ANALYSIS QUALIFYING
EXAMINATION
JANUARY 7, 2011
MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND

Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear which theorem you are using and why its use is justified. You may use any given hint without proving it. In problems with multiple parts, be sure to go on to the rest of the problem even if there is some part you cannot do. In working on any part, you may assume the answer to any previous part, even if you have not proved it.

1. Let $f \in AC[0,1]$ be an absolutely continuous function on $[0,1]$ with $f > 0$. Prove that $1/f \in AC[0,1]$.

2. The functions $\{f_n\}$ are holomorphic on a domain D and converge uniformly on compact subsets to a function f. Show that either f is identically zero or for each open subset U of D with compact closure in D and with f having no zeros on the boundary of U that: there is an integer n_U such that for $n \geq n_U$, f_n and f have the same number of zeros on U.

3. Let $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, $\alpha > 0$, and define

$$E_\alpha(f) = \{x \in \mathbb{R} : |f(x)| > \alpha\}.$$

(i) Show that E_α has finite Lebesgue measure.
(ii) Use (i) to show that every $f \in L^p(\mathbb{R})$, $1 \leq p \leq 2$, can be decomposed as $f_1 + f_2$ where $f_1 \in L^1(\mathbb{R})$ and $f_2 \in L^2(\mathbb{R})$.

4. Let $\mathcal{F}$ be the set of holomorphic maps f with $f(0) = i$ and domain the unit disc $\mathbb{D}$ and range contained in the upper half plane $\mathbb{H}$. Show that the supremum of the imaginary parts $\sup_{f \in \mathcal{F}} \Im f(i/2)$ is bounded (3 points). Find the supremum and justify your answer (7 points).

5. Let $\{f_n\}$ be a sequence of measurable functions which converges $a.\ e.$ to f on $\mathbb{R}$, and suppose there exists $g \in L^1(\mathbb{R})$ such that for all $n \geq 1$, $|f_n| \leq g \quad a.\ e.$ on $\mathbb{R}$. Given $\epsilon > 0$, prove that there is a measurable subset $A \subset \mathbb{R}$ such that $m(A) < \epsilon$ and $f_n \to f$ uniformly on A^c.

6. Suppose R is a region with f a non constant holomorphic function on R. Suppose D is an open disc with $\overline{D} \subset R$ and with $|f|$ constant on the boundary of D. Prove that f has zeros in D.

ANALYSIS QUALIFYING EXAMINATION

AUGUST 6, 2010
MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND

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1. Assume that the function f is of bounded variation on $[0,1]$. For each $x \in [0,1]$, define $v(x)$ to be the total variation of the restriction of f to $[0,x]$ Prove that if f is absolutely continuous, then v is also absolutely continuous. Is the conclusion still true if we remove the assumption that f is absolutely continuous?

2 Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2}\, dx \qquad \text{for } t > 0.$$

Carefully justify convergence.

3. Let $\{f_n\}$ be a sequence of nonnegative Lebesgue measurable functions on $[0,1]$ Show that $\{f_n\}$ converges to zero in measure if and only if

$$\lim_{n\to\infty} \int_0^1 \tfrac{f_n(x)}{1+f_n(x)}\, dx = 0.$$

4. The functions $f(z)$ and $g(z)$ are holomorphic in a domain containing the circle γ and its interior The functions are non vanishing on γ Give a formula for the integral

$$\int_\gamma \frac{f'(z)g(z)}{f(z)}\, dz$$

in terms of the values and zeros of the functions.

5. Assume that $f, g \in L^1(\mathbb{R}, dm) \cap L^\infty(\mathbb{R}, dm)$ where m denotes the Lebesgue measure on $\mathbb{R}$, and define the function h on $\mathbb{R}$ by

$$h(x) = \int_{-\infty}^{\infty} f(x+y)g(y)dm(y)$$

Prove that h is a continuous function on $\mathbb{R}$, and that $\lim_{|x|\to\infty} h(x) = 0$ for all x in $\mathbb{R}$.

6. Statement: a harmonic function $u(x, y)$ on a simply connected domain Ω can be represented in the form $u(x, y) = \log |f(z)|$ for a holomorphic function $f(z)$ on Ω

- Prove the statement beginning with the harmonic conjugate
- Use the statement to show that a harmonic function on Ω cannot have an interior minimum or an interior maximum.

ANALYSIS QUALIFYING
EXAMINATION

JANUARY 11, 2010
MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND

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1. Assume that $n \geq 1$ is an integer, and let $f \in \bigcap_{n=1}^{\infty} L^n([0,1])$. Prove that if $\sum_{n=1}^{\infty} \|f\|_{L^n([0,1])} < \infty$, then $f = 0$ a. e.

2. The function $\sec \pi z$ has a convergent Taylor expansion $\sum_{n=0}^{\infty} a_n (z+i)^n$. Find $\limsup_{n\to\infty} |a_n|^{1/n}$.

