DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
JANUARY, 2003

Statistics (Ph. D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (**NOT YOUR NAME**) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

---

1. Suppose both $X_1, ..., X_m$ and $Y_1, ..., Y_n$ are random samples from the Exponential(1) distribution. With sample means $\overline{X}, \overline{Y}$, let

$$B_{m,n} = \frac{m\overline{X}}{m\overline{X} + n\overline{Y}}$$

where $m/(m+n) \to \alpha$ as $m, n \to \infty$.

a. What is the distribution of $B_{m,n}$?

b. For $0 < \alpha < 1$, derive the asymptotic distribution as $m, n \to \infty$ of

$$\frac{\sqrt{m+n}\left(B_{m,n} - \frac{m}{m+n}\right)}{\sqrt{\alpha(1-\alpha)}}$$

Explain your derivation carefully.

2. Two independent and identically distributed measurements X_1, X_2 are made from a parametric density with parameter $\theta > 0$

$$f(x, \vartheta) = \begin{cases} \frac{1}{2}\,\vartheta\, e^{-\vartheta x} & if \quad x \geq 0 \\ \frac{1}{2}\,\vartheta^{-1}\, e^{x/\vartheta} & if \quad x < 0 \end{cases}$$

(a) Find the MLE of ϑ in terms of X_1, X_2.

(b) Find the most powerful hypothesis test of $\mathrm{H}_0 : \vartheta = 1$ versus the alternative $\mathrm{H}_1 : \vartheta = 2$ of size 0.10, giving an equation to determine the rejection cutoff but not solving it explicitly. Justify that your test either is or is not UMP versus $\mathrm{H}_\mathrm{A} : \vartheta > 1$.

3. Let $X_1, ..., X_n$ be a random sample from a $\mathrm{N}(\mu, 1)$ population, and let the prior pdf of μ be $\mathrm{N}(0, 1)$. Assuming quadratic loss, obtain the Bayes estimator and the corresponding minimum Bayes risk.

4. Let $X_1, ..., X_n$ be iid Poisson(λ), and let $\overline{X}$ and S^2 be the sample mean and sample variance, respectively.

a. Show that $\overline{X}$ is the best unbiased estimator of λ.

b. Show that $\mathrm{E}(\mathrm{S}^2 \mid \overline{\mathrm{X}}) = \overline{\mathrm{X}}$.

c. Use (b) to demonstrate explicitly that $\mathrm{Var}(\mathrm{S}^2) > \mathrm{Var}(\overline{\mathrm{X}})$.

5. Let $X_1, ..., X_n$ be a random sample from a pdf $f(x, \theta)$ where θ is k dimensional, and suppose $\hat{\theta}$ is the maximum likelihood estimator of θ.

a. Under appropriate regularity conditions, show that

$$\sqrt{n}(\hat{\theta} - \theta_0) \to \mathrm{N}(0, \mathrm{I}_1^{-1}(\theta_0))$$

where θ_0 is the true parameter and $\mathrm{I}_1(\theta_0)$ is the Fisher information matrix evaluated at θ_0.

b. Illustrate the result in terms of $X_1, ..., X_n$ from Poisson(λ) where λ is a scalar parameter.

6. Let $X_1, ..., X_n$ denote the times to failure of n pieces of equipment, where $X_1, ..., X_n$ are iid Exponential(λ). Consider the hypothesis

$$\mathrm{H}_0 : \ 1/\lambda = \mu \leq \mu_0$$

a. Show that the test $\overline{X} \geq \mu_0 x(1-\alpha)/2n$, where $x(1-\alpha)$ is the $(1-\alpha)$th quantile of the χ^2_{2n} distribution, is a size α test.

b. Derive the power in terms of the χ^2_{2n} distribution.

c. Approximate the power by appealing to the central limit theorem, and draw the graph of the approximate power function.