DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
January 2011
ALGEBRA (Ph.D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10

b. Use a different booklet for each question. Write the problem number and your **code number** (**not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

---

1. Let G be a finite group of order n There is a homomorphism $G \to S_n$, where $g \in G$ maps to the permutation π_g, where π_g permutes the n elements of G by $\pi_g(x) = gx$.
   (a) Show that π_g is an odd permutation if and only if g has even order and $[G :< g >]$ is odd
   (b) Show that if a Sylow 2-subgroup of G is non-trivial and cyclic then G has a subgroup H with $[G : H] = 2$

2. Let $n \geq 1$ and let B be an $n \times n$ complex matrix. Let M_n be the set of $n \times n$ complex matrices, regarded as a vector space of dimension n^2 over $\mathbf{C}$ Let

$$\phi_B : M_n \longrightarrow M_n, \qquad A \longmapsto AB.$$

   Then ϕ_B is a linear transformation of M_n
   (a) Show that if $m = kn$ for some k with $0 \leq k \leq n$, then there is a matrix B such that $\mathrm{Ker}(\phi_B)$ has dimension kn.
   (b) Let B be arbitrary. Show that the dimension of $\mathrm{Ker}(\phi_B)$ is a multiple of n

3. Let F be a field and suppose that the additive group of F is a finitely generated abelian group. Show that F is a finite field.

4. Let R be a commutative ring with 1.
   (a) Suppose that R contains a non-finitely generated ideal. Show that the set of non-finitely generated ideals contains an ideal I that is maximal for that set That is, I is not finitely generated, and if J is an ideal with $I \subsetneq J$, then J is finitely generated. (*Note:* You may use Zorn's Lemma)
   (b) Let I be as in Part (a). Suppose that there exist $x, y \in R$ with $xy \in I$ but $x, y \notin I$ Let $J = xR + I$. Show that there exist elements $g_1, \ldots, g_n \in I$ (not just in R) such that $x, g_1, \ldots, g_n$ generate J.
   (c) Let x, I be as in Part (b). Let $M = \{r \in R \mid rx \in I\}$. Show that there are $h_1, \ldots, h_m \in M$ that generate the ideal M.
   (d) Let the notation be as in Parts (a), (b), (c) Show that $g_1, \ldots, g_n, xh_1, \ldots, xh_m$ generate I. (This contradicts the choice of I and we conclude that I is prime.)
   (e) Show that if every prime ideal of R is finitely generated, then R is Noetherian

5. Let X_1, X_2, X_3, X_4 be variables. Let S_4 act on $L = \mathbf{Q}(X_1, X_2, X_3, X_4)$ by permuting the variables The fixed field is $K = \mathbf{Q}(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, where

$$\sigma_1 = X_1 + X_2 + X_3 + X_4, \quad \sigma_2 = X_1X_2 + X_1X_3 + X_1X_4 + X_2X_3 + X_2X_4 + X_3X_4,$$

$$\sigma_3 = X_1X_2X_3 + X_1X_2X_4 + X_1X_3X_4 + X_2X_3X_4, \qquad \sigma_4 = X_1X_2X_3X_4.$$

Let

$$\gamma = X_1X_2 + X_3X_4.$$

(a) Show that $\mathrm{Gal}(L/K(\gamma))$ is the dihedral group with 8 elements

(b) Let F be the smallest Galois extension of K containing $K(\gamma)$ Show that $[F : K] = 6$

6. Let G be a finite group and let H be a normal subgroup of G with $[G : H] = n$. Let V be the n-dimensional vector space with basis $\{b_{g_1H}, \quad , b_{g_nH}\}$ corresponding to the cosets of H. Define a representation

$$\rho : G \longrightarrow GL(V),$$

where $\rho(g) : V \to V$ is defined by $\rho(g)(b_{g_iH}) = b_{gg_iH}$. Show that ρ is irreducible if and only if $G = H$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
August 2010
ALGEBRA (Ph D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number** (**not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

---

1. Let p and q be distinct primes with $q \equiv 1 \pmod{p}$. Let r be an integer with $r^{p^2} \equiv 1 \pmod{q}$. Suppose G is the group of order p^2q generated by x and y with relations

$$x^{p^2} = 1, \quad y^q = 1, \quad xyx^{-1} = y^r.$$

(a) Show that G has a unique subgroup N of order q.

(b) Show that G/N is cyclic.

(c) Show that G contains a unique subgroup H of order pq.

2. Let $n \geq 1$ and let T be an $n \times n$ real matrix. Let T^* be the transpose of T. Both T and T^* give linear transformations $\mathbb{R}^n \to \mathbb{R}^n$. Let B_1 be a basis of the nullspace of T and let B_2 be a basis of the image of T^*.

(a) Show that if $b_1 \in B_1$ and $b_2 \in B_2$, then $b_1 \cdot b_2 = 0$ (where $\cdot$ denotes the standard dot product).

(b) Show that $B_1 \cup B_2$ forms a basis of $\mathbb{R}^n$.

3. Let R be a commutative ring with 1 and let M be an R-module. We say that M is *divisible* if, for each nonzero $r \in R$, the map $M \to M$ given by multiplication by r is surjective.

(a) Show that if R is a principal ideal domain that is not a field and if M is nonzero and finitely generated then M is not divisible.

(b) Let A be an abelian group. The group of $\mathbb{Z}$-homomorphisms $\mathrm{Hom}(M, A)$ can be made into an R-module by defining $(r\phi)(m) = \phi(rm)$ for all $r \in R$, $m \in M$, and $\phi \in \mathrm{Hom}(M, A)$. Show that M is divisible if and only if $\mathrm{Hom}(M, A)$ is a torsion-free R-module for all abelian groups A. (*Note:* "torsion-free" means that $r\phi = 0$ if and only if either $r = 0$ or $\phi = 0$.) (*Hint:* the natural map $M \to M/rM$ might be useful.)

4. Let R be a commutative ring with 1. If $I_1, \ldots, I_n$ are ideals of R, the product $I_1 I_2 \cdots I_n$ is the ideal of R generated by all products of the form $j_1 j_2 \cdots j_n$ with $j_i \in I_i$.

(a) Let M be a maximal ideal of R and suppose $I_1, \ldots, I_n$ are ideals of R such that $I_1 I_2 \cdots I_n \subseteq M$. Show that $I_k \subseteq M$ for some k.

(b) Suppose that R satisfies the descending chain condition; that is, if $I_1 \supseteq I_2 \supseteq \cdots$ is a descending chain of ideals then $I_m = I_{m+1} = \cdots$ for some m, or equivalently every non-empty collection of ideals contains an ideal minimal for that set. Show that R contains only finitely many maximal ideals. (*Hint:* Look at the set of products $M_1 M_2 \cdots M_r$ of maximal ideals.) (*Note:* You may not quote theorems about Artinian rings.)

5. Let p be a prime and let K be a subfield of the complex numbers containing a primitive p-th root of unity. Let $a, b \in K$ and let $\alpha, \beta \in \mathbb{C}$ be such that $\alpha^p = a$ and $\beta^p = b$. Assume that $[K(\alpha, \beta) : K] = p^2$.

(a) Show that $\mathrm{Gal}(K(\alpha, \beta)/K) \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$

(b) Show that $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ has exactly $p+1$ subgroups of order p

(c) Show that the fields

$$K(\alpha), K(\alpha\beta), \ldots, K(\alpha\beta^{p-1}), K(\beta)$$

are distinct.

(d) Show that each subextension of $K(\alpha, \beta)/K$ of degree p over K is one of the fields listed in part (c).

6. Let G be a finite group and let $\chi_1, \ldots, \chi_n$ be the irreducible characters (over the complex numbers) of G. Suppose $m_1, \ldots, m_n$ are integers and let

$$\chi = m_1\chi_1 + \cdots + m_n\chi_n.$$

Suppose that $\chi(g) \in \{-1, 0, 1\}$ for all $g \in G$.

(a) Show that either χ is identically 0 or $\chi = \pm\chi_i$ for some i (that is, either χ is 0 or it is $\pm$ an irreducible character).

(b) Show that if χ is not identically 0 then either χ or $-\chi$ is a homomorphism from G to the multiplicative group $\{1, -1\}$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
January 2010
ALGEBRA (Ph.D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number** **(not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

---

1. Let $p < q$ be distinct primes with $q \not\equiv 1 \pmod p$.

(a) Show that every group of order pq is cyclic.

(b) Let G be a group and let H be a subgroup of the center of G. Suppose that G/H is cyclic. Show that G is abelian.

(c) Let r be a prime distinct from p and q, and let G be a nonabelian group of order pqr. Show that the center of G has order 1, p, or q.

