DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
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Statistics (M. A. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (**NOT YOUR NAME**) on the outside cover

c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use

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1. (i) Let $p(x, y; \theta)$, $\theta \in \Theta$ be a family of joint pdf's of random vectors X, Y with θ as a parameter.
Assuming $p(x, y; \theta) > 0$ for all x, y, θ, show that if Y is sufficient for θ and X and Y are independent when $\theta = \theta_0 \in \Theta$, then X and Y are independent for all $\theta \in \Theta$.
(ii) Let (X, Y) be a random vector having bivariate normal distribution with

$$E(X) = \theta_1,\ E(Y) = \theta_2,\ \mathrm{var}(X) = \mathrm{var}(Y) = 1,\ \mathrm{corr}(X,\ Y) = \rho$$

with $(\theta_1,\ \theta_2)$ as a parameter (and ρ known).
Using the factorization theorem, show that if the parameter set is $\Theta = \{(\theta_1,\ \theta_2) : \theta_2 = \rho\theta_1\}$ then Y is sufficient for $(\theta_1,\ \theta_2)$.

2. Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be independent samples from populations with pdf's $f(x - \theta_1)$ and $f(x - \theta_2)$, respectively, with $f(x) = e^{-x}$, $x \geq 0$ and θ_1, θ_2 as parameters.
(i) Find the MLE $\hat{\Delta}_n$ of $\Delta = \theta_1 - \theta_2$ and calculate $E(\hat{\Delta}_n)$ and $\mathrm{var}(\hat{\Delta}_n)$.

(ii) Show that the limiting distribution of $n(\hat{\Delta}_n - \Delta)$ as $n \to \infty$ is double exponential, i. e., given by the pdf $(1/2)\lambda \exp\{-\lambda|x|\}$, $x \in (-\infty, +\infty)$ for some $\lambda > 0$.
(**Hint**: Using the moment generating function may simplify the calculations.)

3. Let $(x_1, \dots, x_n)$, $(y_1, \dots, y_n)$, $(z_1, \dots, z_n)$ be independent samples from exponential populations with densities

$$f(x; \lambda_1),\ f(y; \lambda_2),\ f(z; \lambda_3),$$

respectively, where $f(u; \lambda) = (1/\lambda)e^{-u/\lambda}$, $u > 0$ and $\lambda_i > 0$, $i = 1, 2, 3$ are parameters.
Construct the LR (Likelihood Ratio) test of level α for testing

$$H_0 : \lambda_1 = \lambda_2 = \lambda_3 \text{ vs } H_1 : \lambda_i \neq \lambda_j \text{ for some } i, j$$

(ii) Assuming $\lambda_2/\lambda_1 = c_1$, $\lambda_3/\lambda_1 = c_2$, show that the power function of the LR test depends only on c_1, c_2.

4. Given θ, $(x_1, \dots, x_n)$ is a sample from a population with pdf $e^{(x-\theta)}$, $x \leq \theta$ (notice that the distribution is concentrated on $(-\infty, \theta)$).
(i) Show that the family of pdf's

$$\pi(\theta; a,\ \lambda) = \lambda e^{-\lambda(\theta - a)},\ \theta \geq a$$

with $\lambda > 0$, $a \in (-\infty, +\infty)$ as parameters is a conjugate family of prior pdf's.
(ii) Assuming the prior pdf $\pi(\theta)$ belonging to the conjugate family, find the Bayes estimator of θ for the quadratic loss function $L(\tilde{\theta},\ \theta) = (\tilde{\theta} - \theta)^2$.

5. Let $(x_1, \dots, x_m)$ and $(y_1, \dots, y_n)$ be independent samples of sizes m and n from populations with pdf's

$$f(x; \lambda_1) = (1/2)\lambda_1^3 x^2 e^{-\lambda_1 x},\ x \geq 0$$

and

$$f(y; \lambda_2) = (1/6)\lambda_2^4 y^3 e^{-\lambda_2 y},\ y \geq 0,$$

respectively, with λ_1, λ_2 as parameters.

(i) Based on the sufficient statistics, construct a pivot for λ_1/λ_2.
(ii) Express the distribution of the pivot in terms of the F-distribution and construct a level $1-\alpha$ confidence interval for λ_1/λ_2.

6. Let $(x_1, \ldots, x_n)$ be a sample from a normal population $N(\mu, \sigma^2)$ with parameters μ, σ^2. The $100(1-\alpha)$-th population percentile $\eta_\alpha = \eta(\alpha)$ is a function of μ, σ^2, $\eta_\alpha = \eta_\alpha(\mu, \sigma^2)$.
(i) Find the MLE $\hat{\eta}_n(\alpha)$ of $\eta(\alpha)$ and calculate its variance.
(ii) Find the limiting distribution of $\sqrt{n}(\hat{\eta}_n(\alpha) - \eta(\alpha))$ as $n \to \infty$.