ldots$ be a sequence of independent random variables, X_n having a uniform distribution on $[-n, n]$, $n = 1, 2, \ldots$. Prove that the Central Limit Theorem (CLT) holds: for $S_n = X_1 + \ldots + X_n$, $(S_n - E(S_n))/\sqrt{\mathrm{var}(S_n)}$ converges in distribution to $Z \sim N(0, 1)$ as $n \to \infty$.

4. Let ξ_1, ξ_2, ξ_3 be random variables with finite second moments.
Prove that the relations

$$E(\xi_1|\xi_2) = \xi_2,\ E(\xi_2|\xi_3) = \xi_3,\ E(\xi_3|\xi_1) = \xi_1$$

imply that with probability one $\xi_1 = \xi_2 = \xi_3$.

5. Let X be a random variable with symmetric distribution. Prove the following properties of its characteristic function $f(t) = E(e^{itX})$:
(i) $1 + f(2t) \geq 2[f(t)]^2$,
(ii) if $E|X| < \infty$, then $f(t)$ satisfies the Lipschitz condition, i.e.,

$$|f(t + \Delta) - f(t)| \leq C|\Delta| \text{ for some } C > 0 \text{ and all } t, \Delta.$$

6. For a sequence of random variables $X_1, X_2, \ldots$ set $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ (the σ-algebra generated by $X_1, \ldots, X_n$).
(i) Prove that if $(X_n, \mathcal{F}_n)$, $n = 1, 2, \ldots$ is a martingale with $E|X_n|$ finite for all n, the sequence of variances $\sigma_n^2 = \mathrm{var}(X_n), n = 1, 2, \ldots$ is nondecreasing. You must consider the cases of both finite and infinite variances.
(ii) What are necessary and sufficient conditions for $\sigma_N^2 = \sigma_{N+1}^2 = \sigma_{N+2}^2 = \ldots$ for some N?