2. Let V be a finite dimensional complex vector space and let W be a subspace. Let T be a linear transformation of V such that $T(W) \subseteq W$. Let $m(X)$ be the minimal polynomial of T on V and let $m_1(X)$ and $m_2(X)$ be the minimal polynomials of the linear transformations induced by T on W and V/W, respectively.

(a) Show that $m_1(X)$ divides $m(X)$ and that $m_2(X)$ divides $m(X)$.

(b) Show that $m(X)$ divides $m_1(X)m_2(X)$.

(c) Show that if $m_1(X)$ and $m_2(X)$ are relatively prime then $m(X) = m_1(X)m_2(X)$.

(d) Give an example where $m(X) \neq m_1(X)m_2(X)$.

3. Let R be an integral domain and let K be its field of fractions. For a prime ideal M of R, define the ring

$$R_M = \{r/s \in K \mid r, s \in R,\ s \notin M\}.$$

(a) Let M be a maximal ideal. Show that R_M has a unique maximal ideal.

(b) Show that

$$R = \bigcap_M R_M,$$

where M runs through all maximal ideals of R. (*Hint:* For $\alpha \in K$, look at the ideal $I = \{x \in R \mid x\alpha \in R\}$. *Warning:* Do not write about *the* denominator of α; since R is not necessarily a UFD, there might not be a unique denominator.)

4. Recall that a module I is injective if it satisfies the following condition: Whenever $f : A \to B$ is an injective homomorphism of R-modules and $p : A \to I$ is a homomorphism, then there exists a homomorphism $h : B \to I$ such that $hf = p$.

(a) Show that if $0 \to I \to M \to N \to 0$ is an exact sequence of modules with I injective, then $M \simeq I \oplus N$.

(b) Suppose that we have a diagram of modules

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ & I & & D & \end{tikzcd}$$

where the row is exact and where I is an injective module. Moreover, suppose α and γ are injective homomorphisms Show that there exists a module M, homomorphisms r, s, and an injection β such that the following diagram has exact rows and columns and commutes:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & I \arrow[r, "r"] & M \arrow[r, "s"] & D \arrow[r] & 0 \end{tikzcd}$$

5. Let $n \in \mathbf{Z}$ and let $f(X) = X^3 - nX^2 - (n+3)X - 1$.

(a) Show that $f(X)$ is irreducible in $\mathbf{Q}[X]$.

(b) Show that if ρ is a root of $f(X)$, then $-1/(1+\rho)$ is a root of $f(X)$.

(c) Let K be the splitting field of $f(X)$. Show that $\mathrm{Gal}(K/\mathbf{Q}) \simeq \mathbf{Z}/3\mathbf{Z}$.

6. The following is the character table of a group G:

	I	II	III	IV	V
χ_1	1	1	1	1	1
χ_2	3	0	$(1+\sqrt{5})/2$	$(1-\sqrt{5})/2$	-1
χ_3	3	0	$(1-\sqrt{5})/2$	$(1+\sqrt{5})/2$	x
χ_4	4	1	-1	-1	0
χ_5	5	y	0	0	1

(a) The sizes of the conjugacy classes are as follows:

$$|I| = 1, \quad |II| = 20, \quad |III| = 12, \quad |IV| = 12, \quad |V| = z.$$

Find z.

(b) Compute the entries x and y in the character table.

(c) Show that G is not a solvable group.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
August 2009
ALGEBRA (Ph D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number** (**not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

---

1. Let G be a group of order $2^m k$ with k odd and with $m \geq 1$. Assume that G contains an element g of order 2^m.
(a) Multiplication by $x \in G$ gives a permutation π_x of the elements of G, as in Cayley's theorem. Show that π_g is an odd permutation (where g is the element of order 2^m).
(b) Let H be the subgroup of $h \in G$ such that π_h is an even permutation. Show that $|H| = 2^{m-1}k$ and that H contains an element of order 2^{m-1}.
(c) Show that G contains a subgroup of order k. (*Hint:* Use induction.)

2. Let M be an $n \times n$ complex matrix and let I be the $n \times n$ identity matrix. Show that M is diagonalizable if and only if the rank of $xI - M$ equals the rank of $(xI - M)^2$ for all $x \in \mathbb{C}$.

3. Let R be a commutative ring with 1 and let

$$0 \to A \to B \to C \to 0$$

be an exact sequence of R-modules.
(a) Show that if C is free and B is finitely generated, then A is finitely generated.
(b) Given an example of A, B, C with B finitely generated but C not free such that A is not finitely generated. (*Hint:* Let R be a non-Noetherian ring.)

4. A commutative Artinian ring R is a commutative ring with 1 that satisfies the descending chain condition; that is, if

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

is a descending sequence of ideals, then $I_n = I_{n+1}$ for some n.
(a) Suppose that R is an Artinian integral domain. Show that R is a field. (*Hint:* Consider the ideals (a^n)).
(b) Let P be a prime ideal in a commutative Artinian ring. Show that P is maximal.

CONTINUED ON PAGE 2

5. Let $\alpha \in \mathbb{C}$ be a complex number that is algebraic over $\mathbb{Q}$ Let $f(X) \in \mathbb{Q}[X]$ be the minimal polynomial of α. Let $\sqrt{\alpha}$ denote one of the square roots of α and let $g(X) \in \mathbb{Q}[X]$ be the minimal polynomial of $\sqrt{\alpha}$.
(a) Show that $\deg f(X)$ divides $\deg g(X)$
(b) Show that $\sqrt{\alpha} \in \mathbb{Q}(\alpha)$ if and only if $f(X^2) \in \mathbb{Q}[X]$ is reducible.

6. Let G be a finite group and let $R = \mathbb{C}[G]$ be the group ring of G with complex coefficients. We may regard R as a complex vector space of dimension equal to the order of G. Consider the representations $\rho_i : G \to GL(R)$, $i = 1, 2, 3$, defined by
$\rho_1(g)(r) = grg^{-1}$,
$\rho_2(g)(r) = gr$,
$\rho_3(g)(r) = rg^{-1}$,
where $g \in G$ and $r \in R$
(a) Show that the representations ρ_2 and ρ_3 are equivalent
(b) Show that ρ_1 is equivalent to ρ_2 if and only if G is trivial.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
January 2009
ALGEBRA (Ph.D Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number** (**not** your name) on the outside of the booklet

c. Keep scratch work on separate pages in the same booklet.

---

1. Let $G = \prod_p \mathbf{Z}/p\mathbf{Z}$, where the product is over all prime numbers, so $G = \{(x_2, x_3, x_5, \ldots) \mid x_p \in \mathbf{Z}/p\mathbf{Z}\}$ Let $H = \oplus_p \mathbf{Z}/p\mathbf{Z} = \{(x_2, x_3, x_5, \ldots) \in G \mid x_p = 0$ for all but finitely many $p\}$
(a) Show that H is the torsion subgroup of G That is, H is the set of elements in G of finite order
(b) Let $x \in G$ and let $n \neq 0$ be an integer Show that there exists $y \in G$ such that $x - ny \in H$.
(c) Let $\phi : G/H \to G$ be a homomorphism. Show that $\phi = 0$
(d) Show that $G \not\simeq H \oplus (G/H)$.

2. Let P be an $n \times n$ complex matrix such that $P^2 = P$
(a) Describe the Jordan canonical form of P
(b) Show that $\mathbf{C}^n = \text{Ker } P \oplus \text{Im } P$
(c) Suppose M is an $n \times n$ complex matrix such that $M(\text{Ker } P) \subseteq \text{Ker } P$ and $M(\text{Im } P) \subseteq \text{Im } P$ Show that $MP = PM$

3. Let R be a commutative ring with 1 Let M be a Noetherian R-module. Let $I = \{r \in R \mid rm = 0$ for all $m \in M\}$ Show that R/I is a Noetherian R-module (*Hint:* Consider the map $r \mapsto (rm_1, \ldots, rm_n) \in M^n$ for suitable $m_1, \ldots, m_n \in M$)

4. Let R be a commutative ring with 1 Let N be the set of nilpotent elements of R (that is, the set of $r \in R$ such that $r^n = 0$ for some $n \geq 1$). Prove that the following are equivalent:

(a) R/N is a field.

(b) Every element of R is either a unit or nilpotent.

(c) N is a prime ideal and it is the only prime ideal of R

5. Let L/K be a Galois extension of fields and suppose $G = \text{Gal}(L/K)$ has order pq, where p, q are primes with $p < q$
(a) Show that if $f(X) \in K[X]$ has degree p, then L is not the splitting field of $f(X)$.
(b) Suppose that G is nonabelian. Show that there is a polynomial $g(X) \in K[X]$ of degree q such that L is the splitting field of $g(X)$.

