DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

January 2011

LOGIC (Ph.D./M.A. version)

1. (a) Let T be any L-theory and suppose that $\{\varphi_n(x) : n \in \omega\}$ are L-formulas such that $T \models \forall x(\varphi_n(x) \to \varphi_{n+1}(x))$ for all $n \in \omega$. Suppose further that every element of every model of T realizes some φ_n. Prove that $T \models \forall x \varphi_n(x)$ for some $n \in \omega$.

   (b) Let $\mathfrak{A}$ be an L-structure, let $a \in A$, and assume that a satisfies some complete L-formula in $\mathfrak{A}$. Let $L' = L \cup \{c\}$, and let $\mathfrak{A}'$ be the expansion of $\mathfrak{A}$ to an L'-structure in which $c^{\mathfrak{A}'} = a$. Suppose that $b \in A$ and that b satisfies a complete L'-formula in $\mathfrak{A}'$. Prove that the pair ab satisfies a complete L-formula in $\mathfrak{A}$.

2. A theory T is called *model complete* if every embedding of models of T is an elementary embedding.

   (a) Suppose that $L = \{E\}$ and T is the L-theory asserting that E is an equivalence relation with infinitely many classes, and each class is infinite. Prove that T is model complete.

   (b) Prove that if T is model complete, then for every L-formula $\varphi(x_1, \dots, x_n)$, there is an existential L-formula $\psi(x_1, \dots, x_n)$ such that

$$T \models \forall \overline{x}(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x}))$$

3. (a) Suppose $L = \{U, \leq\}$, where U is a unary predicate and $\leq$ is binary. Let $\mathfrak{A}$ be the L-structure with universe $\mathbb{R}$ (the real numbers), where $U^{\mathfrak{A}} = \mathbb{Q}$ (the rationals) and $\leq^{\mathfrak{A}}$ is the usual ordering on $\mathbb{R}$. Find, with proof, all countable models of $Th(\mathfrak{A})$, up to isomorphism.

   (b) Prove that if T is ω-categorical and $\mathfrak{A}$ is the infinite, countable model, then there is $\mathfrak{B} \preceq \mathfrak{A}$ with $\mathfrak{B} \neq \mathfrak{A}$.

4. (a) Prove that $Th(\mathfrak{N})$, where $\mathfrak{N} = (\omega, +, \cdot, 0, s)$, is not model complete (see Problem #2).

   (b) Assume that $PA + Con(PA)$ is consistent. Use Gödel's Second Incompleteness Theorem to conclude that $PA + \neg Con(PA)$ is consistent.

5. (a) Prove that there is an integer m so that $W_m = \{m\}$.

   (b) Let $Z = \{e : W_e \neq \emptyset\}$. Prove that Z is a many-one complete, recursively enumerable subset of ω.

6. (a) Determine (with proof) whether or not $\mathbf{TOT} = \{e : \{e\} \text{ is total}\}$ is Turing equivalent to $\mathbf{FIN} = \{e : W_e \text{ is finite}\}$.

   (b) Demonstrate that $\{e : W_e \text{ is recursive}\}$ is an arithmetic subset of ω.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

August, 2010

LOGIC (Ph.D./M.A. version)

1. (a) Suppose T is a theory in a language with only finitely many non-logical symbols. Prove that if T has infinitely many non-isomorphic models, then T has an infinite model.

   (b) Suppose $L \subseteq L'$ are languages, $\mathfrak{A}$ is an L-structure, and T' is a consistent L'-theory. Additionally, assume that there is no model of T' whose reduct to L is elementarily equivalent to $\mathfrak{A}$. Prove that there is an L-sentence θ such that $\mathfrak{A} \models \theta$, but $T' \models \neg\theta$.

2. (a) Let $L = \{E\}$, where E is a binary relation, and let T be the L-theory asserting that E is an equivalence relation with infinitely many classes, and that each class is infinite. Prove that T is model complete, i.e., for all models $\mathfrak{A}, \mathfrak{B} \models T$, $\mathfrak{A} \subseteq \mathfrak{B}$ implies $\mathfrak{A} \preceq \mathfrak{B}$.

   (b) Let $\mathfrak{A}$ be any proper elementary extension of $\mathfrak{N} = (\omega, +, \cdot, <)$. An *initial substructure* is a substructure (not necessarily elementary) $\mathfrak{B} \subseteq \mathfrak{A}$ in which the set B is a $<$-initial segment of A. Prove that for any $a \in A$ there is an initial substructure $\mathfrak{B} \subseteq \mathfrak{A}$ with $a \in B$, but $B \neq A$. [Possible hint: Recall that there is an L-formula $\varphi(x, y, z)$ such that $k^\ell = m$ if and only if $\mathfrak{N} \models \varphi(\overline{k}, \overline{\ell}, \overline{m})$ for all $k, \ell, m \in \omega$.]

3. Suppose that T is a complete theory in a countable language.

   (a) Prove directly from the definitions that if $\mathfrak{A} \models T$ is countable and atomic, then it embeds elementarily into every model of T. It is *not* sufficient to simply quote theorems from class.

   (b) Suppose that some atomic $\mathfrak{A} \models T$ has a proper, elementary substructure. Prove that T has an uncountable, atomic model.

4. (a) Assume that $R \subseteq \omega^2$ is recursively enumerable and that the sets $\{R_k : k \in \omega\}$ are all infinite and are pairwise disjoint. Prove that there is a recursive set $C \subseteq \omega$ that intersects each R_k in exactly one point.

   (b) Prove that every decidable theory in a language with finitely many non-logical symbols has a complete, decidable extension.

5. Let Fm_x denote the set of formulas in the language $L = \{+, \cdot, <, s, 0\}$ whose free variables is precisely $\{x\}$. For each $\varphi(x) \in Fm_x$, let $d\varphi$ denote the sentence $\exists x(x = \ulcorner\varphi\urcorner \wedge \varphi(x))$. Let $f : \omega \to \omega$ be the (recursive) function

$$f(n) = \begin{cases} \ulcorner d\varphi \urcorner & \text{if } n = \ulcorner\varphi\urcorner \text{ for some } \varphi \in Fm_x \\ 0 & \text{otherwise} \end{cases}$$

and let T be any theory in which f is represented.

   (a) Prove that for every formula $\theta(x) \in Fm_x$ there is a sentence ψ such that $T \vdash \psi \leftrightarrow \theta(\ulcorner\psi\urcorner)$.

   (b) Prove that if T is a consistent theory in which every recursive function is represented, then T is undecidable.

6. (a) Prove that $\{k \in \omega : \varphi_{2k}(3k) \uparrow\}$ is Π_1 but not Δ_1.

   (b) Prove that **INF** is many-one reducible to **ZERO**, where **INF** $= \{e \in \omega : W_e \text{ is infinite}\}$ and **ZERO** $= \{e \in \omega : \forall n\, \varphi_e(n) = 0\}$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

January 2010

LOGIC (Ph.D./M.A. version)

1. (a) Prove that the class of cyclic groups is not an elementary class. (Recall that a group G is cyclic iff there is some $g \in G$ such that $G = \{g^n : n \in \mathbb{Z}\}$.)

   (b) Prove that every countable linear order embeds isomorphically into $(\mathbb{Q}, \leq)$.

2. (a) Let $L_1 = \{U\}$, where U is a unary predicate symbol. Prove that for any L_1-sentence θ, if θ is true in every finite L_1-structure, then θ is valid.

   (b) Let $L_2 = \{R\}$, where R is a binary predicate symbol. Find (with proof) an L_2-sentence θ such that θ holds in every finite L_2-structure, but θ is not valid.

3. (a) Prove that no complete theory T extending Peano's Axioms can have a countable, saturated model.

   (b) Let T be a complete theory in a countable language, and let $\Gamma(x)$, $\Phi(x)$ be 1-types such that (1) there is a model of T omitting Γ and (2) every model of T that omits Γ realizes Φ. Prove that Φ is realized in every model of T.

4. (a) Prove that there is a model $\mathfrak{A}$ of Peano's Axioms and a formula $\theta(x)$ such that $\mathfrak{A} \models \exists x\theta(x)$, yet $\mathfrak{A} \models \neg\theta(\overline{n})$ for every $n \in \omega$.

   (b) Suppose L has only finitely many nonlogical symbols, and T is a finitely axiomatizable L-theory such that for any L-sentence θ, if θ is not true in every model of T, then θ is false in some finite model of T. Prove that T is decidable.