3. Let $f \in L^1(\mathbb{R})$.
(a) Determine

$$\lim_{x\to 0} \int_{\mathbb{R}} |f(t+x) + f(t)|\, dt.$$

(b) Determine

$$\lim_{x\to \infty} \int_{\mathbb{R}} |f(t+x) + f(t)|\, dt.$$

4. On $\mathbb{C} - \{0\}$ the function $f(z)$ is holomorphic and satisfies $|f(z)| \leq C|z|^{3/2}$. Show that $f \equiv 0$.

5. Suppose that $\{f_n\}$ is a sequence in $L^1(\mathbb{R})$ with $\|f_n\|_1 \leq 1$ for all n and

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ a.e.}$$

(a) Prove that $f \in L^1(\mathbb{R})$ and that $\|f\|_1 \leq 1$

(b) Show that

$$\lim_{n\to\infty} (\|f - f_n\|_1 - \|f_n\|_1 + \|f\|_1) = 0.$$

Hint: The following inequalities might be useful. For any numbers a, and b

$$0 \leq |a-b| - |a| + |b| \leq 2|b|.$$

6. D is a connected, simply connected domain with $\sin z$ never zero on D. Show that $\log|\sin z|$ is the real part of a holomorphic function on D. (Begin with the real part.)

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

August 2009

ANALYSIS (Ph.D./M.A. version)

1. (a) For a bounded Lebesgue-integrable function f on $[0,1]$ define $F(x) = \int_0^x f(t)\,dt$, for $x \in [0,1]$. Prove that F is absolutely continuous on $[0,1]$.

   (b) Give an explicit example of a continuous function F of bounded variation on $[0,1]$, that is not absolutely continuous on $[0,1]$. It is not necessary to verify your assertions.

2. Show that an isolated singularity of an analytic function f cannot be a pole of $\exp(f)$.

3. (a) Suppose that $f \in L^1[0,1]$, and let m denote Lebesgue measure. Prove that, for $c > 0$,

$$m\{|f(x)| \geq c\} \leq \frac{1}{c}\int_0^1 |f(x)|\,dx.$$

   (b) Suppose that $\{f_n\} \subseteq L^p[0,1]$ is a sequence of functions satisfying $||f_n||_p \leq M$, where $1 < p < \infty$. Prove that

$$\lim_{c\to\infty} \int_{|f_n(x)|\geq c} |f_n(x)|\,dx = 0, \text{uniformly in } n.$$

4. Prove that

$$\frac{1}{z\,\sin(z)} = \frac{1}{z^2} + 2\sum_{k=1}^{+\infty} \frac{(-1)^k}{z^2 - k^2\pi^2}.$$

(Suggestion if needed: Check the partial fraction expansion. Then consider magnitudes on certain large rectangles.)

5. Let $\{f_n\} \subseteq L^2[0,1]$ be on the unit sphere, *i.e.*, $||f_n||_2 = 1$ for each n. Suppose that

$$\lim_{m,n\to\infty} ||f_m + f_n||_2 = 2.$$

Prove that there is an $f \in L^2[0,1]$ such that $\lim_{n\to\infty} ||f_n - f||_2 = 0$.

6. Let $T = 3z - 2$ and $\mathcal{F}$ the family of n^{th} iterates $T^n = T \circ T \ldots \circ T$.

   (a) Prove that $\mathcal{F}$ is NOT normal on the Riemann sphere $\hat{\mathbb{C}}$.

   (b) Find the largest domain $D \subset \hat{\mathbb{C}}$ on which $\mathcal{F}$ is a normal family.

Analysis Jan. 2009

1. (a) Let f, g be real valued measurable functions on [0,1] with the property that for every $x \in [0,1]$, g is differentiable at x and

$$g'(x) = (f(x))^2.$$

Prove that $f \in L^1[0,1]$.

(b) Suppose in addition that f is bounded on $[0,1]$. Prove that

$$2\int_0^1 g(x) f^2(x)\,dx = g^2(1) - g^2(0).$$

2. For $n = 1, 2, 3 \ldots$ obtain the explicit formula

$$\int_0^\infty \frac{1}{x^{2n}+1}\,dx = \frac{\pi}{2n} \Big/ \sin\left(\frac{\pi}{2n}\right).$$

3. Let $f \in L^1(-\infty, \infty)$ and suppose $\alpha > 0$. Set $f_n(x) = \frac{f(nx)}{n^\alpha}$ for $n = 1, 2, \ldots$. Prove that for almost every $x \in (-\infty, \infty)$,

$$\lim_{n \to \infty} f_n(x) = 0.$$

4. Suppose that f, g are analytic on $\{|z| \le 1\}$ with $g \ne 0$ on $\{|z| < 1\}$. Prove that $|f(z)| \le |g(z)|$, $\forall z \in \{|z| = 1\}$, implies $|f(0)| \le |g(0)|$.