6. Let S_3 be the group of permutations of 3 objects Let ρ be the irreducible two-dimensional complex representation of S_3 determined by

$$\rho((1,2)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho((1,2,3)) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Define a four-dimensional representation $\widetilde{\rho}$ of S_3 on $\mathbf{C}^2 \otimes \mathbf{C}^2$ by

$$\widetilde{\rho}(g)(v \otimes w) = (\rho(g)v) \otimes (\rho(g)w)$$

Determine the decomposition of the character of $\widetilde{\rho}$ into irreducible characters of S_3

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
August 2008
ALGEBRA (Ph.D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number** (**not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

---

1. Let S_n denote the group of permutations of n objects and let A_n be the subgroup of even permutations. It is known that A_n is simple when $n \geq 5$ and that if $H \subset S_n$ is a simple group with $|H| > 2$ then $H \subseteq A_n$.
(a) (2 points) Let $\phi : A_6 \to S_4$ be a homomorphism. Show that ϕ is trivial.
(b) (3 points) Show that there is no subgroup G of A_6 with $[A_6 : G] = 4$.
(c) (2 points) Let G be a group of order 90 with no normal subgroups of order 5. Show that there is a nontrivial homomorphism $G \to S_6$. (*Hint:* How many Sylow 5-subgroups does G have?)
(d) (3 points) Show that a group of order 90 cannot be simple.

2. Let $n \geq 1$ and let M be an $n \times n$ complex matrix. Let I be the $n \times n$ identity matrix. Show that $\det(I - aM) = 1$ for all $a \in \mathbf{C}$ if and only if $M^n = 0$.

3. Let R be a principal ideal domain (PID).
(a) (6 points) Show that all nonzero prime ideals of R are maximal ideals.
(b) (4 points) Let S be an integral domain that is not a field. Suppose that there is a surjective ring homomorphism $\phi : R \to S$. Show that ϕ is an isomorphism. (*Note:* an integral domain is not allowed to be the zero ring.)

4. Let R be an integral domain contained in a field F. Let I be an ideal of R. Then I and F are R-modules. Let $f : I \to F$ be an R-module homomorphism.
(a) (3 points) Show that if $0 \neq a, b \in I$ then

$$\frac{f(a)}{a} = \frac{f(b)}{b}.$$

(b) (4 points) Define an R-module homomorphism $g : R \to F$ such that $g(a) = f(a)$ for all $a \in I$ (so g extends f to R). (*Note:* don't forget the case $I = 0$.)
(c) (3 points) It can be shown that F is an injective R-module. Use this fact to prove the existence of the map g found in part (b).

5. Let ζ be a primitive 7th root of unity in $\mathbf{C}$ and let $K = \mathbf{Q}(\zeta)$. You may use the fact that ζ is a root of the irreducible polynomial $\Phi(X) = X^6 + X^5 + X^4 + X^3 + X^2 + X + 1 \in \mathbf{Q}[X]$. Let $\alpha = \zeta + \zeta^2 + \zeta^4$.
(a) (3 points) Show that any subset of $\{1, \zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^5, \zeta^6\}$ with 6 elements is linearly

independent over $\mathbf{Q}$.

(b) (3 points) Find a generator of $\mathrm{Gal}(K/\mathbf{Q})$ and describe it explicitly.

(c) (4 points) Show that $[\mathbf{Q}(\alpha) : \mathbf{Q}] = 2$

6. The Klein 4-group V_4 is given by $\{1, a, b, ab\}$, where $a^2 = b^2 = 1$ and $ab = ba$

(a) Find the character table of V_4.

(b) Consider the complex 2-dimensional representation ρ of V_4 determined by

$$\rho(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Find the decomposition of the character of ρ into irreducible characters.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
January 2008
ALGEBRA (Ph.D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number** **(not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

---

1. Let G be a finite group and let H, K be subgroups of G. Suppose that $[G : H] = m$ and $[G : K] = n$.
(a) Show that if H is normal in G then $[G : H \cap K]$ divides mn.
(b) Let S_5 be the group of permutations of 5 objects and let A_5 the subgroup of even permutations. Show that A_5 is the only subgroup of S_5 of index 2.
(c) Suppose that X is a set with 5 elements and that A_5 acts on X (so this gives a homomorphism from A_5 to S_5). Let $x \in X$ and let $H = \{g \in A_5 \mid gx = x\}$. Show that either $H = A_5$ or $H \simeq A_4$.

2. Let $n \geq 1$ and let $f(X)$ and $g(X)$ be monic polynomials with complex coefficients. Assume that $g(X)$ has degree n, that $f(X)$ divides $g(X)$ and that every root of g is a root of f. Show that there is a linear transformation of $\mathbb{C}^n$ with minimal polynomial $f(X)$ and characteristic polynomial $g(X)$.

3. Let R be a ring with 1. Suppose that $\{I_j\}_{j \in J}$ is a linearly ordered family of ideals of R (where J is some index set). This means that, given I_{j_1} and I_{j_2} in the family, either $I_{j_1} \subseteq I_{j_2}$ or $I_{j_2} \subseteq I_{j_1}$. Let $I = \cup_{j \in J} I_j$.
(a) Show that I is an ideal of R.
(b) Show that if each I_j is not finitely generated, then $\cup_{j \in J} I_j$ is not finitely generated.
(c) Suppose that R is not Noetherian. Use Zorn's Lemma to show that there is an ideal K of R that is not finitely generated and such that whenever K_1 is an ideal with $K \subsetneq K_1$ then K_1 is finitely generated (that is, K is maximal among ideals that are not finitely generated).

4. Let R be a commutative ring with 1. Let M be an R-module and let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules. It is known that
$$A \otimes M \to B \otimes M \to C \otimes M \to 0$$
is always exact (where $\otimes$ denotes $\otimes_R$). If $A \otimes M \to B \otimes M$ is an injection for all such exact sequences $0 \to A \to B \to C \to 0$, then M is said to be *flat*. Suppose that $0 \to M_1 \to M_2 \to M \to 0$ is an exact sequence of

R-modules. Let N be an R-module. This problem will show that if M is flat then

$$0 \to M_1 \otimes N \to M_2 \otimes N \to M \otimes N \to 0$$

is exact. This only requires showing that $M_1 \otimes N \to M_2 \otimes N$ is an injection.
(a) Show that a free module is flat.
(b) Write $N = F/K$, where F is free. Use the diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & M_1 \otimes K \arrow[r] \arrow[d] & M_2 \otimes K \arrow[r] \arrow[d] & M \otimes K \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1 \otimes F \arrow[r] \arrow[d] & M_2 \otimes F \arrow[r] \arrow[d] & M \otimes F \arrow[r] \arrow[d] & 0 \\
 & M_1 \otimes N \arrow[r] \arrow[d] & M_2 \otimes N \arrow[r] \arrow[d] & M \otimes N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

to show that $M_1 \otimes N \to M_2 \otimes N$ is an injection when M is flat.

5. Let L/K be a Galois extension of fields such that $\mathrm{Gal}(L/K)$ is cyclic of order n, generated by σ. Write $n = ab$ with $\gcd(a, b) = 1$. Let F_1 be the fixed field of σ^a and F_2 be the fixed field of σ^b. Write $F_1 = K(\alpha)$ and $F_2 = K(\beta)$.
(a) Show that σ^{aj} acts trivially on $\alpha + \beta$ if and only if $j \equiv 0 \pmod{b}$.
(b) Show that b divides the order of $\mathrm{Gal}(K(\alpha+\beta)/K)$ (and, by symmetry, a also divides this order).
(c) Show that $L = K(\alpha + \beta)$

6. The group A_4 of even permutations of 4 objects acts on $\mathbb{C}^4$ by permuting the elements of the standard basis $\{e_1, \ldots, e_4\}$:

$$\sigma(\sum a_i e_i) = \sum a_i e_{\sigma(i)},$$

where $a_i \in \mathbb{C}$. This gives a four-dimensional representation ρ of A_4.
(a) Show that the trivial representation occurs exactly once as a subrepresentation of ρ.
(b) Show that ρ is the sum of the trivial representation and a 3-dimensional irreducible representation.
(*Note:* You may use the fact that the four conjugacy classes of A_4 are $\{1\}$, three products of two disjoint 2-cycles, and two sets of four 3-cycles.)

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
August 2007

ALGEBRA (Ph.D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number** (**not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

---

1. Let S_n be the group of permutations of n objects (call them $1, 2, \ldots, n$). A group G is called a *transitive permutation group of degree* n if G is isomorphic to a subgroup Γ of S_n, and whenever $1 \leq i, j \leq n$, there exists $g \in \Gamma$ such that $g(i) = j$.
(a) (2 points) Suppose that G is a transitive permutation group of degree n. Regard G as a subgroup of S_n. For $1 \leq i \leq n$, let G_i be the set of elements of G fixing i. Show that for each i, j, the groups G_i and G_j are conjugate (that is, there exists $g \in G$ such that $G_i = gG_jg^{-1}$).
(b) (1 point) Show that $G_1 \cap G_2 \cap \cdots \cap G_n = \{1\}$.
(c) (3 points) Suppose $K \subseteq G_1$ and K is normal in G. Show that $K = \{1\}$.
(d) (4 points) Let A be a group containing a subgroup H of index n. Suppose that $K = \{1\}$ is the only subgroup $K \subseteq H$ such that K is normal in A. Show that A is a transitive permutation group of degree n.

