DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
August, 2011

Probability (Ph. D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (**NOT YOUR NAME**) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

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1. Let $X_1, \dots, X_n, \dots$ be independent Gaussian random variables, having mean a and variance σ^2. Define by recurrence the sequence

$$Y_0 = x \in \mathbb{R}, \qquad Y_{n+1} = \lambda Y_n + X_{n+1},$$

for some $\lambda \in (-1, 1)$. Prove that the sequence $\{Y_n\}_{n \in \mathbb{N}}$ is convergent in distribution and determine the limiting distribution.

2. Let Y_n be independent exponential random variables with parameter $\lambda = 1$ and let Z_n be independent $N(0,1)$ random variables.
Find numerical sequences a_n and b_n such that both $\max(Y_1, \dots, Y_n)/a_n$ and $\max(Z_1, \dots, Z_n)/b_n$ converge in probability to 1 as $n \to \infty$.

3. Mike starts watching a match between players $\mathcal{A}$ and $\mathcal{B}$, assuming that it is equally likely that $\mathcal{A}$ or $\mathcal{B}$ is the better player. If $\mathcal{A}$ is the better player, the probability that $\mathcal{A}$ wins a set is .75 independently of the outcomes of the

other sets. If $\mathcal{B}$ is the better player, the probability that $\mathcal{B}$ wins a set is 75 independently of the outcomes of the other sets.
After three sets, the score is 2 sets to 1 with $\mathcal{A}$ leading. What is the probability, in Mike's opinion, that $\mathcal{A}$ will go on to win the match (which is played until either player wins 3 sets)?

4. On the last day of each month a king will add up to 1 kg of gold to his treasury. The actual amount he'll add is uniformly distributed on $[0, 1]$ and the amounts on different months are independent. Currently his treasury is empty. Assume that today is the first day of a month

(a) What is the expected number of months (counting the current month as a full one) that will elapse before the treasury has at least 1 kg of gold?

(b) What is the expected amount of gold that will be in the treasury on the day after the amount reaches 1 kg?

(c) What is the expected number of months (counting the current month as a full one) that will elapse before the treasury has at least 2 kg of gold?

5. Let $X_1, \ldots, X_n, \ldots$ be independent random variables. Assume that for each n the random variable X_n is distributed uniformly on $[0, n]$.

(a) Fine a sequence a_n such that $(X_1^2 + \ldots + X_n^2)/a_n \to 1$ in probability

(b) Find sequences b_n and c_n such that

$$\lim_{n\to\infty} \mathrm{P}\left(\frac{X_1 + \ldots + X_n - b_n}{c_n\sqrt{X_1^2 + \ldots + X_n^2}} \le z\right)$$

exists and differs from zero and one for each $z \in \mathbb{R}$ and identify the limit for each z.

6. Let X_n, $n \ge 0$, be a process adapted to a filtration $\mathcal{F}_n$. Prove that $(X_n, \mathcal{F}_n)$, $n \ge 0$, is a martingale if and only if for any bounded $\mathcal{F}_n$-stopping time τ it holds $\mathrm{E}X_\tau = \mathrm{E}X_0$.