5. Let $f \in L^2(0, \infty)$.

(a) Prove that

$$\left(\int_0^x f(t)\,dt\right)^2 \le 2\sqrt{x}\int_0^x \sqrt{t} f(t)^2\,dt.$$

(b) Let $F : (0, \infty) \mapsto (-\infty, \infty)$ be defined by

$$F(x) = \frac{1}{x}\int_0^x f(t)\,dt.$$

Prove that: $\quad ||F||_2 \le 2||f||_2$.

6. Find a conformal mapping f of $\mathbf{D} = \{|z| < 1\}$ onto the domain $\mathbf{H} = \{|\Im(w)| < \pi/2\}$ so that $f(0) = 0$, $f'(0) = 2$. Hence prove that for every g analytic on $\mathbf{D}$ such that $g(\mathbf{D}) \subset \mathbf{H}$

$$g(0) = 0 \Rightarrow |g'(0)| \le 2.$$

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF MARYLAND

WRITTEN GRADUATE QUALIFYING EXAM

ANALYSIS

August 2008

Instructions

1. Your work on each question will be assigned a grade from 0 to 10. Some problems have multiple parts or ask you to do more than one thing. Be sure to go on to subsequent parts even if there is some part you cannot do. If you are asked to prove a result and then apply it to a given situation you may receive partial credit for a correct application even though you do not give a correct proof.

2. Use a different sheet (or different set of sheets) for each question. Write the problem number and your **code number** (not your name) on the top of every sheet.

3. Keep scratch work on separate sheets.

4. Unless otherwise stated, you may appeal to a "well-known theorem" in your solution to a problem. However, it is your responsibility to make it clear exactly which theorem you are using and to justify its use.

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1. Suppose that $\{f_n\}$ be a sequence of absolutely continuous functions defined on $[0,1]$ such that $f_n(0) = 0$ for every n and

$$\sum_{n=1}^{\infty} \int_0^1 |f_n'(x)|\, dx < +\infty,$$

for every $x \in [0,1]$. Prove that the series $\sum_{n=1}^{\infty} f_n(x)$ converges for each $x \in [0,1]$ pointwise to a function $f(x)$, the function f is absolutely continuous on $[0,1]$, and

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) \quad a\ e.\ x \in [0,1].$$

2. Compute

$$\int_0^{\infty} \frac{1}{x^3+1}\, dx$$

3. Suppose that $\{f_n\}$ is a sequence of nonnegative integrable functions such that $f_n \to f$ a.e., with f integrable, and $\int_R f_n \to \int_R f$. Prove that

$$\int_R |f_n - f| \to 0.$$

4. Suppose $\mathbf{S} = \{z : -\pi/2 < \Im\{z\} < \pi/2\}$, and there is an entire function g with $g(\mathbf{S}) \subset \mathbf{S}$. If $g(-1) = 0,\ g(0) = 1$ prove that $g(z) = z + 1$.

5. (a) Show that if f is absolutely continuous on $[0,1]$ and $p > 1$, then $|f|^p$ is absolutely continuous on $[0,1]$.

(b) Let $0 < p < 1$. Give an example of an absolutely continuous function f on $[0,1]$ such that $|f|^p$ is not absolutely continuous.

6. Let $\mathcal{F}$ be the family of functions f analytic on $\{|z| < 1\}$ so that

$$\iint_{|z|<1} |f(x+iy)|^2\, dx\, dy \le 1.$$

Prove that $\mathcal{F}$ is a normal family on $\{|z| < 1\}$.

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF MARYLAND

WRITTEN GRADUATE QUALIFYING EXAM

ANALYSIS

January 2008

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1. Suppose that $f \in L^1(\mathbb{R})$ is a uniformly continuous function Show that

$$\lim_{|x|\to\infty} f(x) = 0$$

2 Prove there is an entire function f so that for any branch g of $\sqrt{z}$

$$\sin^2(g(z)) = f(z)$$

for all z in the domain of definition of g.

3. Suppose f is absolutely continuous on $\mathbb{R}$, and $f \in L^1(\mathbb{R})$ Show that if in addition

$$\lim_{t\to 0+} \int_{\mathbb{R}} \left| \frac{f(x+t) - f(x)}{t} \right| dx = 0,$$

then $f = 0$ a.e

4 Let $\mathbf{H}$ be the domain $\{z : -\pi/2 < \Re(z) < \pi/2, \Im(z) > 0\}$. Prove that $g = \sin(z)$ is a $1:1$ conformal mapping of $\mathbf{H}$ onto a domain $\mathbf{D}$. What is $\mathbf{D}$?