2. Let V be a finite dimensional vector space over $\mathbb{C}$ and let T be a linear transformation of V. Let W be a subspace of V such that $T(W) \subseteq W$ (this implies that T gives well-defined linear transformations of W and V/W, which we denote by T_W and $T_{V/W}$).
(a) (3 points) Show that $\mathrm{Trace}(T) = \mathrm{Trace}(T_W) + \mathrm{Trace}(T_{V/W})$. (The trace is the sum of the diagonal elements of a matrix representing the linear transformation.)
(b) (2 points) Show that if T_W and $T_{V/W}$ are nilpotent (that is, some power is the 0-transformation), then T is nilpotent.
(c) (2 points) Suppose $\mathrm{Trace}(T^m) = 0$ for all $m \geq 1$. Show that the nullspace of T is non-trivial. (*Hint:* use the Cayley-Hamilton theorem.)
(d) (3 points) Suppose $\mathrm{Trace}(T^m) = 0$ for all $m \geq 1$. Show that T is nilpotent.

3. Let R be a commutative ring with 1. Recall that an R-module P is a projective if whenever $f : M \to P$ is a surjective homomorphism of R-modules, there is an R-module homomorphism $g : P \to M$ such that $f \circ g$ is the identity map on P.
(a) Show that an R-module P is finitely generated projective *if and only if* there is a finitely generated free module F and an R-module P_1 such that $F \simeq P \oplus P_1$.
(b) Let $P^* = \mathrm{Hom}_R(P, R)$ be the set of R-module homomorphisms from P to R. We can make P^* into an R-module by defining $(r\phi)(p) = r(\phi(p))$ for all $r \in R, p \in P, \phi \in P^*$. Show that if P is a finitely generated projective module then P^* is projective.

4. (a) (3 points) Let R be a commutative ring with 1. An element $r \in R$ s called nilpotent if $r^n = 0$ for some integer $n \geq 1$. Show that the nilpotent elements of R form an ideal.
(b) (3 points) Let R be a commutative ring with 1 such that every zero divisor is

nilpotent. Show that the set of all nilpotent elements of R is a prime ideal of R.
(c) (4 points) Find all polynomials $f(x) \in \mathbb{C}[x]$ such that $\mathbb{C}[x]/(f(x))$ has the property that all zero divisors are nilpotent.

5. Let L/K be a Galois extension of fields such that $\mathrm{Gal}(L/K) \simeq S_n$, the group of permutations of n elements. Let $f(X) \in K[X]$ be an irreducible polynomial of degree n whose splitting field is L. Let $\alpha_1, \dots, \alpha_n$ be the roots of f in L.
(a) Show that the fields $K(\alpha_1), \dots, K(\alpha_n)$ are distinct.
(b) Suppose $\sigma \in \mathrm{Gal}(L/K)$ is such that $\sigma(K(\alpha_1)) = K(\alpha_1)$. Show that σ acts as the identity on $K(\alpha_1)$.

6. Let G be a finite group and let $\rho : G \to GL_n(\mathbb{C})$ be a group representation of G. Let S be the set of matrices M such that $M\rho(g) = \rho(g)M$ for all $g \in G$.
(a) Show that if ρ is irreducible then S contains only the scalar matrices. (*Note:* what you are proving is a version of Schur's Lemma, so you may not quote that result. However, you may use the result that a (square) complex matrix has a nonzero eigenspace.)
(b) Show that if S contains only the scalar matrices then ρ is irreducible.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
January 2007

ALGEBRA (Ph.D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number** (**not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

---

1. Let G be a non-abelian group with exactly 3 elements of order 2; call them a, b, c. Assume that G is generated by $\{a, b, c\}$. Let Z be the center of G. Let S_3 be the group of permutations of 3 objects.
(a) Show that there is an injective homomorphism $G/Z \to S_3$.
(b) Show that the cosets Z, aZ, bZ, cZ are distinct. (*Hint:* Use the fact that G is non-abelian.)
(c) Show that $G/Z \simeq S_3$.

2. Let M be an $n \times n$ matrix with complex entries such that M equals the transpose of its complex conjugate: $M = \bar{M}^T$. Let $\langle v, w\rangle = v \cdot \bar{w} = v^T \bar{w}$ be the standard inner product on column vectors $v, w \in \mathbb{C}^n$.
(a) Show that $\langle Mv, w\rangle = \langle v, Mw\rangle$ for all $v, w \in \mathbb{C}^n$.
(b) Let V be a subspace of $\mathbb{C}^n$. Let $N : V \to V$ be a linear transformation of V such that $\langle Nv_1, v_2\rangle = \langle v_1, Nv_2\rangle$ for all $v_1, v_2 \in V$. Let W be a subspace of V and let $W^\perp = \{v \in V \mid \langle v, w\rangle = 0 \text{ for all } w \in W\}$. Suppose $N(W) \subseteq W$. Show that $N(W^\perp) \subseteq W^\perp$.
(c) Show that M is diagonalizable.

3. Let R be a commutative ring with 1 and let m, n be non-negative integers. Let $f : R^m \to R^n$ be a surjective homomorphism of R-modules. Let $K = \mathrm{Ker}(f)$.
(a) Show that $R^m \simeq R^n \oplus K$.
(b) Show that K is finitely generated. (*Hint:* Construct a surjection from R^k to K for some k.)
(c) Show that if R is a principal ideal domain then $K \simeq R^{m-n}$.

4. Let R be a commutative ring with 1 and let P be a prime ideal of R. Let

$$I = \{a_0 + a_1X + \cdots + a_nX^n \in R[X] \mid a_i \in P \text{ for all } i\}.$$

(a) Show that I is a prime ideal of $R[X]$.
(b) Show that I is not a maximal ideal of $R[X]$.

CONTINUED ON NEXT PAGE

5. Let L/K be a Galois extension of degree n, and assume that $\mathrm{Gal}(L/K)$ is cyclic, generated by σ. We can regard the field L as a vector space over the field K.
(a) Show that $\sigma : L \to L$ is a linear transformation of L as a vector space over K.
(b) Let d be a positive integer that is a divisor of n. Find the dimension of $\mathrm{Ker}(\sigma^d - I)$, where I is the identity map on L.

6. Let G be a finite group and let H be a subgroup of index 2. Suppose that there is an element $g_0 \in G$ of order 2 such that $g_0 \notin H$. Let $\rho : H \to GL_n(\mathbb{C})$ be an irreducible representation of H. Define the representation $\rho_1(h) = \rho(g_0 h g_0^{-1})$ for $h \in H$. Suppose that there exists $A \in GL_n(\mathbb{C})$ such that

$$\rho_1(h) = A\rho(h)A^{-1}$$

for all $h \in H$.
(a) Show that $A^2\rho(h) = \rho(h)A^2$ for all $h \in H$.
(b) Show that there exists $c \in \mathbb{C}^\times$ such that $A^2 = cI$, where I is the $n \times n$ identity matrix.
(c) Let $B = A/\sqrt{c}$ (take either choice of square root). Define $\tilde{\rho} : G \to GL_n(\mathbb{C})$ by

$$\tilde{\rho}(h) = \rho(h) \text{ for } h \in H,$$

$$\tilde{\rho}(g_0 h) = B\rho(h) \text{ for } h \in H.$$

Show that $\tilde{\rho}$ is a representation of G.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
August 2006
ALGEBRA (Ph.D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number** (**not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

---

1. In parts (a), (c), and (d), G is a finite group of odd order.

(a) (3 points) Show that every conjugacy class of G contains an odd number of elements.

(b) (2 points) Let H be an arbitrary group and let C be a conjugacy class in H. Suppose that, for some $h \in C$, we also have $h^{-1} \in C$ Show that for all $x \in C$, we also have $x^{-1} \in C$.

(c) (3 points) Let $g \in G$ with g not equal to the identity element. Show that g is not conjugate to g^{-1}.

(d) (2 points) Let k be the number of conjugacy classes of G. Show that $k \equiv |G| \pmod 4$. (*Hint:* If $c_1, \ldots c_j$ are odd integers, then $1 + 2c_1 + \cdots + 2c_j \equiv 1 + 2j \pmod 4$.)