5. (a) Prove that there is no total recursive $f : \omega \to \omega$ such that for all $e \in \omega$, if W_e is finite, then $W_e \subseteq \{0, 1, \ldots, f(e)\}$.

   (b) Construct an r.e. subset $A \subseteq \omega$ such that $\omega \setminus A$ is infinite, but $A \cap B$ is nonempty for every infinite, r.e. set B.

6. (a) Give an example (with justifications) of two sets $A, B \subseteq \omega$ such that A is Turing reducible to B, but A is not many-one reducible to B.

   (b) Exhibit (with proof) two disjoint, r.e. sets A and B that are recursively inseparable, i.e., there is no recursive C such that $A \subseteq C$, but $B \cap C = \emptyset$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

August 2009

LOGIC (Ph.D /M.A. version)

1. Suppose that $L \subseteq L'$ are languages, $\mathfrak{A}$ is an L-structure, and T' is an L'-theory such that $T' \cup Th_L(\mathfrak{A})$ is consistent.

   (a) Prove that there is an L'-structure $\mathfrak{B}' \models T'$ such that the L-reduct, $\mathfrak{B} = \mathfrak{B}'|_L$ elementarily extends $\mathfrak{A}$.

   (b) Prove that there is a model of T' realizing every 1-type $\Gamma(x)$ in the language L consistent with $Th(\mathfrak{A})$.

2. Let $D(x, y)$ denote the divisibility relation on ω, i.e., $D(n, m)$ if and only if n divides m. Let $\mathfrak{A} = (\omega, D)$.

   (a) Prove that the set of primes is definable in $\mathfrak{A}$.

   (b) Prove that $\mathfrak{A}$ has a nontrivial automorphism, i.e., an isomorphism $f : \mathfrak{A} \to \mathfrak{A}$ such that $f(n) \neq n$ for at least one $n \in \omega$.

3. (a) Prove that if $\mathfrak{A}$ is an infinite, countable, saturated model then there is a countable, saturated $\mathfrak{B} \preceq \mathfrak{A}$ with $\mathfrak{B} \neq \mathfrak{A}$.

   (b) Let $\mathfrak{A}_0 \preceq \mathfrak{B}_0 \preceq \mathfrak{A}_1 \preceq \mathfrak{B}_1 \preceq \mathfrak{A}_2 \preceq \ldots$ be an elementary chain of models where each $\mathfrak{A}_n$ is countable and saturated, and each $\mathfrak{B}_n$ is not saturated. Prove that $\bigcup_{n\in\omega} \mathfrak{B}_n$ is countable and saturated.

4. (a) Let $\mathfrak{N} = (\omega, +, \cdot, 0, 1)$ denote the standard model of arithmetic, and let PA denote Peano's axioms. Prove that there is a countable $\mathfrak{A} \models PA$ such that $\mathfrak{N} \subseteq \mathfrak{A}$, but $\mathfrak{N} \not\preceq \mathfrak{A}$.

   (b) Given a binary function $g : \omega \times \omega \to \omega$, let g^* be the partial function defined by

$$g^*(x) = \begin{cases} y & \text{if, for some } n,\ g(m, x) = y \text{ for all } m \geq n \\ \uparrow & \text{otherwise} \end{cases}$$

   Construct a (total) recursive $g : \omega \times \omega \to \omega$ such that the domain of g^* is a non-recursively enumerable set, e.g., $\overline{K}$.

5. Let $E(x, y) = x^y$ denote the exponential function.

   (a) Prove that the graph of multiplication is definable in the structure (ω, E).

   (b) Prove that the structure (ω, E) is strongly undecidable.

6. For $X \subseteq \omega$, let $S_X = \{e \in \omega : W_e = X\}$

   (a) Prove that S_X is Π_3 for every recursive set X.

   (b) Find (with proof) a recursive $X \subseteq \omega$ such that S_X is **not** Π_3-complete.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

January 2009

LOGIC (Ph.D./M.A. version)

1. (a) Let $\mathfrak{A}$ and $\mathfrak{B}$ be elementarily equivalent structures in the same language L. Prove that there is an L-structure $\mathfrak{C}$ and elementary embeddings $f : \mathfrak{A} \to \mathfrak{C}$ and $g : \mathfrak{B} \to \mathfrak{C}$.

   (b) Let $L = \{<, U\}$, where U is unary and $<$ is binary. Let $\mathfrak{A}$ be any L-structure with universe the rationals $\mathbb{Q}$, where $<^{\mathfrak{A}}$ is interpreted as the usual ordering on $\mathbb{Q}$ and $U^{\mathfrak{A}}$ is any dense, codense subset, e.g.,
   $$U^{\mathfrak{A}} = \left\{\frac{n}{2^k} : n, k \text{ are integers}\right\}$$
   Prove that $Th(\mathfrak{A})$ is ω-categorical.

2. (a) Let $L = \{+, \cdot, 0, 1\}$ and let $\mathfrak{N} = (\omega, +, \cdot, 0, 1)$ be the standard model of arithmetic. Let $\varphi(x)$ be any L-formula defining the set of prime numbers in ω. Prove that if $\mathfrak{A}$ is an elementary extension of $\mathfrak{N}$ and $\mathfrak{A} \neq \mathfrak{N}$, then there is $a \in A \setminus \omega$ such that $\mathfrak{A} \models \varphi(a)$.

   (b) Prove that every model (even the uncountable ones) of an ω-categorical theory in a countable language is atomic.

3. Let T be a complete theory in a countable language.

   (a) Prove that if $\mathfrak{A}$ is a countably universal model of T, then $\mathfrak{A}$ has an ω-saturated elementary substructure.

   (b) Prove that if $\mathfrak{A}$ is an infinite, countable, ω-saturated model of T, then $\mathfrak{A}$ has a nontrivial automorphism, i.e., an isomorphism $f : \mathfrak{A} \to \mathfrak{A}$ such that $f(a) \neq a$ for at least one $a \in A$.

4. Let $L = \{f\}$, where f is a binary function symbol, and let $Valid_L$ denote the set of valid sentences in this language.

   (a) Prove that $Valid_L$ is not essentially undecidable.

   (b) Find an L-sentence $\sigma \notin Valid_L$, yet σ holds in every finite L-structure.

5. (a) Suppose that every recursively enumerable set A is many-one reducible to a fixed set $B \subseteq \omega$. Prove that B contains an infinite, recursively enumerable subset.

   (b) Let $A = \{e \in \omega : W_e \text{ is finite}\}$ and $B = \{e \in \omega : W_e \text{ is infinite}\}$. Prove that A is Turing reducible to B, but not many-one reducible to B.

6. (a) Prove or disprove: If a binary relation R is r.e. and $|R_k| \leq 2$ for each k, then R is recursive.

   (b) Let $A \subseteq \omega$ be weakly represented, but not represented by a formula $\varphi(x)$ with respect to Q. Prove that there is a consistent, recursively axiomatizable theory $T \supseteq Q$ such that A is not weakly represented by $\varphi(x)$ with respect to T.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

August 2008

LOGIC (Ph.D./M.A. version)

1. (a) Prove that if $\mathfrak{A} \preceq \mathfrak{B}$ and A is finite, then $\mathfrak{A} = \mathfrak{B}$.

   (b) Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are structures in the same language L that satisfy the same universal sentences. Prove that there is an L-structure $\mathfrak{C}$ into which both $\mathfrak{A}$ and $\mathfrak{B}$ embed isomorphically.

2. (a) Find (with proof) all automorphisms of the structure $\mathfrak{A} = (\mathbb{Z}, +)$.

   (b) Recall that a countable $\mathfrak{A} \models T$ is *ω-homogeneous* iff for all $n \in \omega$ and all $a_0, \ldots, a_n, b_0, \ldots, b_n \in A$ there is an automorphism h of $\mathfrak{A}$ such that $h(a_i) = b_i$ for all $0 \leq i \leq n$ whenever $tp_{\mathfrak{A}}(a_0, \ldots, a_n) = tp_{\mathfrak{A}}(b_0, \ldots, b_n)$.

   Prove that if $\mathfrak{A}$ and $\mathfrak{B}$ are both countable, ω-homogeneous models of T, $\mathfrak{A}$ embeds elementarily into $\mathfrak{B}$, and $\mathfrak{B}$ embeds elementarily into $\mathfrak{A}$, then $\mathfrak{A} \cong \mathfrak{B}$.