5 Suppose that $L^{1/2}(\mathbb{R})$ is the set of all equivalence classes of measurable functions for which

$$\int_{\mathbb{R}} |f(x)|^{1/2}\, dx < \infty.$$

(a) Show that it is a metric linear space with the metric

$$d(f,g) = \int_{\mathbb{R}} |f(x) - g(x)|^{1/2}\, dx, \text{ where } f, g \in L^{1/2}(\mathbb{R}).$$

(b) Show that with this metric $L^{1/2}(\mathbb{R})$ is complete.

6. Suppose that for a sequence $a_n \in \mathbb{R}$ and any $z, \Im(z) > 0$, the series

$$h(z) = \sum_{n=1}^{\infty} a_n \sin(nz)$$

is convergent. Show that h is analytic on $\{\Im(z) > 0\}$ and has analytic continuation to $\mathbb{C}$

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF MARYLAND

WRITTEN GRADUATE QUALIFYING EXAM

ANALYSIS
August 2007

Instructions

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3. Keep scratch work on separate sheets.

4. Unless otherwise stated, you may appeal to a "well-known theorem" in your solution to a problem. However, it is your responsibility to make it clear exactly which theorem you are using and to justify its use.

1. Suppose that f is a continuous real valued function with domain $(-\infty, \infty)$ and that f is absolutely continuous on every finite interval $[a, b]$.

   Prove: If f and f' are both integrable on $(-\infty, \infty)$ then $\int_{-\infty}^{\infty} f' = 0$.

2. Find all domains D so that for

$$h(z) = \frac{z^2}{z^4 - 1}$$

   there is a function H with $H' = h$ on D.

3. Suppose that $\{f_n\}$ is a sequence of real valued measurable functions defined on the interval $[0, 1]$ and suppose that $f_n(x) \to f(x)$ for almost every $x \in [0, 1]$. Let $p > 1$ and $M > 0$ and suppose that $||f_n||_p \leq M$ for all n.

   (a) Prove that $||f||_p \leq M$

   (b) Prove that $||f - f_n||_1 \to 0$ as $n \to \infty$.

4. Consider the n^{th} iterates $g^n = g \circ g.. \circ g$ of the rational function

$$g(z) = \left\{\frac{z + 1/2}{1 + z/2}\right\}^2.$$

   Prove that $g^n \to 1$, uniformly on compact subsets of $\{|z| < 1\}$.

5. Suppose $f(x), xf(x) \in L^2(R)$. Prove that $f(x) \in L^1(R)$, and that

$$||f||_1 \leq \sqrt{2}(||f||_2 + ||xf||_2).$$

6 Suppose that there are entire functions f_n so that for all complex numbers $x + \mathbf{i}y$

$$\sum_{n=1}^{\infty} |f_n(x + \mathbf{i}y)|^{1/n} \leq e^x.$$

   Show that $f(z) = \sum_{n=1}^{\infty} f_n(x + \mathbf{i}y)$ is analytic on $\{\Re(z) < 0\}$ and has period $2\pi\mathbf{i}$.

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF MARYLAND

WRITTEN GRADUATE QUALIFYING EXAM

ANALYSIS

January 2007

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1. Suppose that $f : [0, \infty) \mapsto [0, \infty)$ is measurable and that $\int_0^1 f(x)\,dx < \infty$. Prove that

$$\lim_{n\to\infty} \int_0^\infty \frac{x^n f(x)}{1+x^n}\,dx = \int_1^\infty f(x)\,dx .$$

2. Compute the partial fractions decomposition of $f(z) = \dfrac{z^7}{z^8+1}$.

3. Let $f \in L^p[0,1]$, $g \in L^q[0,1]$, $h \in L^r[0,1]$, where $1 \le p, q, r \le \infty$, $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} = 1$. Prove that $fgh \in L^1[0,1]$, and $||fgh||_1 \le ||f||_p ||g||_q ||h||_r$.

4. Consider the series:

$$g(z) = \sum_{n=1}^\infty \frac{z^{2^n}}{n!}$$

   (a) Find the domain $\mathbf{D}$ where the series is convergent.

   (b) Prove that for any $k, n \in \mathbb{N}$, $g(e^{2\pi i k/2^n} z) = g(z) + p(z)$, for some polynomial p.

   (c) Prove that if g has analytic extension from $\mathbf{D}$ then it has analytic extension to $\partial\mathbf{D}$.

   (d) Prove that g has no analytic extension from $\mathbf{D}$.

5. Suppose E_n are measurable sets, and there is an integrable function $f \in L^1(R)$ such that $\lim_{n\to\infty} ||\chi_{E_n} - f||_1 = 0$. Prove that there is a measurable set E such that $f = \chi_E$ a.e.

6. Let f be a conformal mapping of the domain

$$\Omega = \{z : \Re(z) > 0\} - (0, 1]$$

onto the domain $\{z : \Re(z) > 0\}$ so that $f(2) = 2$, $f'(2) > 0$. Prove that we have real $f(3) > 3$.