2. Let V be a finite–dimensional vector space over some field and let A be a linear transformation from V to V. Let $m(X)$ be the minimal polynomial of A and suppose $m(X) = m_1(X)m_2(X)$ where m_1 and m_2 are relatively prime, so there exist polynomials f and g such that

$$f(X)m_1(X) + g(X)m_2(X) = 1.$$

Let $W_i = \mathrm{Ker}(m_i(A))$.

(a) (3 points) Show that $W_2 = \mathrm{Im}(m_1(A))$, the image of the linear transformation $m_1(A)$. (Of course, the same proof yields that $W_1 = \mathrm{Im}(m_2(A))$.)

(b) (3 points) Show that $V = W_1 \oplus W_2$ (*Note:* This is an internal direct sum, not just an isomorphism. Do not show only that V is isomorphic to the direct sum of W_1 and a space isomorphic to W_2.)

(c) (1 point) Show that A maps W_1 to W_1.

(d) (3 points) Show that $m_1(X)$ is the minimal polynomial of A restricted to W_1.

3. Let R be an integral domain.

(a) (5 points) Suppose that every finitely generated R-module is free Show that R is a field.

(b) (5 points) Suppose that R is Noetherian and that every finitely generated R-module with no R-torsion is free. Show that R is a principal ideal domain.

4. Recall that a module P is projective if it satisfies the following condition: Whenever $f : A \to B$ is a surjective homomorphism of R-modules and $p : P \to B$ is a homomorphism, then there exists a homomorphism $h : P \to A$ such that $fh = p$.

(a) (3 points) Show that the direct sum of two projective modules is projective.

(b) (7 points) Suppose that we have a diagram of modules

$$\begin{tikzcd}
 & P_1 \arrow[d, "\alpha"] & & P_2 \arrow[d, "\gamma"] & \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

where the row is exact, where P_1 and P_2 are projective, and where α and γ are surjective. Show that there exists a projective module P, homomorphisms r and s, and a surjection β such that the following diagram has exact rows and commutes:

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r, "r"] \arrow[d, "\alpha"] & P \arrow[r, "s"] \arrow[d, "\beta"] & P_2 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

You should prove that the top row is exact, that the diagram commutes, and that β is surjective.

5. Let $f(X) \in \mathbf{Q}[X]$ be irreducible of degree p, where p is prime. Let L be the splitting field of f and suppose that there are roots α, β of f such that $L = \mathbf{Q}(\alpha, \beta)$. Let $G = \mathrm{Gal}(L/\mathbf{Q})$, which we regard as a subgroup of S_p, the group of permutations of p objects.
(a) (2 points) Show that $|G| \leq p(p-1)$.
(b) (2 points) Show that G contains an element of order p. Explain where you are using the fact that p is prime.
(c) (2 points) Show that G contains a p-cycle π. Explain where you are using the fact that p is prime.
(d) (2 points) Show that the subgroup $\langle \pi \rangle$ of G generated by π is a normal subgroup of G.
(e) (2 points) Show that $G/\langle \pi \rangle$ is cyclic of order dividing $p-1$. You may use without proof the following (well known and/or easily proved) facts: the only elements in S_p that commute with π are powers of π, and the automorphism group of the cyclic group of order p is cyclic of order $p-1$.
(*Remark*: This shows that G is solvable. Galois also proved the converse that if G is solvable for an irreducible polynomial of prime degree, then the splitting field can be generated by two roots.)

6. Let A_4, the group of even permutations of 4 objects, act on $\mathbf{C}^4$ by permuting the entries of the vectors. For example, the 3-cycle $(1, 2, 4)$ maps the vector (a, b, c, d) to (d, a, c, b). This yields a representation $\rho : A_4 \to GL_4(\mathbf{C})$.
(a) (3 points) Show that ρ contains the trivial representation as a subrepresentation.
(b) (7 points) Show that ρ is the sum of two irreducible representations. (*Note*: A_4 contains the identity, eight 3-cycles, and three products of two disjoint 2-cycles.) (*Hint*: A character is irreducible if and only if its inner product with itself is 1.)

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
January 2006

ALGEBRA (Ph.D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number (not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

---

1. Let G be a group of order $168 = 3 \cdot 8 \cdot 7$, and assume that G has no nontrivial normal subgroups.
(a) Show that G has eight Sylow 7-subgroups.
(b) Let H be a Sylow 7-subgroup of G and let $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$. Show that $|N_G(H)| = 21$.
(c) Show that G has no subgroup of order 14. (*Hint:* There is no element of order 2 in $N_G(H)$.)

2. Let A and B be linear transformations from $\mathbb{C}^n$ to $\mathbb{C}^n$. Let N_A and N_B be the nullspaces of A and B.
(a) Show that if $N_A \cap N_B \neq 0$, then $\det(xA - B) = 0$ for all complex numbers x.
(b) Assume A and B are diagonalizable and $AB = BA$. Suppose that $\det(xA - B) = 0$ for all complex numbers x. Show that $N_A \cap N_B \neq 0$. (You may use without proof the well-known fact that there is a basis for which both A and B are diagonal.)

3. Let R be a commutative ring with 1. If $P_1, \ldots, P_n$ are ideals of R, the product $P_1 \cdots P_n$ is defined to be the ideal of R generated by products $p_1 \cdots p_n$ with $p_i \in P_i$ for all i. An empty product of ideals is defined to be the ring R.
(a) Let $I \neq R$ be an ideal of R and suppose that I does not contain any nonzero prime ideals and does not contain any products of nonzero prime ideals. Moreover, assume that for each ideal $J \supset I$ with $J \neq I$, there is a product of nonzero prime ideals contained in J (a product is allowed to be empty or contain only one factor). Show that I is a prime ideal (hence contains a prime, namely I), and therefore $I = 0$.
(b) Assume that R is Noetherian. Show that every nonzero ideal $I \neq R$ contains a product of nonzero prime ideals.

4. (a) Give an element of $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$ that is not of the form $a \otimes b$ with $a, b \in \mathbb{R}^2$. Justify your answer.
(b) Give an example of $\mathbb{Z}$-modules A, B, C, D such that there is an exact sequence $0 \to A \to B \to C \to 0$, but such that the induced sequence $0 \to A \otimes_{\mathbb{Z}} D \to B \otimes_{\mathbb{Z}} D \to C \otimes_{\mathbb{Z}} D \to 0$ is not exact. Justify your answer.

5. Let K be a field of characteristic 0, let n be a positive integer, and assume that K contains the nth roots of unity. Let $X^n - a \in K[X]$ be an irreducible polynomial and let L be its splitting field.
(a) Show that $\mathrm{Gal}(L/K) \simeq \mathbb{Z}/n\mathbb{Z}$.

(b) Let $K \subseteq F \subseteq L$. Show that there exists an integer d dividing n such that $F = K(a^{1/d})$.

6. Let G be a finite group. Suppose that for each normal subgroup $K \neq 1$ of G, the quotient group G/K is abelian. Let $\rho : G \to GL_n(\mathbb{C})$ be an irreducible representation of G with $n > 1$. Prove that ρ is injective.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
January 2006

ALGEBRA (M.A. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number (not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

---

1. Let G be a group of order $168 = 3 \cdot 8 \cdot 7$, and assume that G has no nontrivial normal subgroups.
(a) Show that G has eight Sylow 7-subgroups.
(b) Let H be a Sylow 7-subgroup of G and let $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$. Show that $|N_G(H)| = 21$.
(c) Show that G has no subgroup of order 14. (*Hint:* There is no element of order 2 in $N_G(H)$.)

2. Let A and B be linear transformations from $\mathbb{C}^n$ to $\mathbb{C}^n$. Let N_A and N_B be the nullspaces of A and B.
(a) Show that if $N_A \cap N_B \neq 0$, then $\det(xA - B) = 0$ for all complex numbers x.
(b) Assume A and B are diagonalizable and $AB = BA$. Suppose that $\det(xA - B) = 0$ for all complex numbers x. Show that $N_A \cap N_B \neq 0$. (You may use without proof the well-known fact that there is a basis for which both A and B are diagonal.)

3. Let R be a commutative ring with 1. If $P_1, \ldots, P_n$ are ideals of R, the product $P_1 \cdots P_n$ is defined to be the ideal of R generated by products $p_1 \cdots p_n$ with $p_i \in P_i$ for all i. An empty product of ideals is defined to be the ring R.
(a) Let $I \neq R$ be an ideal of R and suppose that I does not contain any nonzero prime ideals and does not contain any products of nonzero prime ideals. Moreover, assume that for each ideal $J \supset I$ with $J \neq I$, there is a product of nonzero prime ideals contained in J (a product is allowed to be empty or contain only one factor). Show that I is a prime ideal (hence contains a prime, namely I), and therefore $I = 0$.
(b) Assume that R is Noetherian. Show that every nonzero ideal $I \neq R$ contains a product of nonzero prime ideals.

