Probability (Ph. D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (NOT YOUR NAME) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a “well known” theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

1. $X, Y$ are independent exponentially distributed with $\lambda = 1$. Derive the distribution of $U = \exp\{-2 \min(X, Y)\}$.

2. $X, Y$ are independent $N(0, 1)$ random variables.
   (a) Prove that $X + Y$ and $X - Y$ are independent.
   (b) Calculate $E(X + 2Y \mid X + Y)$.
   (c) Compute $E(Y \mid Y > 0)$. 


3. Suppose both \( X_1, \ldots, X_m \) and \( Y_1, \ldots, Y_n \) are iid sequences from the Exponential(1) distribution. Define \( \bar{X} = \frac{1}{m} \sum_{i=1}^{m} X_i \), \( \bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i \). Let

\[
B_{m,n} = \frac{m\bar{X}}{m\bar{X} + n\bar{Y}}
\]

where \( m/(m + n) \to \alpha \) as \( m, n \to \infty \).

For \( 0 < \alpha < 1 \), derive the asymptotic distribution as \( m, n \to \infty \) of

\[
\frac{\sqrt{m + n} \left( B_{m,n} - \frac{m}{m+n} \right)}{\sqrt{\alpha(1-\alpha)}}
\]

4. Let \( S_n \) be the number of successes in \( n \) Bernoulli trials with success probability \( p = 1/2 \). Prove that

(i) \( \lim_{n \to \infty} P \{ \max_{1 \leq k \leq n} |S_k - \frac{k}{2}| > \sqrt{n \ln n} \} = 0 \)

(ii) \( \lim_{n \to \infty} \inf P \{ \max_{1 \leq k \leq n} |S_k - \frac{k}{2}| > \sqrt{n} \} > 0 \)

5. Consider the Markov chain \( X_n \) with two states 0 and 1, and transition matrix

\[
P = \begin{pmatrix}
\frac{1}{4} & \frac{3}{4} \\
\frac{3}{4} & \frac{1}{4}
\end{pmatrix}
\]

Determine the limit

\[
\lim_{n \to \infty} P(X_n = 0, X_{n+1} = 0)
\]

6. Let \( U_1, U_2, \ldots \) be independent Uniform(0,1) random variables. Define a random variable \( X \) by

\[
X + 1 = \min \left\{ n : \prod_{i=1}^{n} U_i < e^{-\lambda} \right\}, \quad \text{where} \quad \prod_{i=1}^{0} U_i \equiv 1.
\]

Show that the distribution of \( X \) is Poisson(\( \lambda \)).