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF MARYLAND

WRITTEN GRADUATE QUALIFYING EXAM

ANALYSIS

August 2006

Instructions

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1. (a) Prove the following version of the Riemann-Lebesgue Lemma: Let $f \in L^2[-\pi, \pi]$. Prove in detail that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \to 0 \qquad \text{as } n \to \infty.$$

Here n denotes a positive integer You may use any of a variety of techniques, but you cannot simply cite another version of the Riemann-Lebesgue Lemma.

(b) Let n_k be an increasing sequence of positive integers.
Show that $\{x | \liminf_{k \to \infty} \sin(n_k x) > 0\}$ has measure 0.
Notes: You may take it as granted that the above set is measurable.

2. For real s (only) consider the integral

$$\int_{-\infty}^{+\infty} \frac{e^{ist}}{t-i}\, dt$$

(a) Compute the Cauchy Principal Value of the integral (when it exists).

(b) For which values of s is the integral convergent?

3. Suppose $(x^p + \frac{1}{x^p}) f \in L^2(0, \infty)$, where $p > \frac{1}{2}$. Show that $f \in L^1(0, \infty)$.

4. Let $\mathbf{D} = \{|z| < 1\}$ have boundary $\mathbf{S} = \{|z| = 1\}$. For $\zeta \in \mathbf{D}$ define

$$f(z) = \frac{\zeta - z^2}{1 - \overline{\zeta} z^2}$$

(a) Show that $f(z) \in \mathbf{S}$ if and only if $z \in \mathbf{S}$.

(b) Show that f has at least one fixed point $\omega \in \mathbf{S}$, i.e. $f(\omega) = \omega$.

5. Let $f \in L^1(R)$,

$$F(x) = \int_R f(t) \frac{\sin xt}{t}\, dt.$$

(a) Show that F is differentiable a. e. and find $F'(x)$.

(b) Is F absolutely continuous on closed bounded intervals $[a, b]$?

6. Let $\mathcal{F}$ be a family of entire functions. For $n = 0, \pm 1, \pm 2, \ldots$ define the domains

$$\mathbf{D}_n = \{n - 2 < \Re(z) < n + 2\} .$$

If $\mathcal{F}$ is normal (i.e. convergence to ∞ is allowed) on each $\mathbf{D}_n$ show that $\mathcal{F}$ is normal on $\mathbf{C}$.

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF MARYLAND

WRITTEN GRADUATE QUALIFYING EXAM

ANALYSIS

January 2006

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1. Let $f \in L^1[0,\infty)$, $f \geq 0$. Show that

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n x f(x)\, dx = 0.$$

2. Let u be (real valued) harmonic in the annulus $\mathbf{A} = \{1 < |z| < 2\}$. Prove there is $f(z)$ analytic on $\mathbf{A}$ and real b so that

$$u = b \log |z| + \Re\{f(z)\}$$

3. Let $k \in L^1(R), k \geq 0, \int_R k = 1$.

(a) For each $\delta > 0$, prove that

$$\lim_{n\to\infty} \int_{|x|\geq\delta} nk(nx)\, dx = 0.$$

(b) If g is real-valued, bounded, and continuous on R, show that

$$\lim_{n\to\infty} \int_R nk(nx)g(x)\, dx = g(0).$$

4. (a) Given a constant K so that a sequence of complex a_n, satisfies

$$\left|\sum_{n=1}^{m} a_n\right| \leq K,\ \forall m.$$

Suppose another monotone sequence of positive $b_n \to 0$.
Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges. (Hint: summation by parts)

(b) Hence prove that for $|z| = 1$, except for $z = 1$,

$$\sum_{n=1}^{\infty} \frac{z^n}{n} = \text{Log}\left\{\frac{1}{1-z}\right\}$$

where the principal branch of logarithm is used.

5. Suppose f is a measurable real-valued function of two real variables $x > 0$ and $y > 0$. Suppose

(i) for every $y > 0, x \mapsto f(x, y)$ on $(0, \infty)$ decreases monotonically to zero as $x \to \infty$.

(ii) for every $x > 0, F(x) = \int_0^\infty f(x, y)\, dy < \infty$.

(iii) $f_1(x, y) = \frac{\partial f}{\partial x}(x, y)$ is a continuous function of (x, y).

Prove

$$F(x) = -\int_x^\infty \int_0^\infty f_1(t, y)\, dy\, dt, \text{ for every } \quad x > 0 \text{ and,}$$

$$F'(x) = \int_0^\infty f_1(x, y)\, dy \text{ for almost every } \quad x > 0.$$

6. For the function $T(z) = \tan(z)$ define the nth iterate $T^n = T \circ T \circ T \ldots \circ T$.

(a) Prove that for $y > 0$

$$\lim_{n \to \infty} T^n(\mathrm{i}y) \to 0$$

(b) Show $\{T^n\}$ is not a normal family in any neighborhood of $z = 0$.