4. Let A be a 6×6 complex matrix. Suppose that $A^4 = 0$ but $A^3 \neq 0$. Find all possibilities for the Jordan canonical form of A. Justify your answer.

5. Let m and n be positive integers. Show that $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = 0$ if and only if $\gcd(m, n) = 1$. (Here, Hom means homormorphisms as abelian groups).

6. Let G be a finite group of order p^n, where p is prime and $n \geq 1$.
(a) Let $y \in G$. Show that the set $\{gyg^{-1} \mid g \in G\}$ has p^m elements for some $m \geq 0$.
(b) Show that the center of G (that is, $\{g \in G \mid gx = xg, \forall x \in G\}$) is nontrivial.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
August 2005

ALGEBRA (Ph.D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number** (**not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

---

1. Let p be a prime number and let G be a group of order p^n for some $n \geq 1$. Let G' be the commutator subgroup of G. You may assume without proof the fact that every subgroup of G of index p is normal.
(a) Let H be a subgroup of G with $[G : H] = p$. Show that $G' \subseteq H$.
(b) Let
$$F_G = \bigcap_{[G:H]=p} H$$
be the intersection of all subgroups of G of index p. Show that F_G is a normal subgroup of G and that G/F_G is an elementary abelian p-group (an elementary abelian p-group A is an abelian group such that $a^p = 1$ for all $a \in A$).
(c) Let A be an elementary abelian p-group. Show that $F_A = 1$ (where F_A, as in (b), is the intersection of all subgroups of A of index p)
(d) Let B be a normal subgroup of G such that G/B is an elementary abelian p-group. Show that $F_G \subseteq B$.

2. Consider the following matrices in $M_4(\mathbb{C})$:
$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$
Show that A and B are similar but that A and C are not similar.

3. Let R be a commutative ring with 1
(a) Let Q be a prime ideal of R and let $I_1, \ldots, I_n$ be ideals of R such that
$$I_1 \cap I_2 \cap \cdots \cap I_n \subseteq Q.$$
Show that $I_k \subseteq Q$ for some k.
(b) A prime ideal M is called a minimal prime if there are no prime ideals properly contained in M (for example, in $\mathbb{Z}$, the only minimal prime is 0). Suppose there are prime ideals $P_1, \ldots, P_n$ of R such that
$$P_1 \cap P_2 \cap \cdots \cap P_n = 0$$

Show that if M is a minimal prime of R then $M = P_i$ for some i.
(c) Let m be a composite squarefree integer and let $R = \mathbb{Z}/m\mathbb{Z}$. Show that R contains prime ideals $P_1, \ldots, P_n$ of R such that $P_1 \cap P_2 \cap \cdots \cap P_n = 0$.

4. Let R be a commutative ring with 1. Suppose there is an exact sequence of R-modules

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

Let $r, s \in R$ be such that the ideal generated by r, s is R. Suppose that $rA = sC = 0$.
(a) Show that the map $C \to C$ given by $c \mapsto rc$ is an isomorphism.
(b) Show that the restriction of g to rB gives an isomorphism $rB \simeq C$.
(c) Show that $B \simeq A \oplus C$.

5. Let L/K be a Galois extension of degree 15.
(a) Show that every subextension F/K with $K \subseteq F \subseteq L$ is normal.
(b) Suppose that $f(X) \in K[X]$ is irreducible and its splitting field is L. Show that $\deg f(X) = 15$.

6. A certain group G or order 24 has the following character table:

χ_1 :	1	1	1	1	1
χ_2 :	1	-1	1	1	-1
χ_3 :	2	0	2	-1	0
χ_4 :	3	1	-1	0	-1
χ_5 :	3	A	-1	0	1

(a) Find the value of A.
(b) Show that the center of G is trivial. (*Hint:* What does Schur's Lemma tell you about the image of the center of G under an irreducible representation?)
(c) Show that G contains a normal subgroup H of index 2.
(d) Show that if $K \subset G$ is a proper normal subgroup such that G/K is abelian, then $K = H$, where H is as in part (c). (*Hint:* If you have a representation of G/K, how can you get a representation of G?)

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
August 2005

ALGEBRA (M.A. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number (not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

---

1. Let p be a prime number and let G be a group of order p^n for some $n \geq 1$. Let G' be the commutator subgroup of G. You may assume without proof the fact that every subgroup of G of index p is normal.
(a) Let H be a subgroup of G with $[G : H] = p$. Show that $G' \subseteq H$.
(b) Let

$$F_G = \bigcap_{[G:H]=p} H$$

be the intersection of all subgroups of G of index p. Show that F_G is a normal subgroup of G and that G/F_G is an elementary abelian p-group (an elementary abelian p-group A is an abelian group such that $a^p = 1$ for all $a \in A$).
(c) Let A be an elementary abelian p-group. Show that $F_A = 1$ (where F_A, as in (b), is the intersection of all subgroups of A of index p).
(d) Let B be a normal subgroup of G such that G/B is an elementary abelian p-group. Show that $F_G \subseteq B$.

2. Consider the following matrices in $M_4(\mathbb{C})$:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Show that A and B are similar but that A and C are not similar

3. Let R be a commutative ring with 1.
(a) Let Q be a prime ideal of R and let $I_1, \dots, I_n$ be ideals of R such that

$$I_1 \cap I_2 \cap \cdots \cap I_n \subseteq Q$$

Show that $I_k \subseteq Q$ for some k.
(b) A prime ideal M is called a minimal prime if there are no prime ideals properly contained in M (for example, in $\mathbb{Z}$, the only minimal prime is 0). Suppose there are prime ideals $P_1, \dots, P_n$ of R such that

$$P_1 \cap P_2 \cap \cdots \cap P_n = 0.$$

Show that if M is a minimal prime of R then $M = P_i$ for some i.
(c) Let m be a composite squarefree integer and let $R = \mathbb{Z}/m\mathbb{Z}$. Show that R contains prime ideals $P_1, \ldots, P_n$ of R such that $P_1 \cap P_2 \cap \cdots \cap P_n = 0$.

4. Let R be a commutative ring with 1. Suppose there is an exact sequence of R-modules

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

Let $r, s \in R$ be such that the ideal generated by r, s is R. Suppose that $rA = sC = 0$.
(a) Show that the map $C \to C$ given by $c \mapsto rc$ is an isomorphism.
(b) Show that the restriction of g to rB gives an isomorphism $rB \simeq C$.

5. Let p and q be odd primes with $p < q$ and $q \not\equiv 1 \pmod{p}$. Show that every group of order pq is cyclic.

6. Let V and W be finite-dimensional vector spaces over $\mathbb{C}$ and let $T : V \to W$ be a linear transformation. Show that there exists a linear transformation $S : W \to V$ such that $TS : W \to W$ is the identity map if and only if T is surjective.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
January 2005

ALGEBRA (Ph.D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number (not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

---

1. (a) Let p be an odd prime and let G be a group of order $2p$. Show that the set $G^2 = \{x^2 \mid x \in G\}$ is a subgroup of G.
(b) Let A_4 be the group of even permutations of 4 objects. Show that the set $A_4^2 = \{x^2 \mid x \in A_4\}$ is not a subgroup of A_4.

2. Let m and n be positive integers. Let V be the set of $m \times n$ matrices over $\mathbb{C}$ and let W be the set of $n \times m$ matrices over $\mathbb{C}$. Both V and W are vector spaces of dimension mn over $\mathbb{C}$. Let $A \in V$. Consider the linear functional on W:

$$\phi_A : W \to \mathbb{C}, \quad \phi_A(B) = \text{Trace}(AB),$$

where the trace of a square matrix is the sum of the diagonal elements.
(a) Show that if $A \neq 0$ then ϕ_A is not the zero map.
(b) Show that every linear functional on W is of the form ϕ_A for some $A \in V$.

3. Let R be a commutative Noetherian ring with 1 and let M be a non-zero R-module. If $m \in M$, define the annihilator of m to be $\text{Ann}(m) = \{r \in R \mid rm = 0\}$.
(a) Suppose I is an ideal of R that is maximal among ideals that are annihilators of nonzero elements (that is, $I = \text{Ann}(m)$ for some $0 \neq m \in M$, and if $I \subseteq \text{Ann}(x)$ for some $0 \neq x \in M$, then $I = \text{Ann}(x)$). Show that I is a prime ideal.
(b) Let $r \in R$ and $0 \neq m \in M$ be such that $rm = 0$. Show that there exists a prime ideal P of R and a nonzero element $x \in M$ such that $r \in P$ and $P = \text{Ann}(x)$.

4. Let A, B, C, D, E be modules over a ring R. Suppose we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & D \arrow[r] & E \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

with exact rows.
(a) Show that $D/f(A)$ is isomorphic to $E/g(B)$.
(b) Suppose $X \to Y \to Z \to 0$ is an exact sequence of R-modules and that X and Z are finitely generated. Show that Y is finitely generated.
(c) Suppose that A and E are finitely generated. Show that D is finitely generated.

