Statistics (Ph. D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (NOT YOUR NAME) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a “well known” theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

1. Suppose both $X_1, ..., X_m$ and $Y_1, ..., Y_n$ are random samples from the Exponential(1) distribution. With sample means $\bar{X}, \bar{Y}$, let

$$B_{m,n} = \frac{m\bar{X}}{m\bar{X} + n\bar{Y}}$$

where $m/(m + n) \to \alpha$ as $m, n \to \infty$.

   a. What is the distribution of $B_{m,n}$?

   b. For $0 < \alpha < 1$, derive the asymptotic distribution as $m, n \to \infty$ of

$$\sqrt{m + n} \left( B_{m,n} - \frac{m}{m+n} \right) \over \sqrt{\alpha(1-\alpha)}$$

Explain your derivation carefully.
2. Two independent and identically distributed measurements \( X_1, X_2 \) are made from a parametric density with parameter \( \theta > 0 \)

\[
f(x, \theta) = \begin{cases} 
\frac{1}{2} \theta e^{-\theta x} & \text{if } x \geq 0 \\
\frac{1}{2} \theta^{-1} e^{x/\theta} & \text{if } x < 0 
\end{cases}
\]

(a) Find the MLE of \( \theta \) in terms of \( X_1, X_2 \).

(b) Find the most powerful hypothesis test of \( H_0 : \theta = 1 \) versus the alternative \( H_1 : \theta = 2 \) of size 0.10, giving an equation to determine the rejection cutoff but not solving it explicitly. Justify that your test either is or is not UMP versus \( H_A : \theta > 1 \).

3. Let \( X_1, ..., X_n \) be a random sample from a \( N(\mu, 1) \) population, and let the prior pdf of \( \mu \) be \( N(0, 1) \). Assuming quadratic loss, obtain the Bayes estimator and the corresponding minimum Bayes risk.

4. Let \( X_1, ..., X_n \) be iid Poisson(\( \lambda \)), and let \( \overline{X} \) and \( S^2 \) be the sample mean and sample variance, respectively.

   a. Show that \( \overline{X} \) is the best unbiased estimator of \( \lambda \).

   b. Show that \( E(S^2 \mid \overline{X}) = \overline{X} \).

   c. Use (b) to demonstrate explicitly that \( \text{Var}(S^2) > \text{Var}(\overline{X}) \).

5. Let \( X_1, ..., X_n \) be a random sample from a pdf \( f(x, \theta) \) where \( \theta \) is \( k \)-dimensional, and suppose \( \hat{\theta} \) is the maximum likelihood estimator of \( \theta \).

   a. Under appropriate regularity conditions, show that

\[
\sqrt{n}(\hat{\theta} - \theta_0) \to N(0, I^{-1}_1(\theta_0))
\]

where \( \theta_0 \) is the true parameter and \( I_1(\theta_0) \) is the Fisher information matrix evaluated at \( \theta_0 \).

   b. Illustrate the result in terms of \( X_1, ..., X_n \) from Poisson(\( \lambda \)) where \( \lambda \) is a scalar parameter.
6. Let $X_1, \ldots, X_n$ denote the times to failure of $n$ pieces of equipment, where $X_1, \ldots, X_n$ are iid Exponential($\lambda$). Consider the hypothesis

$$H_0 : 1/\lambda = \mu \leq \mu_0$$

a. Show that the test $\bar{X} \geq \mu_0 x(1 - \alpha)/2n$, where $x(1 - \alpha)$ is the $(1 - \alpha)$th quantile of the $\chi^2_{2n}$ distribution, is a size $\alpha$ test.

b. Derive the power in terms of the $\chi^2_{2n}$ distribution.

c. Approximate the power by appealing to the central limit theorem, and draw the graph of the approximate power function.