3. Let T be a complete theory in a countable language.

   (a) Prove that if T does not have a prime model, then T has uncountably many nonisomorphic countable models.

   (b) Let X be a countable set of 1-types such that for every finite $F \subseteq X$ there is a model $\mathfrak{A}_F \models T$ omitting every $\Phi \in F$. Prove that there is a model $\mathfrak{B} \models T$ omitting every $\Phi \in X$.

4. (a) Suppose that T is a recursively axiomatizable theory in a finite language L that has no infinite models. Prove that T is decidable.

   (b) Let $L = \{+, \cdot, 0, s, <\}$ and let $Valid_L$ denote the set of valid L-sentences. Prove that $Valid_L$ is undecidable, but not essentially undecidable.

5. (a) Let T be any consistent, recursively axiomatizable extension of Robinson's Q and let $Thm_T = \{\ulcorner\sigma\urcorner : T \vdash \sigma\}$. Prove that Thm_T is weakly represented in Q, but is not represented in Q.

   (b) Let PA denote Peano's Axioms. Use Gödel's 2^{nd} Incompleteness Theorem to prove that if PA is consistent, then

$$PA \cup \{Con(PA + \neg Con(PA))\}$$

   has a model.

6. Let $K = \{e \in \omega : \{e\}(e) \downarrow\}$ and $Even = \{e \in \omega : W_e = \{2n : n \in \omega\}\}$.

   (a) Prove that there is an infinite, r.e. B such that K and B are recursively inseparable.

   (b) Prove that $Even \leq_T \mathbf{0}''$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

January 2008

LOGIC (Ph.D./M.A. version)

1. (a) Let T be any theory in a language L that has an infinite model. Prove that T has a model $\mathfrak{A}$ with an element $a \in A$ such that $a \neq c^{\mathfrak{A}}$ for every constant symbol $c \in L$.

   (b) Suppose that $\mathfrak{A}$ is a saturated model of $Th(\mathfrak{A})$, and that a complete 1-type $\Phi(x)$ is realized by only finitely many elements of $\mathfrak{A}$. Prove that there is a formula $\varphi(x) \in \Phi(x)$ such that φ is realized by only finitely many elements of $\mathfrak{A}$.

2. (a) Let $L^{\text{nl}} = \{+, \cdot, 0, 1, \leq\}$. Prove that any proper elementary extension $\mathfrak{B} \succ (\mathbb{R}, +, \cdot, 0, 1, \leq)$ contains an element $b \in B$ such that $\mathfrak{B}_B \models \bar{b} > \bar{r}$ for every $r \in \mathbb{R}$.

   (b) Recall that a countable model $\mathfrak{A}$ is *ω-homogeneous* iff for all $n \in \omega$ and all $a_0, \ldots, a_n, b_0, \ldots, b_n \in A$ there is an automorphism h of $\mathfrak{A}$ such that $h(a_i) = b_i$ for all $0 \leq i \leq n$ whenever $tp_{\mathfrak{A}}(a_0, \ldots, a_n) = tp_{\mathfrak{A}}(b_0, \ldots, b_n)$.

   Prove that every countable model in a countable language has a countable, ω-homogeneous elementary extension.

3. Let $L^{\text{nl}} = \{E\}$, where E is a binary relation symbol. Let T be the theory asserting that E is an equivalence relation with exactly two classes, both of which are infinite.

   (a) Prove that T is a complete L-theory.

   (b) Prove that if $\mathfrak{A}$ and $\mathfrak{B}$ are models of T and $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \prec \mathfrak{B}$.

4. (a) Suppose that T is a recursively axiomatizable theory with a model $\mathfrak{A} \models T$ that embeds elementarily into every model of T. Prove that T is decidable.

   (b) Assume that $A \subseteq \omega$ is recursive, $R \subseteq \omega \times \omega$ is r.e., and that $\bigcup_{k\in\omega} R_k = A$. Prove that there is a recursive $S \subseteq R$ such that $\bigcup_{k\in\omega} S_k = A$.

5. Let $\mathcal{F} = \{$all functions $f : \omega \to \omega$ such that $f(n+1) = nf(n)$ for all but finitely many $n \in \omega\}$.

   (a) Prove that every $f \in \mathcal{F}$ is recursive.

   (b) Prove that there is a recursive function $g : \omega \to \omega$ such that for every $f \in \mathcal{F}$ there is an $N \in \omega$ such that $g(n) \geq f(n)$ for every $n \geq N$.

6. (a) Let T be a consistent, recursively axiomatizable theory containing the axioms for Q. Prove that for every formula $\varphi(x)$ of the language for Q there is a sentence σ such that $T \vdash \sigma \leftrightarrow \varphi(\ulcorner \sigma \urcorner)$.

   (b) Recall that $K = \{e : \{e\}(e)\downarrow\}$ and $\overline{K} = \omega \setminus K$. Prove that K is not many-one reducible to $\overline{K}$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

August 2007

LOGIC (Ph.D./M.A. version)

1. a) Prove or disprove: $(\mathbb{Z}, <)$ has a proper elementary substructure.

   b) Let $L^{nl} = \{E\}$ where E is a binary relation symbol. Let $\mathfrak{A}$ be the countable L-structure in which $E^{\mathfrak{A}}$ is an equivalence relation such that $E^{\mathfrak{A}}$ has no infinite equivalence classes and for every $n \geq 1$ there is exactly one $E^{\mathfrak{A}}$-class with exactly n elements. Prove that $Th(\mathfrak{A})$ has exactly one countable model with infinitely many infinite equivalence classes.

2. a) Let T be a theory in a language L. Assume that whenever θ_1 and θ_2 are universal sentences of L and $T \models (\theta_1 \vee \theta_2)$ then either $T \models \theta_1$ or $T \models \theta_2$. Prove that for any $\mathfrak{A}, \mathfrak{B} \models T$ there is some $\mathfrak{C} \models T$ such that both $\mathfrak{A}$ and $\mathfrak{B}$ can be embedded in $\mathfrak{C}$. [Recall that θ is *universal* iff it has the form $\forall x_1 \ldots \forall x_n \varphi$ where φ is an open formula]

   b) Let T be an ω-categorical theory in a countable language L. Prove that every uncountable model of T is ω-saturated.

3. a) Let T be a complete theory in a countable language L. Let $\mathfrak{A}$ be a countable ω_1-universal model of T. Prove that there is some ω-saturated $\mathfrak{B}$ such that $\mathfrak{B} \prec \mathfrak{A}$.

   b) Let T be a complete theory in a countable language L and let $\Phi(x)$ be an L-type. Assume that Φ is realized by at most two elements in every model of T. Prove that there is some formula $\varphi(x)$ of L such that for every $\mathfrak{A} \models T$, $\Phi^{\mathfrak{A}} = \varphi^{\mathfrak{A}}$.

4. a) Assume that $R \subseteq \omega \times \omega$ is r.e. but not recursive and that $R_k \cap R_l = \emptyset$ for all $k \neq l$. Prove that $(\omega \setminus \bigcup_{k \in \omega} R_k)$ is infinite.

b) Prove that $\{\ulcorner \sigma \urcorner : \sigma$ is an open sentence and $\mathfrak{N} \models \sigma\}$ is recursive.

5. a) Let $A, B \subseteq \omega$ be recursively inseparable r.e. sets. Assume that $A \leq_m C$ for some $C \subseteq \omega$. Prove that $(\omega \setminus C)$ contains an infinite r.e. subset.

b) Let f, g be total recursive functions of one argument. Let

$I_f = \{e \in \omega : \{e\} = f\}$ and $I_g = \{e \in \omega : \{e\} = g\}$.

Prove that $I_f \equiv_m I_g$.

6. a) Let $R \subseteq \omega \times \omega$ be r.e. Let $A = \{k \in \omega : R_k$ is cofinite$\}$. Prove that A is arithmetic.

b) Prove that there are infinitely many $e \in \omega$ such that $\{e\}(2e) = 3e$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

January 2007

LOGIC (Ph.D./M.A. version)

1. Let L be a countable language and let $\{T_n\}_{n\in\omega}$ be L-theories such that $T_n \subseteq T_{n+1}$ for all $n \in \omega$. Let $T^* = \bigcup_{n\in\omega} T_n$ and let $\Phi(x)$ be an L-type. Prove or disprove (with a counterexample) each of the following.

   a) If each T_n has a model realizing Φ then T^* has a model realizing Φ.

   b) If each T_n has a model omitting Φ then T^* has a model omitting Φ.