CONTINUED ON NEXT PAGE

5. Let p be an odd prime. Let K be a field of characteristic 0 containing a nontrivial pth root of unity ζ. Let $\alpha \in K$ and assume α is not a pth power in K.
(a) Let $\beta^p = \alpha$ and let $L = K(\beta)$. Show that L/K is a Galois extension and that its degree is p. (*Hint:* To show the degree is p, explicitly show that any nontrivial automorphism in $\mathrm{Gal}(L/K)$ has order p. Do not assume that the Galois group has order p, since this is what you are proving.)
(b) If $\gamma \in L$, let

$$N(\gamma) = \prod_{\sigma \in \mathrm{Gal}(L/K)} \sigma(\gamma).$$

Show that $N(\gamma) \in K$. (*Note:* N is the norm mapping. You may not quote results about the norm to do this problem.)
(c) Show that $N(\beta) = \alpha$.
(d) Show that β is not the pth power of an element in L (and therefore the polynomial $X^p - \beta$ is irreducible in $L[X]$, by the previous results).
(e) Show that $X^{p^2} - \alpha$ is irreducible in $K[X]$.

6. Let V be a vector space over $\mathbb{C}$ of dimension n and let $\{\beta_1, \cdots, \beta_n\}$ be a basis of V. The group S_n acts on V by permuting the basis elements:

$$\pi(x_1\beta_1 + \cdots + x_n\beta_n) = x_1\beta_{\pi(1)} + \cdots + x_n\beta_{\pi(n)}.$$

This gives a representation ρ of S_n.
(a) Show that the trivial representation occurs exactly once in ρ. (*Hint:* What are the fixed vectors?)
(b) Let $\mathrm{Fix}(\pi)$ be the number of elements of $\{1, 2, \cdots, n\}$ fixed by the permutation π. Show that

$$n! = \sum_{\pi \in S_n} \mathrm{Fix}(\pi).$$

(*Note:* This is a special case of what is known as Burnside's Lemma. You may not simply deduce the present result from Burnside's Lemma.)

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
August 2004

ALGEBRA (Ph.D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number (not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

---

1. (a) Suppose that A is a subgroup of a group B with $[B : A] = 2$. Let C be a subgroup of B of odd order. Show that $C \subseteq A$.

(b) Let G be a finite group and suppose that there exist subgroups

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_j = H$$

with $[G_i : G_{i+1}] = 2$ for all i. Suppose H has odd order. Show that H is normal in G.

(c) Let G be a group of order $2^j k$, where k is odd. Suppose G contains a normal subgroup H of order k. Show that there exist subgroups

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_j = H$$

with $[G_i : G_{i+1}] = 2$ for all i.

2. Let A be an invertible square matrix over $\mathbb{C}$. Suppose A^n is diagonalizable for some $n \geq 1$. Show that A is diagonalizable.

3. Let R be a commutative ring with 1. Let N be the intersection of all maximal ideals of R.

(a) Suppose R is a PID with infinitely many nonassociated irreducibles. Show that $N = 0$.

(b) Let $n \in R$. Show that $n \in N$ if and only if $1 + nr$ is a unit of R for all $r \in R$.

(c) Let $R = K[X, Y]$ be the ring of polynomials in two variables over a field K. Show that $N = 0$.

4. Let A, B, C, D be modules over a ring R. Suppose we have a diagram

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "h"'] & C \\ & D & \end{tikzcd}$$

with exact row, and suppose that h is surjective. Show that hf is surjective if and only if $g(\mathrm{Ker}(h)) = g(B)$

CONTINUED ON NEXT PAGE

5. Let $\mathbb{F}_p$ denote the finite field with p elements, where p is a prime. Let $0 \neq a \in \mathbb{F}_p$. Define $f(X) = X^p - X + a \in \mathbb{F}_p[X]$, and let L be the splitting field of $f(X)$.
(a) (1 point) Show that $f(X)$ has no roots in $\mathbb{F}_p$
(b) (1 point) Show that there exists $\sigma \in \mathrm{Gal}(L/\mathbb{F}_p)$ with $\sigma \neq 1$.
(c) (2 points) Show that if $\alpha \in L$ is a root of $f(X)$, then the roots of $f(X)$ are exactly the set $\{\alpha, \ \alpha+1, \ \alpha+2, \ldots, \alpha+p-1\}$.
(d) (3 points) Show that $\sigma^p = 1$ for all $\sigma \in \mathrm{Gal}(L/\mathbb{F}_p)$.
(e) (3 points) Show that $f(X)$ is irreducible in $\mathbb{F}_p[X]$.

6. Let $\rho : G \to GL_n(\mathbb{C})$ be a representation of the finite group G. Define $\tilde{\rho} : G \times (\mathbb{Z}/2\mathbb{Z}) \to GL_{2n}(\mathbb{C})$ by

$$\tilde{\rho}(g,0) = \begin{pmatrix} \rho(g) & 0 \\ 0 & \rho(g) \end{pmatrix}, \qquad \tilde{\rho}(g,1) = \begin{pmatrix} 0 & \rho(g) \\ \rho(g) & 0 \end{pmatrix}.$$

(a) Show that $\tilde{\rho}$ is a representation of $G \times (\mathbb{Z}/2\mathbb{Z})$
(b) Show that the number of times that the trivial representation of G occurs in ρ equals the number of times that the trivial representation of $G \times (\mathbb{Z}/2\mathbb{Z})$ occurs in $\tilde{\rho}$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
January 2004

ALGEBRA (Ph.D Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10

b. Use a different booklet for each question Write the problem number and your **code number (not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

---

1. **(a)** Let p be a prime and let $H \neq 1$ be a finite abelian p-group. Let N be the number of elements of H with order equal to p. Show that $N \equiv -1 \pmod p$

(b) Let p be a prime and let G be a finite group. Let S be a Sylow p-subgroup of G Suppose $x \in G$ has order p and commutes with all elements of S. Show that $x \in S$. (*Hint*: If $x \notin S$, what is the order of $\langle x, S \rangle$?)

(c) Let the notation be as in part (b). Let X the the set of elements of G of order equal to p. Then S acts on X by conjugation (that is, $s \in S$ maps $x \in X$ to $sxs^{-1} \in X$). Let $x \in X$. Show that the orbit of x has only one element if and only if x is in the center of S Also, show that if the orbit of x has more than one element, then the number of elements in the orbit is a multiple of p.

(d) Let G be a finite group and let p be a prime number dividing the order of G. Let M be the number of elements of G with order equal to p. Show that $M \equiv -1 \pmod p$.

2. Let A and B be non-zero $n \times n$ matrices over $\mathbb{C}$ and suppose $AB = BA$ Show that if the *characteristic* polynomial of A has no multiple roots then the *minimal* polynomial of B has no multiple roots.

3. Let R be a commutative ring with 1 Let $N = \{r \in R \mid r^n = 0 \text{ for some } n \geq 1\}$.

(a) Show that N is an ideal of R.

(b) Suppose that N is a maximal ideal of R Show that N is then the unique maximal ideal of R.

(c) Suppose $r, s \in R$ with $rs = 0$. Show that if N is a maximal ideal and $r \notin N$, then $s = 0$.

4. Recall that a module N is called injective if whenever there is a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & M_1 & \xrightarrow{g} & M_2 \\ & & {\scriptstyle\beta}\big\downarrow & & \\ & & N & & \end{array}$$

with exact row then there is a map $\gamma : M_2 \to N$ such that $\gamma g = \beta$

Consider the following commutative diagram of modules, where the top row is exact:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \phi\downarrow & & \psi\downarrow & & \\ M & \xrightarrow{h} & N & & \end{array}$$

Suppose that N is injective and $h\phi = 0$. Show that there is a well-defined homomorphism $\alpha : C \to N$ such that $\alpha g = \psi$.

5. Let $\mathbb{F}_q$ denote the finite field with q elements. Let $n \geq 1$. The extension $\mathbb{F}_{q^n}/\mathbb{F}_q$ is Galois with Galois group generated by the map ϕ, where $\phi(x) = x^q$ for all $x \in \mathbb{F}_{q^n}$ (you may assume these facts). Consider the sequence of additive groups

$$0 \longrightarrow \mathbb{F}_q \xrightarrow{i} \mathbb{F}_{q^n} \xrightarrow{f} \mathbb{F}_{q^n} \xrightarrow{g} \mathbb{F}_{q^n},$$

where i is inclusion and

$$f(x) = x^q - x, \qquad g(x) = \sum_{j=0}^{n-1} \phi^j(x)$$

You may assume that $\mathrm{Im}(f) = \mathrm{Ker}(g)$ (this is the additive form of Hilbert's Theorem 90)

(a) Show that $\mathrm{Im}(g) \subseteq \mathbb{F}_q$.
(b) Show that $\mathrm{Im}(i) = \mathrm{Ker}(f)$.
(c) Show that $\mathrm{Ker}(g)$ has order q^{n-1}.
(d) Show that $\mathrm{Im}(g) = \mathbb{F}_q$.