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF MARYLAND

WRITTEN GRADUATE QUALIFYING EXAM

ANALYSIS

August 2005

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1. Let f be a bounded measurable function on R for which there is a constant C such that

$$\forall \epsilon > 0, m(x \in R : |f(x)| > \epsilon) \leq C/\sqrt{\epsilon}.$$

Prove that $f \in L^1(R)$.

2. Compute the radius of convergence R for the power series:

$$\sum_{n=0}^{\infty} \frac{(3n)!}{n!} \frac{(2n)!}{(4n)!} z^n.$$

For which $z \in \mathbf{C}, |z| = R$, is the series convergent?

3. Let f be a real-valued function on R which is square integrable on all finite intervals $[c, d] \subseteq [0, \infty)$. Suppose that $\exists a > 0$ such that

$$|f(x)| \geq \int_0^x |f(t)|^2 \, dt, \text{ for all } \quad x \geq a.$$

Prove that $f = 0$ a.e. on $[0, \infty)$.

Hint: Set $G(x) = \int_0^x |f(t)|^2 \, dt$ and consider G'/G^2.

4. Let $\mathbf{D}$ be the domain $\{z : 1 < |z - 1|, \ |z - 2| < 2\}$. Beginning with the circle $C_0 = \{z : |z - 3| = 1\}$ we consider the "inscribed" circles C_n, $n \in \mathbf{Z}$, with disjoint interiors contained in $\mathbf{D}$ so that C_n is tangent to C_{n-1} and tangent to both circles $\{z : |z - 1| = 1\}$ and $\{z : |z - 2| = 2\}$. Let $N = N(t)$ be the number of circles $C_n \subset \{z : |z| > t\}$. Show that as $t \to 0+$

$$N(t) \sim \frac{8}{t}.$$

5. Suppose that f is absolutely continuous on $[\epsilon, 1]$ for each $\epsilon > 0$. Does the continuity f at 0 imply that f is absolutely continuous on $[0, 1]$? What if f is also of bounded variation on $[0, 1]$.

6. Consider any function f analytic on $\{0 < |z| < \infty\}$:

$$|f(z)| \leq |z|^{1/2}, \forall z : 0 < |z| < \infty.$$

Show that $f \equiv 0$.

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1. Suppose $f \in L^1(0,\infty)$. Show that there is a sequence $\{t_n\} \to \infty$ such that $t_n f(t_n) \to 0$.

2. Compute the integral

$$\int_0^1 \frac{\sqrt{x-x^2}}{x+2}\,dx$$

3. Throughout this problem $f \in L^1(\mathbf{R})$, $g \in L^p(\mathbf{R})$, $h \in L^q(\mathbf{R})$ where $1 \leq p < \infty$ and $\frac{1}{p}+\frac{1}{q} = 1$. Take it as given that the $(x,y) \mapsto f(x-y)g(y)h(x)$ and $(x,y) \mapsto f(y)g(x-y)h(x)$ are measurable functions in the plane $\mathbf{R}\times\mathbf{R}$.

(a) Prove that the function $(x,y) \mapsto f(y)g(x-y)h(x)$ is integrable over the plane $\mathbf{R}\times\mathbf{R}$.

(b) Prove that the function $x \mapsto \int_{\mathbf{R}} f(y)g(x-y)\,dy$ is defined for almost all x.

(c) Prove that the function $f * g$, defined by $(f*g)(x) = \int_{\mathbf{R}} f(y)g(x-y)\,dy$ is in L^p and show that $\|f*g\|_p \leq \|f\|_1\|g\|_p$.

4. A function defined for $x \in \mathbf{R}$ by

$$f(x) = \frac{1}{1+x^2+x^4+x^6},$$

has Taylor expansion $\sum_{n=0}^{\infty} a_n\,(x-2)^n$. Find

$$\limsup_{n\to\infty} |a_n|^{1/n}$$

5. Let f be absolutely continuous on $[0,x], \forall x > 0$, $f, f' \in L^2[0,\infty)$, $f(0) = 0$.

(a) Prove

$$\int_0^x |ff'| \leq \frac{1}{2}\left(\int_0^x |f'|\right)^2$$

(b) Prove

$$\lim_{x\to\infty} f(x) = 0.$$

6. Prove that a function f, analytic and 1:1 on the domain

$$D = \{z : |z+2| > 1,\ |z-2| > 1\}$$

and satisfying $f(D) = D$, is a bilinear transformation.

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1. Let $E \subseteq R$ be a measurable set of positive finite measure. For each $t \geq 0$, define $f(t) = m(E \cap E_t)$ where $E_t = \{t + x : x \in E\}$. Prove that

a) f is continuous on $[0, \infty)$.

b) $\lim_{t \to \infty} f(t) = 0$.