2. a) Let T be a theory in a language L and let $\mathfrak{B}$ be an L-structure. Assume that whenever θ is a universal sentence of L and $T \models \theta$ then $\mathfrak{B} \models \theta$. Prove that $\mathfrak{B}$ can be embedded in some model of T. [Recall that θ is *universal* iff it has the form $\forall x_1 \ldots \forall x_n \varphi$ where φ is an open formula]

   b) Let T be a complete theory of L. Assume that T has some model which realizes just finitely many complete types in one variable. Prove that every model of T realizes just finitely many complete types in one variable.

3. a) Let T be a complete theory in a countable language L and let $\Phi(x)$ be an L-type. Assume that any two countable models of T omitting Φ are isomorphic. Prove that every countable model of T omitting Φ is prime. [Warning: you are not given that T has a prime model]

   b) Recall that a countable model $\mathfrak{A}$ is *ω-homogeneous* iff for all $n \in \omega$ and all $a_0, \ldots, a_n, b_0, \ldots, b_n \in A$ there is an automorphism h of $\mathfrak{A}$ such that $h(a_i) = b_i$ for all $0 \leq i \leq n$ whenever $tp_{\mathfrak{A}}(a_0, \ldots, a_n) = tp_{\mathfrak{A}}(b_0, \ldots, b_n)$.

Let T be a complete theory of a countable language L, and assume that $\mathfrak{A} \models T$ is countable, ω-homogeneous, and ω_1-universal. Prove that $\mathfrak{A}$ is ω-saturated.

4. a) Let $f : \omega \to \omega$ be a (total) function. Assume that there is some finite $X \subseteq \omega$ such that for all $n \in (\omega \setminus X)$ we have $f(n+1) = f(n) + 1$. Prove or disprove (with a counterexample): f is recursive.

   b) Let T be a recursively axiomatizable theory containing the axioms for Q such that $\mathfrak{N} \models T$. Prove that there is some formula $\varphi(x)$ (of the language for Q) such that $T \vdash \varphi(\bar{n})$ for all $n \in \omega$ but $T \not\vdash \forall x \varphi(x)$.

5. a) Let $A \subseteq \omega$ be infinite and r.e. Prove that there are infinite recursive sets $B_0, B_1 \subseteq A$ such that $(B_0 \cap B_1) = \emptyset$.

   b) Define sets $A, B \subseteq \omega$ such that A is r.e. in B but $(\omega \setminus A)$ is not r.e. in $(\omega \setminus B)$. [You must prove the sets you define have these properties]

6. a) Let $I = \{e : |W_e| = 1\}$. Prove that $A \leq_m I$ for every r.e. $A \subseteq \omega$.

   b) Prove that there is some $n \in \omega$ such that W_n is the set whose only element is n.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

August 2006

LOGIC (Ph.D./M.A. version)

1. a) Prove or disprove: $\{1\}$ is definable (by an L-formula) in the structure $(\mathbb{Q}, <, +)$ for the language L with $L^{nl} = \{<, +\}$.

b) Assume that $\{T_n : n \in \omega\}$ is a sequence of consistent theories in a language L such that $T_n \subseteq T_{n+1}$ for all $n \in \omega$ and $T_n \not\models T_{n+1}$ for all $n \in \omega$. Prove that $T^* = \bigcup_{n\in\omega} T_n$ is a consistent theory and that T^* is not finitely axiomatizable.

2. a) Let L be the language whose only non-logical symbol is the binary relation symbol E. An L-structure $\mathfrak{A}$ is called a *graph* provided

$$\mathfrak{A} \models \forall x \forall y (Exy \rightarrow Eyx) \text{ and } \mathfrak{A} \models \forall x \neg Exx.$$

A graph $\mathfrak{A}$ is *connected* iff for all $a \neq a^*$ in A either $E^{\mathfrak{A}}(a, a^*)$ holds or there are $a_1, \ldots, a_n \in A$ for some positive integer n such that

$$E^{\mathfrak{A}}(a, a_1),\ E^{\mathfrak{A}}(a_i, a_{i+1}) \text{ for all } 1 \leq i < n, \text{ and } E^{\mathfrak{A}}(a_n, a^*)$$

all hold. Prove or disprove each of the following:

a) Every elementary substructure of a connected graph $\mathfrak{A}$ is connected.

b) Every elementary extension of a connected graph $\mathfrak{A}$ is connected.

3. a) Let T be a complete theory in a countable language L which has a prime model $\mathfrak{A}$. Assume further that $\mathfrak{A}$ realizes every L-type (in finitely many variables) consistent with T. Prove that T is ω-categorical.

b) Let T be a complete theory in a countable language L and let $\Phi(x)$ be an L- type consistent with T which is omitted in some model of T. Prove that Φ is realized by infinitely many elements in some model of T.

4. a) Let L be the language with $L^{nl} = \{+, \cdot, <, \bar{0}, s\}$ and let $\mathfrak{N} = (\omega, +, \cdot, <, 0, s)$. Let T be a recursively axiomatizable L-theory such that $\mathfrak{N} \models T$, let $\varphi(x)$ be a Σ-formula of L, and let $D = \varphi^{\mathfrak{N}}$. Assume that D is not recursive. Prove that there is some $\mathfrak{A} \models T$ and some $n \in (\omega \setminus D)$ such that $\mathfrak{A} \models \varphi(\bar{n})$.

b) Let $A, B \subseteq \omega$ be disjoint r.e., non-recursive sets. Prove that $(A \cup B)$ is not recursive.

5. a) Let $R \subseteq (\omega \times \omega)$ be r.e., and assume that R_k is infinite for all $k \in \omega$. Prove that there is some recursive $C \subseteq \omega$ such that $(C \cap R_k) \neq \emptyset$ for all $k \in \omega$ and such that $(\omega \setminus C)$ is infinite.

b) Prove that there is some $f : \omega \rightarrow \omega$ such that for every recursive $g : \omega \rightarrow \omega$ there is some $n \in \omega$ such that $g(k) < f(k)$ for all $k \geq n$.

6. a) Let $A = \{e : \{e\}(k) = 0 \text{ for all } k \in \omega\}$ and let $B = \{e : \{e\}(k) = 1 \text{ for all } k \in \omega\}$. Prove that $A \equiv_m B$.

b) Let $\mathfrak{N}$ be the standard model for arithmetic on the natural numbers, and let $T = \{\ulcorner\sigma\urcorner : \mathfrak{N} \models \sigma\}$. Prove that $A \leq_m T$ for every arithmetic set A.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

January 2006

LOGIC (Ph.D./M.A. version)

1. a) Let L be a language containing (at least) the binary relation symbol E. Let $\mathfrak{A}$ be an L-structure such that $E^{\mathfrak{A}}$ is an equivalence relation on A. Prove that every $E^{\mathfrak{A}}$-equivalence class is finite iff every proper elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ conatins an element which is not $E^{\mathfrak{B}}$-equivalent to any element of $\mathfrak{A}$.

   b) Let T be a theory in a language L and let $\Phi(x)$ and $\Psi(y)$ be L-types. Assume that no model of T realizes both $\Phi(x)$ and $\Psi(y)$. Prove that there is some $\theta \in Sn_L$ such that whenever $\mathfrak{A} \models T$ and $\mathfrak{A}$ realizes $\Phi(x)$ then $\mathfrak{A} \models \theta$, and whenever $\mathfrak{A} \models T$ and $\mathfrak{A}$ realizes $\Psi(y)$ then $\mathfrak{A} \models \neg\theta$.

2. a) Let $\mathfrak{A}$ be an L-structure. Assume that $Th(\mathfrak{A}_A)$ is axiomatized by some $\Sigma \subseteq Sn_{L(A)}$ such that every sentence in Σ is either universal or the negation of a universal sentence. Prove that $Th(\mathfrak{A}_A)$ is axiomatized by some $\Sigma^* \subseteq Sn_{L(A)}$ consisting solely of universal sentences. [Recall that θ is *universal* iff it has the form $\forall x_0 \ldots \forall x_k\ \varphi$ where φ is an open formula.]

   b) Let T be a complete theory in a countable language L. Assume that there is some complete non-principal 1-type consistent with T. Prove that every model of T realizes infinitely many complete 1-types.

