Graduate Complex Analysis **January 2007**

Problem solving in Complex Analysis is based on following model arguments. A model argument is a set outline of steps, and details are filled in for the specific application.

The model for Schwarz's Lemma is: divide by the comparison function, use the given boundary bound and Maximum Principle to conclude that the quotient is bounded, then restate the bound for the original function. The comparison function is selected to match the geometry of the domain of the function.

Cauchy's Theorem, the Cauchy Integral Formula, Cauchy's estimate, Local Mapping Principle, the Argument Principle and Residue Theorem are all examples with model arguments. Many proofs of theorems are applications of model arguments and so provide models for solving problems. Graduate Complex Analysis questions commonly involve only a *single* model argument. The *first step* in parsing a question is to *match up* the given information with the conditions (hypotheses) for a single model argument.

Graduate Complex analysis includes the following topics.

Basics

Vector calculus, topology of sets in $\mathbb{C}$ (compact, connected, simply connected, Jordan Curve Theorem), trigonometric form $re^{\iota\theta}$, powers & roots of complex numbers, logarithm, definition of arg and its principal branch, $\sin \pi z$ and $\tan \pi z$, uniform and absolute convergence of power series, including the *radius of convergence*.

Holomorphic functions

The complex derivative $f'(z)$, Cauchy-Riemann equations, zeros, poles and Cauchy's Theorem. Partial fraction decomposition. Geometry of Möbius transformations (mapping circle/lines to circle lines; symmetry about a line/circle; determined by three points.)

Cauchy Integral Formula

Winding number, Cauchy Integral Formula for $f^{(n)}(z)$, Cauchy's estimate $|f^{(n)}(a)| \leq Mn!r^{-n}$, Morera's Theorem, Liouville's Theorem, Residue Theorem (evaluations for rational functions, and with branches of $\log z$ and z^{α}.)

Local properties

Isolated singularities, removable conditions, Casorati-Weierstrass Theorem on essential singularities, open mappings, Local Representations when $f(z_0) = w_0$: $f(z) - w_0 = (z - z_0)^n k(z) = g((z - z_0)^n) = (h(z - z_0))^n$ with $k(z_0) \neq 0$, $g'(0) \neq 0$, $h'(0) \neq 0$.

Maximum Principle and Schwarz's Lemma

Facility with comparison functions for simple domains.

Argument/Winding Principle

The range of a holomorphic function contains those values for which (the image of a closed curve in the domain by) the function winds around the value. Rouché's Theorem. Fundamental Theorem of Algebra.

Harmonic functions

Definition, Mean Value Property, harmonic conjugate, Poisson Integral Formula, Schwarz Reflection.

Series of functions

Taylor and Laurant series, uniform and absolute convergence, infinite sums. (Not on the Qualifying Exam: infinite products, absolute convergence for $\prod_n (1 + a_n)$, Mittag-Leffler Theorem, $\frac{\pi^2}{\sin^2 \pi z}$, Riemann zeta $\zeta(s)$.)

Normal Families

Equicontinuous and normal families, the Riemann Mapping Theorem.

References

[1] Lars V. Ahlfors. *Complex analysis.* McGraw-Hill Book Co., New York, third edition, 1978. An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics.

[2] John B. Conway. *Functions of one complex variable*, volume 11 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1978.

[3] Walter Rudin. *Real and complex analysis.* McGraw-Hill Book Co., New York, third edition, 1987.