2. Compute the explicit value of

$$\int_0^\infty \frac{1}{x^9 + 1}\, dx.$$

3. Suppose $\{f_n\} \subseteq C^1[0,1]$, and suppose that the sequence $\{f_n\}$ satisfies a, b, and c below:

a) $f_n(0) = 0, \quad$ all n,

b) $|f_n'(x)| \leq \frac{1}{\sqrt{x}}, \quad$ a.e., $\quad 0 < x \leq 1,$

c) There exists a measurable function h on $[0,1]$ such that $f_n'(x) \to h(x)$ for all $x \in [0,1]$

Prove that $\{f_n\}$ converges uniformly on $[0,1]$ to an absolutely continuous function f.

4. Let $u(x,y)$ be a real valued function harmonic on $\mathbf{C}$. Show that, unless u has constant gradient ∇u, for all nonzero vectors ω there is a point where ∇u is parallel to ω.

5. Suppose $f_0 \in L^1[0,1]$, that $f_0(x) \geq 0$ *a.e.* and that

$$f_{n+1}(x) = \left(\int_0^x f_n\right)^{1/2}, \quad \forall \quad n \geq 0.$$

Assume further that $f_1(x) \leq f_0(x)$ *a.e.*

a) Prove that for each $x \in [0,1]$ the sequence $\{f_n(x)\}_{n=1}^\infty$ converges monotonically to a non-negative number $f(x)$

b) Prove that $f(x) = (\int_0^x f)^{1/2}$ for all $x \in [0,1]$

c) Prove that f is differentiable at all x for which $f(x) > 0$ and calculate $f'(x)$ at these points.

d) In particular, find a simple explicit formula for the function f in the case that $f(x) > 0$ for all $x \in (0,1]$.

6. Let $f(z) = 1 + a_1 z + a_2 z^2 + \ldots$ be analytic on $\mathbf{D} = \{|z| < 1\}$ so that $f(\mathbf{D}) \subset \{z : \Re(z) > 0\}$. Prove that for all $z \in \mathbf{D}$,

$$|f(z)| \leq \frac{1 + |z|}{1 - |z|}.$$

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1. Let $f \geq 0$ be measurable on $\mathbf{R}$ and let $P(f) = \{p > 0 : \int_{\mathbf{R}} f^p < \infty\}$.

   (a) Prove: If $p_1, p_2 \in P(f)$, then $[p_1, p_2] \subseteq P(f)$.

   (b) Prove: If I is any open interval contained in $(0, \infty)$, then there is a function $f \geq 0$ such that $P(f) = I$.

2. Consider the function $\cot(\pi z)$ with its Laurent expansion:

$$\sum_{m=-\infty}^{m=+\infty} a_{n\,m}\, z^m \,, \text{ over the annulus } n < |z| < n+1 .$$

   Compute the (explicit) values of $a_{n,m}$ for $m < 0$, with $n = 0$ and $n = 1$.

3. (a) Suppose that f is a measurable real valued function on $[0, 1]$ such that $0 \leq f(x) < \infty$ for almost all x.
   Prove: For every $\epsilon > 0$ there is a $T > 0$ such that $m(\{x \in [0,1] : f(x) > T\}) < \epsilon$.

   (b) Suppose that $\{f_n\}$ is a sequence of measurable real valued functions on $[0, 1]$ and that $M > 0$ is a constant such that for all n, $0 \leq f_n(x) \leq M$ for almost all x.
   Suppose additionally that $\int_0^1 f_n = 1$ for all $n \geq 1$.
   Prove: If $\{a_n\}$ is a sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n f_n(x) < \infty$ a.e., then $\sum_{n=1}^{\infty} a_n < \infty$.

4. On the sphere find a domain containing ∞ where there is an analytic branch $f(z)$ of

$$\log\left\{\frac{z^2+1}{z^2-1}\right\},$$

   (i.e. it has analytic continuation to a function analytic at ∞)

5. (a) Let f be in $L^1[0, 1]$.
   Prove: For every $\epsilon > 0$ there is a $\delta > 0$ such that $m(E) < \delta$ implies $\int_E |f| < \epsilon$

   (b) Let $\{f_n\}$ be a Cauchy sequence in $L^1[0, 1]$.
   Prove: For every $\epsilon > 0$ there is a $\delta > 0$ such that $m(E) < \delta$ implies $\int_E |f_n| < \epsilon$ for all $n \geq 1$.