3. Let $\mathfrak{A}$ be an L-structure and let $\Phi(x)$ be a complete L-type. Assume that $\Phi(x)$ is realized by exactly three elements in $\mathfrak{A}$.

a) Assuming, in addition, that $\Phi(x)$ is principal, prove that $\Phi(x)$ is realized by exactly three elements in every L-structure $\mathfrak{B}$ elementarily equivalent to $\mathfrak{A}$.

b) Assuming, in addition, that $\mathfrak{A}$ is ω-saturated (but not that Φ is principal), prove that $\Phi(x)$ is realized by exactly three elements in every L-structure $\mathfrak{B}$ elementarily equivalent to $\mathfrak{A}$.

c) Give an example of L, L-structures $\mathfrak{A}$ and $\mathfrak{B}$, and a complete L-type $\Phi(x)$ such that $\Phi(x)$ is realized by exactly three elements in $\mathfrak{A}$ and $\mathfrak{A} \equiv \mathfrak{B}$, but $\Phi(x)$ is not realized by exactly three elements in $\mathfrak{B}$.

4. a) Let $S \subseteq (\omega \times \omega)$ be r.e., and assume that $\bigcup_{k\in\omega} S_k$ is recursive. Prove that there is some recursive $R \subseteq (\omega \times \omega)$ such that $R_k \subseteq S_k$ for all $k \in \omega$ and $\bigcup_{k\in\omega} R_k = \bigcup_{k\in\omega} S_k$.

b) Let T be a consistent theory in a language with just finitely many non-logical symbols, including at least the unary function symbol s and the constant $\bar{0}$. Assume that every recursive relation is representable in T. Prove that T is undecidable.

5. a) Let $A_0 = \{e \in \omega : \forall k(\{e\}(k) = 0)\}$ and $A_1 = \{e \in \omega : \forall k(\{e\}(k) = 1)\}$. Prove or disprove: there is some recursive $B \subseteq \omega$ such that $A_0 \subseteq B$ and $(A_1 \cap B) = \emptyset$.

b) Let $A, B \subseteq \omega$. Explicitly define some $C \subseteq \omega$ such that the Turing degree of C is the least upper bound of the Turing degree of A and the Turing degree of B. You must prove that C has these properties.

6. a) Recall that $\mathbf{INF} = \{e \in \omega : W_e \text{ is infinite}\}$. Prove that
$\mathbf{INF} \leq_m \{e \in \omega : \forall k(\{e\}(k) = 0)\}$.

b) Define $E \subseteq (\omega \times \omega)$ by $E = \{(e_1, e_2) : \{e_1\} = \{e_2\}\}$. Place E in the arithmetic hierarchy, that is determine (with proof) some $n \in \omega$ such that either $E \in \Sigma_n$ or $E \in \Pi_n$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

August 2005

LOGIC (Ph.D./M.A. version)

1. a) Let L be a language containing (at least) the unary function symbol s. An L-structure $\mathfrak{A}$ is *periodic* iff for every $a \in A$ there is some positive integer n such that $(s^{\mathfrak{A}})^n(a) = a$. Prove that there is no L-theory T such that for all L-structures $\mathfrak{A}$, $\mathfrak{A} \models T$ iff $\mathfrak{A}$ is periodic.

b) Let T be a complete ω-categorical theory in a countable language L. Let $\varphi(x, y) \in Fm_L$ and let $\mathfrak{A}$ be any model of T. Prove that there is some $n \in \omega$ such that for every $a \in A$ either $|\varphi^{\mathfrak{A}}(x, \bar{a})| < n$ or $\varphi^{\mathfrak{A}}(x, \bar{a})$ is infinite.

2. a) Let L be the language with $L^{nl} = \{+, \cdot, <, \bar{0}, s\}$, let $\mathfrak{N} = (\omega, +, \cdot, <, 0, s)$, and let $\mathfrak{A}$ be any proper elementary extension of $\mathfrak{N}$. Let $\varphi(x) \in Fm_L$. Prove that $\varphi^{\mathfrak{N}}$ is infinite if and only if there is some $a \in A$ such that $a \in (\varphi^{\mathfrak{A}} \setminus \omega)$.

b) Let T be a complete theory in a countable language L. Let $\Phi(x)$ and $\Psi(x)$ be types consistent with T. Assume that every model of T realizes either Φ or Ψ (or both). Prove that either every model of T realizes Φ or every model of T realizes Ψ.

3. Let T be a complete theory in a countable language L with infinite models.

a) Prove that every countable model of T has a proper countable elementary extension.

b) Assume that $\mathfrak{A} \models T$ is countable and ω_1-universal. Prove that $\mathfrak{A}$ is isomorphic to some proper elementary extension of itself.

c) Assume that $\mathfrak{A} \models T$ is countable and isomorphic to every countable elementary extension of itself. Prove that $\mathfrak{A}$ is ω-saturated.

4. Let L be the language with $L^{nl} = \{+, \cdot, <, \bar{0}, s\}$ and let $\mathfrak{N} = (\omega, +, \cdot, <, 0, s)$.

a) Define the function $\pi : \omega \to \omega$ by $\pi(n) =$ the number of primes $\leq n$. Prove or disprove: there is some $\varphi(x, y) \in Fm_L$ which defines the graph of π (that is, the relation $\pi(n) = l$) in $\mathfrak{N}$.

b) Prove that there is some $\theta(y) \in Fm_L$ such that for every Σ-formula $\varphi(x)$ and for every $n \in \omega$ we have $\mathfrak{N} \models \theta(\ulcorner\varphi(\bar{n})\urcorner)$ iff $\mathfrak{N} \models \varphi(\bar{n})$.

5. a) Assume that $R \subseteq \omega \times \omega$ is r.e., R_k is infinite for all $k \in \omega$, and $(R_k \cap R_l) = \emptyset$ whenever $k \neq l$. Prove that there is some recursive $C \subseteq \omega$ such that $|C \cap R_k| = 1$ for all $k \in \omega$.

b) Give an example of a theory T in a language L with just finitely many non-logical symbols which is undecidable but not essentially undecidable (you must establish these properties of T).

6. a) Prove or disprove: there is some arithmetic relation $R \subseteq \omega \times \omega$ such that for every arithmetic $X \subseteq \omega$ there is some $k \in \omega$ such that $X = R_k$.

Let $A = \{e \in \omega : 0 \in W_e\}$, $B = \{e \in \omega : 1 \in W_e\}$, and let $C = \{e \in \omega : 0 \notin W_e\}$. Prove that

b) $A \leq_m B$, but

c) $A \not\leq_m C$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

January 2005

LOGIC (Ph.D./M.A. version)

1. a) Let T be a theory in a language L and let $\varphi(x), \psi_k(x) \in Fm_L$ for all $k \in \omega$. Assume that $T \models \forall x(\psi_k \to \psi_{k+1})$ for all $k \in \omega$. Assume further that for every $\mathfrak{A} \models T$ and every $a \in A$ we have

$\mathfrak{A}_A \models \varphi(\bar{a})$ iff there is some $k \in \omega$ such that $\mathfrak{A}_A \models \psi_k(\bar{a})$.

Prove that there is some $k \in \omega$ such that $T \models \forall x(\varphi \leftrightarrow \psi_k)$.

b) Prove that there is some $\mathfrak{A} \equiv (\omega, <)$ such that $(\mathbb{R}, <)$ can be isomorphically embedded into $\mathfrak{A}$.

2. a) Let L be the language whose only non-logical symbol is a binary relation symbol $<$ and let $\mathfrak{B}$ be the L-structure $(\mathbb{Q}, <)$. Let $X \subseteq \mathbb{Q}$ be finite. Prove that the set $\mathbb{Z}$ is not definable in the $L(X)$-structure $\mathfrak{B}_X$.

b) Let L be the language whose only non-logical symbol is a binary relation symbol E. Let $\mathfrak{A}$ be the L-structure such that

$E^{\mathfrak{A}}$ is an equivalence relation on A,

there is exactly one n-element equivalence class for every positive integer n, and

there are no infinite equivalence classes.

Is there is some proper substructure $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{A} \equiv \mathfrak{B}$? Prove or disprove.

