DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
JANUARY, 2004

Statistics (Ph. D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (**NOT YOUR NAME**) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

---

1. Let $\mathcal{P} = \{P_\theta\}$ be a family of distributions of a random element $X \in \mathcal{X} \subset R^n$ parametrized by a discrete parameter $\theta \in \{\theta_0, \theta_1, \theta_2\}$. Assuming P_θ is given by a density $p(x; \theta)$ which is everywhere positive on $\mathcal{X}$, develop the minimal sufficient statistic for θ.

2. Let $(x_1,\ x_2, \ldots,\ x_n)$ be a sample from a population X with density $e^{-(x-\theta)}$, $x \geq \theta$. Find the uniformly minimum variance unbiased estimator of the value of the parameter function $g(\theta) = P_\theta(1 < X < 2)$.

3. Given a parameter θ, $X_1,\ X_2, \ldots,\ X_n$ are independent identically distributed random variables with $E(X_i|\theta) = \theta$, $\text{var}(X_i|\theta) = \sigma^2$. The prior distribution of θ has known mean a and variance b^2. The linear Bayesian estimator of θ, by definition, minimizes the overall risk within the class of estimators of the form $c_0 + \sum_{i=1}^n c_i X_i$ where the constant coefficients $c_0, c_1, \ldots, c_n$ do not depend upon the data. Find the linear Bayesian estimator of θ with respect to the squared-error loss function.

4. Let $(x_1,\ x_2, \ldots,\ x_n)$ be a sample from a population with density

$$f(x;\theta) = \frac{1}{B(3,3)}(x-\theta)^2(1-x+\theta)^2,\ \theta \le x \le 1+\theta.$$

with θ as a parameter. Here $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$ is the beta function, $B(\alpha,\beta) = (\alpha-1)!(\beta-1)!/(\alpha+\beta-1)!$ for integer $\alpha > 0,\ \beta > 0$. Develop the method of moments estimator of θ and calculate its efficiency.

5. Let $(x_1,\ x_2,\ldots,x_m)$ and $(y_1,\ y_2,\ldots,y_n)$ be two independent samples from normal populations with parameters $(\mu_1,\ \sigma_1^2)$ and $(\mu_2,\ \sigma_2^2)$, respectively. Assuming $\sigma_1^2 = 4\sigma^2$, $\sigma_2^2 = \sigma^2$, σ^2 unknown, develop a test of a constant level α for all $\sigma^2 > 0$ of the null hypothesis $H_0 : \mu_1 = \mu_2$ versus the alternative $H_1 : \mu_1 \neq \mu_2$.

6. Let $(x_1, y_1), \ldots, (x_n, y_n)$ be a sample from a bivariate normal population with mean vector (θ_1, θ_2), regarded as a parameter and covariance matrix I, the identity matrix. For the maximum likelihood estimator $\hat{g}_n$ of the value of the parameter function $g(\theta_1,\theta_2) = (\theta_1-\theta_2)^2$, find the nondegenerate limiting distribution of $a_n(\hat{g}_n - g(\theta_1,\theta_2))$ as $n \to \infty$ for a properly chosen a_n.