DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
August, 2011

Probability (M. A. Version)

Instructions to the Student

a. Answer all six questions Each will be graded from 0 to 10.

b. Use a different booklet for each question Write the problem number and your code number (**NOT YOUR NAME**) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

---

1. Let X and Y be independent Poisson random variables with parameters λ_1 and λ_2, respectively. Calculate $\mathrm{E}(X^2 + Y^2 | X + Y = n)$.

2. There are two hunters waiting for geese to fly by. When a flock of geese appears, each hunter aims at a randomly chosen goose independently of the other hunter and shoots it down (the hunters never miss). Both hunters may hit the same goose, and a goose is shot down if it is hit by at least one hunter). What is the expected number of geese that will be shot down if the size of the flock is a random variable uniformly distributed on $\{1, 2, \ldots, 10\}$.

3. Mike starts watching a match between players $\mathcal{A}$ and $\mathcal{B}$, assuming that it is equally likely that $\mathcal{A}$ or $\mathcal{B}$ is the better player. If $\mathcal{A}$ is the better player, the probability that $\mathcal{A}$ wins a set is .75 independently of the outcomes of the other sets If $\mathcal{B}$ is the better player, the probability that $\mathcal{B}$ wins a set is 75 independently of the outcomes of the other sets.

After three sets, the score is 2 sets to 1 with A leading What is the probability, in Mike's opinion, that A will go on to win the match (which is played until either player wins 3 sets)?

4. There are 5 points a_1, a_2, a_3, a_4 and a_5 on the circle (listed in the clockwise order). There is a frog at a_1 and a grasshopper at a_2 at time $t = 0$ After one second, and at the end of each second thereafter, both the frog and the grasshopper jump independently, with probability 1/2 to each of their respective two nearby points (e.g., at $t = 1$ the frog may jump to a_2 or a_5) When they end up at the same point, the frog will eat the grasshopper.
(a) Find the expectation of the time until the frog eats the grasshopper.
(b) What is the probability that the grasshopper survives for 4 seconds or longer?

5. Let $X_1, \ldots, X_n, \ldots$ be independent random variables distributed uniformly on $[0, 1]$. Find a number a such that

$$\lim_{n\to\infty} P\left(\frac{X_1^2 + \ldots + X_n^2 - na}{\sqrt{X_1 + \ldots + X_n}} \leq z\right)$$

exists and differs from zero and one for each $z \in \mathbb{R}$.

6. Let $X_1, \ldots, X_n, \ldots$ be independent random variables. Assume that for any $n \in \mathbb{N}$ the law of X_n is exponential of parameter c^n, for some $c > 0$. Define

$$Y_n = \min(X_1, \ldots X_n).$$

Prove that the sequence $\{Y_n\}_{n\in\mathbb{N}}$ is convergent almost surely (i. e., with probability 1) and determine the limit, for each $c > 0$.