3. a) Let T be a complete theory in a countable language L. Assume that T has no countable ω-saturated model. Prove that every type consistent with T is realized on at least two non-isomorphic countable models of T.

b) Let T be a complete theory in a countable language L. Let $\Phi(x)$ be a complete non-principal type consistent with T. Let $\mathfrak{A}$ be an ω-saturated model of T. Prove that Φ is realized by infinitely many elements of A.

4. a) Let T be a consistent recursively axiomatizable theory in the language L for arithmetic, let $\varphi(x) \in Fm_L$, and let $A \subseteq \omega$. Assume that A is weakly representable in T by φ and A is not recursive. Prove that there is some $k \in \omega$ such that $k \notin A$, $T \nvdash \neg\varphi(\bar{k})$, and $T \nvdash \varphi(\bar{k})$.

b) Let L be a language with just finitely many non-logical symbols which contains at least the unary function symbol s and the constant symbol $\bar{0}$. Let T be a consistent theory of L such that all recursive functions and relations are representable in T. Prove that T is undecidable.

5. a) Let $A \subseteq \omega$ be an infinite r.e. set. Prove that there are infinite recursive sets B_0 and B_1 contained in A such that $(B_0 \cap B_1) = \emptyset$.

b) Let $A, B \subseteq \omega$. Prove that B is r.e. in A iff $B \leq_m A'$.

6. a) Let $A = \{e \in \omega : \{e\}(e) = e\}$. Prove that A is not recursive.

b) Let $A = \{e \in \omega : |W_e| \leq 1\}$ and let $B = \{e \in \omega : |W_e| \geq 2\}$. Prove that $A \equiv_T B$ but $A \not\equiv_m B$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

August 2004

LOGIC (Ph.D./M.A. version)

1. a) Let T be a theory in a language L containing at least the binary relation symbol E. Assume that for every $\mathfrak{A} \models T$, $E^{\mathfrak{A}}$ is an equivalence relation on A. Assume further that whenever $\mathfrak{A} \models T$, $\mathfrak{A} \prec \mathfrak{B}$, and $a \in A$ then $\{b \in B : E^{\mathfrak{B}}(a, b) \text{ holds}\} \subseteq A$. Prove that there is some $n \in \omega$ such that for every $\mathfrak{A} \models T$ every $E^{\mathfrak{A}}$-class has $< n$ elements.

b) Let T be a theory of L and let $\Phi(x)$ and $\Psi(x)$ be L-types. We say that a formula $\theta(x)$ of L *separates* Φ and Ψ if in every model of T every element realizing Φ satisfies θ and every element realizing Ψ satisfies $\neg\theta$. Assume that no formula of L separates Φ and Ψ. Prove that T has a model realizing $(\Phi \cup \Psi)$.

2. a) Prove that there is no formula $\varphi(x)$ which defines $\{1\}$ in the structure $(\mathbf{Q}, <, +)$.

b) Prove or disprove: $\mathrm{Th}((\mathbf{Q}, +, \cdot, <, 0, 1))$ has a countable ω-saturated model.

3. a) Let T be a complete theory in a countable language. Assume that there is some complete, non-principal type in one variable consistent with T. Prove that there are infinitely many complete types in one variable consistent with T.

b) Let L be the language whose only non-logical symbol is the binary relation symbol $<$. An L-structure $\mathfrak{A}$ is a *linear order* provided $<^{\mathfrak{A}}$ is a linear order of A. Prove that there is some infinite linear order $\mathfrak{A}$ such that every L-sentence true on $\mathfrak{A}$ is also true on some finite linear order.

4. a) Let $A \subseteq \omega$ be an infinite r.e. set. Prove that there is some infinite recursive set $B \subseteq A$.

b) Let L be the language for arithmetic on the natural numbers, that is, $L^{nl} = \{+, \cdot, <, \bar{0}, s\}$. Let $A = \{\ulcorner\sigma\urcorner : \models \sigma\}$. Prove that A is an m-complete r.e. set.

5. a) Let $A = \{e \in \omega : W_e = \emptyset\}$ and let $B = \{e \in \omega : W_e = \omega\}$. Prove that A and B are recursively inseparable, that is there is no recursive $C \subseteq \omega$ such that $A \subseteq C$ and $(B \cap C) = \emptyset$.

b) Prove that there is some $B \subseteq \omega$ such that $A \leq_m B$ for every arithmetic set $A \subseteq \omega$.

6. a) Define a partial recursive function g of one argument which cannot be extended to a total recursive function, i.e., there is no total recursive $f : \omega \to \omega$ such that $f(n) = g(n)$ whenever $g(n) \downarrow$.

b) Prove that there are infinitely many $e \in \omega$ such that $\{e\}(e+1) = 2e$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

January 2004

LOGIC (Ph.D./M.A. version)

1. a) Let L be a countable language containing at least the binary relation symbol E. Let T be a theory of L such that in every model $\mathfrak{A}$ of T, $E^{\mathfrak{A}}$ is an equivalence relation on A. Let $\varphi(x) \in Fm_L$. Assume that no model $\mathfrak{A}$ of T contains an element satisfying φ whose $E^{\mathfrak{A}}$-class is infinite. Prove that there is some $n \in \omega$ such that no model $\mathfrak{A}$ of T contains an element satisfying φ whose $E^{\mathfrak{A}}$-class has $> n$ elements.

   b) Let $\mathfrak{A} = (\omega, +, \cdot)$ and let $\mathfrak{B}$ be a proper elementary extension of $\mathfrak{A}$. Prove that there are infinitely many primes in $(B \setminus \omega)$. [An element b of B is prime if it cannot be expressed in $\mathfrak{B}$ as the product of two elements of B each of which is different than b]

2. a) Let $L^{nl} = \{E\}$ where E is a binary relation symbol. Let $\mathfrak{A}$ be the L-structure such that $E^{\mathfrak{A}}$ is an equivalence relation on A with exactly one n-element equivalence class for every positive integer n and with no infinite equivalence classes. Let $\mathfrak{B}$ be a countable elementary extension of $\mathfrak{A}$. Prove that $tp_{\mathfrak{B}}(b_1) = tp_{\mathfrak{B}}(b_2)$ for all $b_1, b_2 \in (B \setminus A)$.

   b) Let $L = (L_1 \cap L_2)$ and assume that $(L_i \setminus L)$ contains just constant symbols, for $i = 1, 2$. Let T be a complete theory of L and let T_i be a theory of L_i for $i = 1, 2$. Assume that some model of T can be expanded to a model of T_1, and also that some model of T can be expanded to a model of T_2. Prove that there is some model $\mathfrak{A}$ of T such that $\mathfrak{A}$ can be expanded to a model $\mathfrak{A}_1$ of T_1 and $\mathfrak{A}$ can also be expanded to a model $\mathfrak{A}_2$ of T_2.

3. a) Let L be a countable language containing at least the binary relation symbol E. Let T be a theory of L such that $T \models \forall x \forall y(Exy \to Eyx)$. If $\mathfrak{A} \models T$ and $a, a^* \in A$ with $a \neq a^*$ we say that a, a^* are *connected* if either $E^{\mathfrak{A}}(a, a^*)$ holds or there are $a_1, ..., a_n \in A$ for some positive integer n such that

$$E^{\mathfrak{A}}(a, a_1),\ E^{\mathfrak{A}}(a_i, a_{i+1}) \text{ for all } 1 \leq i < n, \text{ and } E^{\mathfrak{A}}(a_n, a^*)$$

all hold. Assume that in every model of T there is a pair of distinct elements that is not connected. Prove that there is some $\psi(x, y) \in Fm_L$ consistent with T such that for every $\mathfrak{A} \models T$ and every $a, a^* \in A$, if $\mathfrak{A}_A \models \psi(\bar{a}, \bar{a^*})$ then $a \neq a^*$ and a, a^* are not connected.

b) Let T be a complete theory in a countable language L. Let $\mathfrak{A}$ be a prime model of T and let $\Phi(x)$ be a complete type of L. Assume that Φ is realized by exactly two elements in $\mathfrak{A}$. Prove that Φ is realized by exactly two elements in every model of T.