6. Let G be a group and let H be a normal subgroup. Let $a_1H, \ldots, a_nH$ be the cosets of H. Let V be the vector space over $\mathbb{C}$ with basis $\{e_1, \ldots, e_n\}$. For $g \in G$, define the linear transformation T_g by $T_g(e_i) = e_j$ if $ga_iH = a_jH$. The map

$$\rho : G \longrightarrow GL(V), \qquad g \longmapsto T_g$$

gives a representation of G (you do not need to show this). Show that when ρ is decomposed as a direct sum of irreducible representations, the trivial representation of G appears exactly once.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
August 2003

ALGEBRA (Ph.D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number** (**not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

---

1. (a) Let G be a group and let H be a cyclic normal subgroup of G. Let K be a subgroup of H. Show that K is normal in G.
(b) Let G be the nth dihedral group, which is generated by a and b with relations $a^n = e$, $b^2 = e$, $ba = a^{-1}b$. Let p be an odd prime dividing n. Show that G contains exactly one p-Sylow subgroup.

2. Let V be a finite dimensional vector space over a field F and let $T : V \to V$ be a linear transformation.
(a) Suppose that every nonzero $v \in V$ is an eigenvector of T. Show that T is a scalar multiple of the identity.
(b) A cyclic vector for T is a vector $v \in V$ such that $\{v, Tv, T^2v, \ldots\}$ spans V. Suppose that every nonzero vector $v \in V$ is a cyclic vector for T. Show that the characteristic polynomial of T must be irreducible over F.
(c) Suppose that V is 2-dimensional over F and that the characteristic polynomial of T is irreducible over F. Show that every nonzero $v \in V$ is a cyclic vector for T.

3. Let A, B, C be modules over a ring R (commutative with 1), and suppose there is an exact sequence
$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$
(a) Show that if A and C are free R-modules, then B is a free R-module.
(b) Let I be an ideal of R. Show that if I is free as an R-module then I is a principal ideal.
(c) Suppose that R is not a PID. Show that there is an exact sequence as in part (a) where B is free but neither A nor C is free.

4. Let R be a commutative ring with 1. The ring R is said to be Artinian if every descending chain of ideals stops. That is, for each sequence
$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$
of ideals of R, there exists n such that $I_n = I_{n+1} = \cdots$.
(a) Show that if R is Artinian and I is an ideal of R, then R/I is an Artinian ring.

CONTINUED ON NEXT PAGE

(b) Show that if R is an Artinian integral domain, then R is a field. (*Hint:* Consider the ideals $(x) \supseteq (x^2) \supseteq (x^3) \cdots$)
(c) Show that if R is Artinian and P is a prime ideal of R, then P is a maximal ideal.
(d) Show that if $P_1, \ldots, P_n$ are distinct prime ideals of an Artinian ring R, then there exists an element $x \in P_1 \cap \cdots \cap P_{n-1}$ with $x \notin P_n$.
(e) Show that an Artinian ring has only finitely many prime ideals.

5. (a) Let K be a field and suppose $K/\mathbb{Q}$ is a finite Galois extension of odd degree. Let $\sigma : K \hookrightarrow \mathbb{C}$ be an embedding of K into the complex numbers $\mathbb{C}$. Show that $\sigma(K) \subset \mathbb{R}$.
(b) Let ζ be a primitive 7th root of unity. Show that $\mathbb{Q}(\zeta)$ contains a subfield $K \neq \mathbb{Q}$ that is Galois of odd degree over $\mathbb{Q}$.

6. The character table of a group G is given below. The cardinalities of the conjugacy classes are as follows:

$$|I| = 1, \quad |II| = 20, \quad |III| = 12, \quad |IV| = 12, \quad |V| = A.$$

(a) Compute A, B, C.
(b) Use the character table to show that G is not solvable.
(c) Use the character table to show that there does not exist a homomorphism $\rho : G \longrightarrow GL_4(\mathbb{C})$ ($= 4 \times 4$ invertible complex matrices) such that

$$\rho(g) = \begin{pmatrix} 1 & 26 & -3 & 56 \\ 0 & -30 & 0 & -67 \\ 1 & 0 & -2 & -2 \\ 0 & 13 & 0 & 29 \end{pmatrix}$$

for some g in conjugacy class II. (*Remark:* The matrix $\rho(g)$ satisfies $\rho(g)^3 = I$, and also $g^3 = e$, so elementary group theory does not immediately imply the result.)

	I	II	III	IV	V
χ_1	1	1	1	1	1
χ_2	3	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	-1
χ_3	3	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	B
χ_4	4	1	-1	-1	0
χ_5	5	C	0	0	1

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM
January 2003

ALGEBRA (Ph.D. Version)

Instructions to the student

a. Answer all six questions; each will be assigned a grade from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your **code number** (**not** your name) on the outside of the booklet.

c. Keep scratch work on separate pages in the same booklet.

---

1. (a) Let H be a group of order 9. Show that the order of the automorphism group of H divides 48. (*Hint:* You may assume, without proving it, that the group of 2 by 2 invertible matrices with entries in $\mathbb{Z}/3\mathbb{Z}$ has order 48.).
(b) Let G be a group of order $153 = 3^2 \times 17$. Show that the center of G contains a group of order 9.
(c) Find all groups of order 153.
(In this problem, you may not quote any results about groups of order p^2q.)

2. Let M be an $m \times n$ matrix (with entries in some field). Show that there exist an invertible $m \times m$ matrix A and an invertible $n \times n$ matrix B such that $AMB = (c_{ij})$, with $c_{ii} = 0$ or 1 for all i and $c_{ij} = 0$ whenever $i \neq j$.

3. Let R be a principal ideal domain and let A and B be finitely generated R-modules.
(a) Show that if $0 \neq a \in A$ and $0 \neq b \in B$ are not torsion elements (that is, there is no $0 \neq r \in R$ with $ra = 0$ and similarly for b), then $a \otimes b \neq 0$ in $A \otimes_R B$.
(b) Give an example where $a \neq 0$ is a non-torsion element, $b \neq 0$ is a torsion element, and $a \otimes b = 0$.

4. Let R be a commutative ring with 1. Let $x, y \in R$ be nonzero. Assume that x is not a zero divisor in R, that y is not in the ideal xR. Consider the R-module homomorphisms
$$f: R \to R \oplus R, \quad f(r) = (xr, yr)$$
$$g: R \oplus R \to R, \quad g(a, b) = ay - bx.$$
(a) Show that $\mathrm{Im}(f)=\mathrm{Ker}(g)$ if and only if the congruence class of y, namely $y + xR$, is not a zero divisor in the ring R/xR.
(b) Show that there is an R-module homomorphism $h: R \to R \oplus R$ such that gh is the identity map of R if and only if the ideal $xR + yR$ generated by x and y equals R.

5. (a) Show that the polynomial $f(X) = X^4 + 1$ is irreducible in $\mathbb{Q}[X]$.
(b) The roots of $f(X)$ are the primitive 8th roots of unity. Show that the splitting field F of $f(X)$ has Galois group (over $\mathbb{Q}$) isomorphic to $(\mathbb{Z}/8\mathbb{Z})^\times$ (= the multiplicative group mod 8). (You may not deduce this from general results about Galois groups of the fields generated by roots of unity. Your solution should explain how you use the result in part (a).)

CONTINUED ON NEXT PAGE

(c) Show that there are exactly three fields L with $L \subset F$ and $[L : \mathbb{Q}] = 2$.

6. Let G be a finite group. Let $\rho : G \to GL_m(\mathbb{C})$ be a complex representation of G and let χ be the character of ρ. Let $g \in G$ have order n.
(a) Show that the matrix $\rho(g)$ is diagonalizable and that $\chi(g)$ can be written as a sum of nth roots of unity.
(b) Let $\gcd(a, n) = 1$ and let σ be the automorphism of the field of nth roots of unity such that $\sigma(\zeta) = \zeta^a$ for all nth roots of unity ζ. Show that $\chi(g^a) = \sigma(\chi(g))$.
(c) Show that if g is conjugate in G to g^a for all a with $\gcd(a, n) = 1$, then $\chi(g) \in \mathbb{Q}$.
(d) Let G be a finite group and suppose $g \in G$ is such that $\chi(g) \in \mathbb{Q}$ for all characters χ of G. Let n be the order of g and let $\gcd(a, n) = 1$. Show that g is conjugate to g^a.