DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
January, 2004

Probability (Ph. D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (**NOT YOUR NAME**) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

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1. X, Y are independent exponentially distributed with $\lambda = 1$. Derive the distribution of

$$U = \exp\{-2\min(X, Y)\}.$$

2. X, Y are independent $\mathrm{N}(0, 1)$ random variables.
(a) Prove that $X + Y$ and $X - Y$ are independent.
(b) Calculate $\mathrm{E(X + 2Y \mid X + Y)}$.
(c) Compute $\mathrm{E(Y \mid Y > 0)}$.

3. Suppose both $X_1, ..., X_m$ and $Y_1, ..., Y_n$ are iid sequences from the Exponential(1) distribution. Define $\overline{X} = \frac{1}{m}\sum_{i=1}^m X_i$, $\overline{Y} = \frac{1}{n}\sum_{i=1}^n Y_i$. Let

$$B_{m,n} = \frac{m\overline{X}}{m\overline{X} + n\overline{Y}}$$

where $m/(m+n) \to \alpha$ as $m, n \to \infty$.
For $0 < \alpha < 1$, derive the asymptotic distribution as $m, n \to \infty$ of

$$\frac{\sqrt{m+n}\left(B_{m,n} - \frac{m}{m+n}\right)}{\sqrt{\alpha(1-\alpha)}}$$

4. Let S_n be the number of successes in n Bernoulli trials with success probability $p = 1/2$. Prove that

(i) $\lim_{n\to\infty} P\left\{\max_{1\le k\le n}\left|S_k - \frac{k}{2}\right| > \sqrt{n\ln n}\right\} = 0$

(ii) $\liminf_{n\to\infty} P\left\{\max_{1\le k\le n}\left|S_k - \frac{k}{2}\right| > \sqrt{n}\right\} > 0$

5. Consider the Markov chain X_n with two states 0 and 1, and transition matrix

$$\mathbf{P} = \begin{pmatrix} 1/4 & 3/4 \\ 3/4 & 1/4 \end{pmatrix}$$

Determine the limit

$$\lim_{n\to\infty} P(X_n = 0, X_{n+1} = 0)$$

6. Let $U_1, U_2, ...$ be independent Uniform(0,1) random variables. Define a random variable X by

$$X + 1 = \min\left\{n : \prod_{i=1}^n U_i < e^{-\lambda}\right\}, \quad \text{where} \quad \prod_{\mathrm{i=1}}^{0} \mathrm{U_i} \equiv 1.$$

Show that the distribution of X is Poisson(λ).