4. a) Let $R \subseteq \omega \times \omega$ be r.e. and assume that $\bigcup_{k \in \omega} R_k$ is recursive. Prove that there is some recursive $S \subseteq \omega \times \omega$ such that $S_k \subseteq R_k$ for all $k \in \omega$ and $\bigcup_{k \in \omega} S_k = \bigcup_{k \in \omega} R_k$.

b) A total function $f : \omega \to \omega$ is *monotone* iff for all $m, n \in \omega$, if $m \leq n$ then $f(m) \leq f(n)$. Let f be a recursive monotone function. Prove that the range of f is recursive. [Warning: f need not be strictly increasing]

5. a) Give an example of a theory T which is undecidable but not essentially undecidable. [You must prove both assertions about T]

b) Prove that there are r.e. sets $A, B \subseteq \omega$ such that $(A \cap B) = \emptyset$ but there is no recursive $C \subseteq \omega$ such that $A \subseteq C$ and $(B \cap C) = \emptyset$.

6. a) Prove that $\{e : 2 \in W_e\} \equiv_m \{e : 3 \in W_e\}$.

b) Let $I = \{e \in \omega : W_e = \{3\}\}$. Determine some $n \in \omega$ such that either $I \in \Sigma_n$ or $I \in \Pi_n$. [You need not prove your choice of n is minimal]

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

August 2003

LOGIC (Ph.D./M.A. version)

1. a) Let T be a theory of a language L. Assume that there is some $\theta \in Sn_L$ such that for every model $\mathfrak{A}$ of T, $\mathfrak{A}$ is infinite iff $\mathfrak{A} \models \theta$. Prove that there is some $n \in \omega$ such that every finite model of T has at most n elements.

   b) Prove that $(\mathbf{Q}, +, \cdot, 0, 1)$ is a prime model of its complete theory.

2. a) Let $\mathfrak{N} = (\omega, +, \cdot, <, 0, s)$ be the standard model for arithmetic on ω and let $\mathfrak{B}$ be some fixed proper elementary extension of $\mathfrak{N}$. Let $\varphi(x) \in Fm_L$ and assume that $\varphi^{\mathfrak{N}} = \varphi^{\mathfrak{B}}$. Prove that $\varphi^{\mathfrak{N}}$ is finite.

   b) Let $L^{nl} = \{E\}$ where E is a binary relation symbol. An L-structure $\mathfrak{A}$ is a *graph* provided $\mathfrak{A} \models \forall x \forall y (Exy \rightarrow Eyx)$. A graph $\mathfrak{A}$ is *connected* iff for all $a, a^* \in A$ with $a \neq a^*$ there are $a_1, ..., a_n \in A$ for some $n \in \omega$ such that

   $$E^{\mathfrak{A}}(a, a_1),\ E^{\mathfrak{A}}(a_i, a_{i+1}) \text{ for all } 1 \leq i < n, \text{ and } E^{\mathfrak{A}}(a_n, a^*)$$

   all hold. Let T be an L-theory such that every connected graph is a model of T. Prove that there is some graph which is a model of T but is not connected.

3. a) Let T be a complete theory in a countable language L. Assume that for every $\varphi(x) \in Fm_L$ consistent with T there is some $\psi(x) \in Fm_L$ such that both $(\varphi \wedge \psi)$ and $(\varphi \wedge \neg\psi)$ are consistent with T. Prove that T does not have a prime model.

b) Let T be a complete theory in a countable language L. Let $\mathfrak{A} \models T$ be countable and assume that $\mathfrak{A}$ is isomorphic to each of its countable elementary extensions. Prove that T has a countable ω-saturated model and that $\mathfrak{A}$ itself is ω-saturated.

4. a) Let L be a language with just finitely many non-logical symbols, including at least the unary function symbol s and the constant 0. Let T be a theory of L such that every recursive relation is representable in T. Prove that T is undecidable.

b) Let $A = \{\ulcorner\sigma\urcorner : \sigma \text{ is a } \Sigma\text{-sentence and } \mathfrak{N} \models \sigma\}$, where $\mathfrak{N}$ is the usual model for arithmetic on ω. Prove that A is not Π_1.

5. a) Let $R \subseteq \omega \times \omega$ be r.e. Assume that $R_k \neq \emptyset$ for all $k \in \omega$, $\bigcup_{k\in\omega} R_k = \omega$, and for all $k, l \in \omega$ either $R_k = R_l$ or $R_k \cap R_l = \emptyset$. Assume further that there is some recursive $C \subseteq \omega$ such that for all $k \in \omega$, $|R_k \cap C| = 1$. Prove that R is recursive.

b) Let $A = \{e \in \omega : W_e \text{ is either finite or cofinite}\}$. Find an n so that $A \in \Delta_n$. [You need not prove your n is the least possible]

6. a) Let $A, B \subseteq \omega$ be recursively inseparable r.e. sets (so $A \cap B = \emptyset$ and there is no recursive set A^* with $A \subseteq A^*$ and $A^* \cap B = \emptyset$.) Assume that $A \leq_m C$ where $C \subseteq \omega$. Prove that there is some infinite r.e. set $D \subseteq \omega$ such that $C \cap D = \emptyset$.

b) Let $I = \{e \in \omega : |W_e| = 1\}$. Prove that every r.e. set is many-one reducible to I.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

January 2003

LOGIC (Ph.D./M.A. version)

1. a) Prove or disprove: $(\mathbb{Z}, +)$ has a proper elementary substructure.

b) Assume that $\mathfrak{A}$ and $\mathfrak{B}$ are L-structures and $\mathfrak{A} \equiv \mathfrak{B}$. Prove that there is some $\mathfrak{C}$ such that both $\mathfrak{A}$ and $\mathfrak{B}$ can be elementarily embedded in $\mathfrak{C}$.

2. a) Let L be a countable language containing (at least) the binary relation symbol E. Let T be a complete ω-categorical L-theory, let $\mathfrak{A}$ be a countable model of T, and assume that $E^{\mathfrak{A}}$ is an equivalence relation on A. Prove that there is some $n \in \omega$ such that for every $a \in A$ the $E^{\mathfrak{A}}$-class of a is either infinite or has fewer than n elements.

b) Let T be a complete theory in a countable language L and let $\Phi(x)$ be a complete L-type. Assume that T has some model which contains exactly one element realizing Φ and also some model which contains exactly two elements realizing Φ. Prove that T has a model omitting Φ.

3. a) Let $L^{nl} = \{c_n : n \in \omega\}$. Let $\mathfrak{A}$ be an L- structure such that ${c_n}^{\mathfrak{A}} \neq {c_m}^{\mathfrak{A}}$ for all $n \neq m$ and such that there is exactly one element $a^* \in A$ such that $a^* \neq {c_n}^{\mathfrak{A}}$ for all $n \in \omega$. Prove that there is no formula $\varphi(x)$ of L such that $\varphi^{\mathfrak{A}} = \{a^*\}$.

b) Let $\mathfrak{A}$ be a countable ω-saturated structure for a countable language L. Let $a_0 \in A$ be such that $h(a_0) = a_0$ for every automorphism h of $\mathfrak{A}$. Prove that there is some formula $\varphi(x)$ of L such that $\varphi^{\mathfrak{A}} = \{a_0\}$.

4. a) Let T be a recursively axiomatizable theory true on $\mathfrak{N}$, the standard model for arithmetic on the natural numbers. Let $X \subseteq \omega$ be r.e. but not recursive, and assume that $X = \varphi^{\mathfrak{N}}$ for some Σ-formula $\varphi(x)$. Prove that there is some $\mathfrak{B} \models T$ such that $\mathfrak{B} \models \varphi(\bar{n})$ for some $n \in (\omega \setminus X)$.

   b) Let $R \subseteq (\omega \times \omega)$ be r.e. Assume the R_n's are infinite and pairwise disjoint. Prove that there is some recursive $C \subseteq \omega$ such that $|R_n \cap C| = 1$ for all $n \in \omega$.

5. a) Let $L^{nl} = \emptyset$. Give an example of a theory T of L which is undecidable but all its complete extensions (in L) are decidable.

   b) Let T be a recursively axiomatizable theory in a language L with just finitely many non-logical symbols. Assume that T has just finitely many complete extensions (in L). Prove that T is decidable.