6. Let $\mathcal{H}$ be the family of functions h analytic on $\mathbf{D} = \{|z| < 1\}$ so that

$$h(\mathbf{D}) \subset \mathbf{C} - [-\infty, 0]$$

   (a) Show that $\mathcal{H}$ is a normal family

   (b) Suppose we have a sequence $h_n \in \mathcal{H}$ satisfying $h_n(0) \to 0$
   Prove that $h_n(z) \to 0$ uniformly on compact subsets of $\mathbf{D}$

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1 (a) Prove that the set $E = \{f \in L^1[0,1] : ||f||_2 \le 1\}$ has empty interior in $L^1[0,1]$, i e. if $f \in E$, there does not exist $r > 0$ such that $\{g \in L^1 : ||g - f||_1 < r\} \subseteq E$

(b) Show that E is closed in $L^1[0,1]$, i e. if $f_n \in E$ and $f_n \to f$ in L^1, then $f \in E$

2. For real $\alpha > 1$, prove that the improper integral

$$\int_0^\infty \sin(x^\alpha)\, dx$$

converges to a positive number, with value equal to

$$\sin\left(\frac{\pi}{2\alpha}\right) \int_0^\infty \exp(-r^\alpha)\, dr\,.$$

3 (a) State the Vitali Covering Theorem.

(b) Let A be an uncountable index set, and let

$$E = \bigcup_{\alpha \in A} [a_\alpha, b_\alpha]$$

with $b_\alpha - a_\alpha > 0$, for each $\alpha \in A$ (i e. E is a union of closed intervals of positive length). Prove that E is measurable.

4. Find all polynomials P such the sequence of nth iterates

$$P^n = P \circ \dots \circ P$$

converges uniformly on compact subsets of the plane $\mathbf{C}$.

5 Let F be a measurable function on $(-\infty, \infty)$ which grows at most linearly, i. e $|F(x)| \le C|x|$, and is differentiable at zero, $F'(0) = a$. Show that

$$\lim_{n\to\infty} \int_{-\infty}^\infty \frac{nF(x)}{x(1+n^2x^2)}\, dx = \pi a.$$

6 Prove that for any polynomial

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

we have

$$\max_{|z|=1} |P(z)| \geq 1 .$$

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4. Unless otherwise stated, you may appeal to a "well-known theorem" in your solution to a problem. However, it is your responsibility to make it clear exactly which theorem you are using and to justify its use.

1 Prove that the function $f : [0,1] \mapsto \mathbf{R}$ is absolutely continuous on $[0,1]$ if and only if there is a sequence $\{f_n\}$ of Lipschitz functions on $[0,1]$ such that $\lim_{n\to\infty} T_0^1(f_n - f) = 0$

NOTES: $T_0^1(g)$ is the total variation of a function g on $[0,1]$, and a function h is Lipschitz on $[0,1]$ if there exists a constant M such that $|h(x) - h(y)| \leq M|x - y|$ for all $x, y \in [0,1]$.

2. Let T be a Möbius transformation of $\mathbf{D} = \{|z| < 1\}$ onto itself.
(a) Find all T so that

$$T(T(z)) \equiv z .$$

(b) If T has no fixed points in $\mathbf{D}$ show $\exists \omega \in \partial\mathbf{D}$ such that the nth iterate

$$T^n(z) \to \omega \ , \forall z \in \mathbf{D}$$

as $n \to \infty$.

3 Let $\{f_n\}$ be a sequence in $L^2[0,1]$ such that $||f_n||_2 \leq 1$ for $n = 1, 2, \cdots$, and let $f \in L^2[0,1]$. Suppose that $f_n \to f$ a.e. on $[0,1]$. Prove that for any $g \in L^2[0,1]$,

$$\lim_{n\to\infty} \int_0^1 f_n g = \int_0^1 fg.$$

4. (a) Let $K = \{\ldots, -\pi, 0, \pi, 2\pi, \ldots\}$. Prove that as $N \to \infty$

$$F_N(z) = \sum_{n=-N}^{N} \frac{1}{(z - \pi n)^2} ,$$

converges uniformly on compact subsets of $\mathbf{C} - K$ to a meromorphic function $F(z)$ with poles of order two at $\omega \in K$.

(b) Prove that

$$\frac{1}{\sin^2(z)} = F(z) .$$

5. Evaluate and justify your steps.

$$\int_0^\infty \int_0^{2\pi} \frac{e^{-y/x}(\sin x)y}{x}\, dx dy$$

6. Let $D = \{|z| < 1\}$ be the unit disk. Suppose that

$$f : \mathbf{D} \overset{\text{onto}}{\to} \mathbf{D}$$

is a rational mapping of degree n.

(a) Prove that

$$\int\int_{\mathbf{D}} |f'(x+iy)|\, dx\, dy \leq \pi\sqrt{n}.$$

(b) Hence prove that there is some circle

$$C : z = re^{it}, 0 \leq t \leq 2\pi$$

with $1/2 < r < 1$ so that the curve $f(C)$ has length

$$L(r) \leq 2\pi\sqrt{n},$$

which is surprising considering that $f(\partial\mathbf{D})$ winds n times around $\partial\mathbf{D}$.