DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
August, 2011

Statistics (Ph. D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (**NOT YOUR NAME**) on the outside cover

c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

---

1. Let $\{p(x;\theta_1,\,\theta_2),\,(\theta_1,\,\theta_2)\in\Theta_1\times\Theta_2\}$ be a family of pdf's on a measurable space $(\mathcal{X},\,\mathcal{A})$ with $(\theta_1,\,\theta_2)$ as a parameter. Assume that there exists $\theta_1^*\in\Theta_1$ such that $T_1=T_1(X)$ is sufficient for the family $\{p(x;\theta_1^*,\,\theta_2),\,\theta_2\in\Theta_2\}$. Assume also that for **any** $\tilde{\theta}_2\in\Theta_2$, a statistic $T_2=T_2(X)$ is sufficient for the family $\{p(x;\theta_1,\,\tilde{\theta}_2),\,\theta_1\in\Theta_1\}$.
(i) Write the versions of the factorization theorem expressing the above properties.
(ii) Assuming $p(x;\theta_1,\,\theta_2)>0$ for all $x,\,\theta_1,\,\theta_2$, show that the statistic $T=(T_1,\,T_2)$ is sufficient for $(\theta_1,\,\theta_2)$ (i. e., for the family $\{p(x;\theta_1,\,\theta_2),$ $(\theta_1,\,\theta_2)\in\Theta_1\times\Theta_2\}$).

2. Let $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_n)$ be independent samples from populations with pdf's $f(x-\theta_1)$ and $f(x-\theta_2)$, respectively, with $f(x)=e^{-x}$, $x\ge 0$ and $\theta_1,\,\theta_2$ as parameters.
(i) Find the MLE $\hat{\Delta}_n$ of $\Delta=\theta_1-\theta_2$ and calculate $E(\hat{\Delta}_n)$ and $\mathrm{var}(\hat{\Delta}_n)$.
(ii) Find the normalizing constants $a_n>0$ and the explicit formula of the pdf of the nondegenerate limiting distribution of $a_n(\hat{\Delta}_n-\Delta)$ as $n\to\infty$.
(**Hint:** Using the moment generating function may simplify the calculations.)

3. Let $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$, $(z_1, \ldots, z_n)$ be independent samples from exponential populations with densities

$$f(x; \lambda_1),\ f(y; \lambda_2),\ f(z; \lambda_3),$$

respectively, where $f(u; \lambda) = (1/\lambda)e^{-u/\lambda}$, $u > 0$ and $\lambda_i > 0$, $i = 1, 2, 3$ are parameters.
(i) Construct the LR (Likelihood Ratio) test of level α for testing

$$H_0 : \lambda_1 = \lambda_2 = \lambda_3 \text{ vs } H_1 : \lambda_i \neq \lambda_j \text{ for some } i, j.$$

(ii) Show that the power function of the LR test depends only on $\lambda_2/\lambda_1, \lambda_3/\lambda_1$.

4. Given θ, $(x_1, \ldots, x_n)$ is a sample from a population with pdf $e^{x-\theta}$, $x \leq \theta$ (notice that the distribution is concentrated on $(-\infty, \theta)$)
(i) Construct the conjugate family of prior pdf's parameterized by a two-dimensional parameter.
(ii) Assuming the prior pdf $\pi(\theta)$ belongs to the conjugate family, find the Bayes estimator of θ for the quadratic loss function $L(\tilde{\theta}, \theta) = (\tilde{\theta} - \theta)^2$.

5. Let $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$ be independent samples of sizes m and n from populations with pdf's

$$f(x; \lambda_1) = (1/2)\lambda_1^3 x^2 e^{-\lambda_1 x},\ x \geq 0$$

and

$$f(y; \lambda_2) = (1/6)\lambda_2^4 y^3 e^{-\lambda_2 y},\ y \geq 0,$$

respectively, with λ_1, λ_2 as parameters.
(i) Based on the sufficient statistics, construct a pivot for λ_1/λ_2.
(ii) Express the distribution of the pivot in terms of the F-distribution and construct a level $1 - \alpha$ confidence interval for λ_1/λ_2.

6. Let $(x_1, \ldots, x_n)$ be a sample from a normal population $N(\mu, \sigma^2)$ with parameters μ, σ^2. The $100(1-\alpha)$-th population percentile $\eta_\alpha = \eta(\alpha)$ is a function of μ, σ^2, $\eta_\alpha = \eta_\alpha(\mu, \sigma^2)$
(i) Find the MLE $\hat{\eta}_n(\alpha)$ of $\eta(\alpha)$ and show that it is biased.
(ii) Find the limiting distribution of $\sqrt{n}(\hat{\eta}_n(\alpha) - \eta(\alpha))$ as $n \to \infty$