DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
JANUARY, 2001

Statistics (Ph. D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (**NOT YOUR NAME**) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

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1. Let $\mathcal{P} = \{P_\theta, \theta \in \Theta\}$ be a family of distributions on a measurable space $(\mathcal{X}, \mathcal{A})$, and let $T : (\mathcal{X}, \mathcal{A}) \to (\mathcal{T}, \mathcal{B})$ be a *statistic*. Let $Q_\theta(B) = P_\theta(T^{-1}B)$, $B \in \mathcal{B}$, $\mathcal{Q} = \{Q_\theta, \theta \in \Theta\}$.

Prove that if T is sufficient for $\mathcal{P}$ and $S : (\mathcal{T}, \mathcal{B}) \to (\mathcal{S}, \mathcal{C})$ is sufficient for $\mathcal{Q}$ then S is sufficient for $\mathcal{P}$.

2. Let $X_1, \ldots, X_n$ be a random sample from a population with pdf

$$f(x - \theta) = \frac{1}{3\sqrt{2\pi}}(x - \theta)^4 \exp\{-(x - \theta)^2/2\}, \quad \theta \in R$$

Compute the Fisher information on θ contained in the sample and find the efficiency of $\bar{X} = (X_1 + \cdots + X_n)/n$ as an estimator for θ.

3. Two identical coins with $P(H) = 1 - P(T) = p$ are tossed independently n times. Let n_0, n_1, n_2 denote the number of times when both coins show T, one shows T and the other shows H, both coins show H, respectively.

Find whether n_2/n is an admissible estimator for p^2 assuming quadratic loss.

4. Let $X_1, \ldots, X_n$ be of the form

$$X_i = \theta a_i + \epsilon_i, \quad i = 1, ..., n$$

where $a_1, ..., a_n$ are known constants, θ is a parameter to be estimated, and $\epsilon_1, ..., \epsilon_n$ are independent random variables with mean 0 and variance $\mathrm{Var}\,(\epsilon_i) = \sigma_i^2 < \infty$.

Find the minimum variance linear unbiased estimator of θ and compute its variance.

Assuming $\epsilon_1, \epsilon_2, ...$ are normally distributed, study the consistency of the aforementioned estimator as $n \to \infty$.

5. Let the prior distribution of a parameter θ be Beta(α, β) with pdf,

$$\pi(\theta; \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}, \quad \theta \in (0, 1)$$

where $\alpha > 0, \beta > 0$ and Γ is the gamma function. Given θ, the observations $X_1, \ldots, X_n$ are independent binary random variables with

$$P(X_i = 1|\theta) = 1 - P(X_i = 0|\theta) = \theta$$

(i) Find the Bayes estimator of θ assuming quadratic loss. (ii) Are $X_1, \ldots, X_n$ independent ?

6. Let $(X_1', ..., X_{n_1}')$, $(X_1'', ..., X_{n_2}'')$ be two independent random samples from normal populations $\mathrm{N}(\mu_1, \sigma^2)$ and $\mathrm{N}(\mu_2, \sigma^2)$, respectively, with unknown σ^2. Develop a t-test at significance level α for testing the null hypothesis $\mathrm{H}_0 : \mu_2 = 2\mu_1$.