6. a) Recall that

   $$FIN = \{e : W_e \text{ is finite}\} \text{ and } INF = \{e : W_e \text{ is infinite}\}.$$

   Prove that $FIN \leq_T INF$ but $FIN \not\leq_m INF$.

   b) Recall that $REC = \{e : W_e \text{ is recursive}\}$. Prove that REC is arithmetic, that is, that REC is in Σ_n or Π_n for some $n \in \omega$. Although you should try to make n as small as possible, you do **not** need to prove your choice of n is minimal.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

August 2002

LOGIC (Ph.D./M.A. version)

1. a) Let a theory T and sentences σ_n of a language L be given. Assume that $T \models (\sigma_n \to \sigma_{n+1})$ for all $n \in \omega$. Assume further that for every $\mathfrak{A} \models T$ there is some $n \in \omega$ such that $\mathfrak{A} \models \sigma_n$. Prove that there is some $n_0 \in \omega$ such that $T \models (\sigma_{n_0+1} \to \sigma_{n_0})$. [In fact, $T \models (\sigma_m \to \sigma_{n_0})$ will hold for all $m > n_0$.]

   b) Let L_0 be the language containing just the binary relation symbol $<$, let L be a language containing L_0, and let T be a theory of L. Assume that $(\omega, <)$ embeds into the L_0-reduct of some model of T. Prove that $(\mathbf{Q}, <)$ can be embedded into the L_0-reduct of some model of T.

2. a) Let $\mathfrak{A}$ be $(\omega, +, \cdot, <, 0, s)$. In $\mathfrak{A}$ the set of primes is definable by the following formula $\varphi(x)$:

$$(s\bar{0} < x) \wedge \forall y \forall z (x = y \cdot z \to (x = y) \vee (x = z))$$

   Let $\mathfrak{B}$ be any proper elementary extension of $\mathfrak{A}$. Prove that $\mathfrak{B}$ contains a new prime, that is,.some element b satisfying $\varphi(x)$ which is not in ω.

   b) Let L be the language whose only non-logical symbol is the binary relation E and let T be the L-theory axiomatized by sentences saying that E is an equivalence relation on the universe with infinitely many equivalence classes, each of which is infinite. Prove that T is model complete, that is, for all models $\mathfrak{A}$ and $\mathfrak{B}$ of T, if $\mathfrak{A} \subseteq \mathfrak{B}$ then $\mathfrak{A} \prec \mathfrak{B}$.

3. a) Let T be a complete theory in a countable language L. Assume that there is some non-principal complete type in one variable consistent with T. Prove that every model of T realizes (at least) three different complete types in one variable. [In fact each model of T will realize infinitely many, but you need not prove this.]

b) Let $\mathfrak{A}$ be an ω-saturated L-structure and let $\varphi(x, y)$ be an L-formula. Assume that for every $a \in A$ the set $\varphi^{\mathfrak{A}}(x, \bar{a})$ is finite. Prove that there is some $n \in \omega$ such that for every $a \in A$ the set $\varphi^{\mathfrak{A}}(x, \bar{a})$ contains at most n elements.

4. a) Assume that $R \subseteq \omega \times \omega$ is r.e. and that R_n is infinite for every $n \in \omega$. Let $g : \omega \to \omega$ be any recursive function. Prove that there is some recursive function $f : \omega \to \omega$ such that $f(n) \in R_n$ and $g(n) < f(n)$ for all $n \in \omega$.

b) Let L be the language whose only non-logical symbol is the binary relation E and let T_0 be the L-theory axiomatized by sentences stating that E is an equivalence relation on the universe. Prove that T has a complete undecidable extension.

5. a) Define $f : \omega \to \omega$ by

$$f(n) = (\mu k)[\{n\} = \{k\}].$$

Prove that f is not recursive.

b) Assume that $B \subseteq \omega$ is such that $A \leq_m B$ for all r.e. sets A. Prove that B contains some infinite r.e. subset.

6. a) Let $A_n \subseteq \omega$ be given for all $n \in \omega$. Prove that there is some $B \subseteq \omega$ such that $A_n \leq_T B$ holds for all $n \in \omega$.

b) Let $A = \{e \in \omega : \{e\}(5) = 7\}$. Prove that $A \equiv_m K$. [Recall that $K = \{e : \{e\}(e) \downarrow\}$]

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

January 2002

LOGIC (Ph.D./M.A. version)

1. a) Let T be a theory of L, let $\Phi(x)$ and $\Psi(x)$ be types of L. Assume that for every $\mathfrak{A} \models T$ and all $a \in A$, a realizes Φ iff a does not realize Ψ Prove that there is some $\varphi(x) \in Fm_L$ such that $\Phi^{\mathfrak{A}} = \varphi^{\mathfrak{A}}$ for every model $\mathfrak{A}$ of T

   b) Let L be a language containing (at least) the binary relation symbol E Let $\mathfrak{A}$ be an ω-saturated L-structure in which $E^{\mathfrak{A}}$ is an equivalence relation on A with exactly one infinite equivalence class. Prove that there is some $n \in \omega$ such that every finite $E^{\mathfrak{A}}$-class has at most n elements.

2. a) Prove or disprove: $(\omega, +)$ has a proper elementary substructure.

   b) Let T be an L-theory. Let $\mathfrak{A}$ be an L-structure which cannot be embedded in any model of T. Prove that there is an existential sentence θ of L (that is, θ has the form $\exists x_1 ... \exists x_n \alpha$ where α is an open formula of L) such that $\mathfrak{A} \models \theta$ but $T \models \neg\theta$.

3. a) Prove that the structure $(\omega, |)$ has uncountably many automorphisms (where $n|k$ iff $k = n \cdot l$ for some $l \in \omega$).

   b) Let T be a complete theory in a countable language L and let $\Phi(x)$ be an L-type which is omitted on some model of T. Assume further that any two countable models of T omitting Φ are isomorphic. Prove that every countable model of T omitting Φ is prime.

   [Warning: You cannot assume that T has a prime model]

4. a) Assume that $R \subseteq \omega \times \omega$ is r.e. and that $\bigcup_{k \in \omega} R_k = \omega$. Prove that there is some recursive $S \subseteq R$ such that $\bigcup_{k \in \omega} S_k = \omega$.

   b) Let L be a language with only finitely many non-logical symbols and let $L' = L \cup \{c\}$ where c is a constant symbol not in L. Let T'be a finitely axiomatizable undecidable theory of L' and let $T = T' \cap Sn_L$. Prove that T is also undecidable.

5. Recall that subsets A and B of ω are called *recursively inseparable* if there is no recursive $C \subseteq \omega$ such that $A \subseteq C$ and $B \cap C = \emptyset$.

   a) Prove that there are disjoint r.e. subsets A and B of ω which are recursively inseparable.

   b) Assume that A and B are disjoint r.e. subsets of ω which are recursively inseparable. Prove that $\omega \setminus (A \cup B)$ is infinite.

6. a) Let $A = \{\ulcorner\sigma\urcorner : \sigma \in Sn_L \text{ and } Q \vdash \sigma\}$ (where L is the usual language for arithmetic on the natural numbers). Prove that A is an m-complete r.e. set.

   b) Prove that there is some $A \subseteq \omega$ such that $A \in \Sigma_3$ but $A \notin \Pi_2$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

August 2001

LOGIC (Ph.D./M.A. version)

1. a) Let T be a theory of a language L, and let $\varphi_i(x)$ be formulas of L for all $i \in \omega$. Assume that for all $i \in \omega$
$T \models \forall x(\varphi_{i+1}(x) \rightarrow \varphi_i(x))$ and $T \models \neg\forall x(\varphi_i(x) \rightarrow \varphi_{i+1}(x))$.
Prove that T has a model $\mathfrak{A}$ with an element a such that $\mathfrak{A} \models \varphi_i(\bar{a})$ for all $i \in \omega$.

b) Let T be a complete theory in a countable language L, and assume that for each $n > 0$ there are just countably many complete types in n free variables consistent with T. Prove that T has a prime model.

2. a) Prove or disprove: $(\mathrm{Z}, <)$ has a proper elementary submodel.

b) Does $Th((\mathrm{Z}, +, 1))$ have a countable ω- saturated model? Prove your answer.

3. a) Let $\mathfrak{A}$ be the unique countable model of a a complete ω-categorical theory T in a countable language L, and let $\varphi(x, y) \in Fm_L$. Prove that there is some $n \in \omega$ such that for every $a \in A$, either $|\varphi^{\mathfrak{A}}(x, \bar{a})| < n$ or $\varphi^{\mathfrak{A}}(x, \bar{a})$ is infinite.

b) Let T be a complete theory in a countable language L having infinite models. Assume that for every $\varphi(x) \in Fm_L$ and for every $\mathfrak{A} \models T$, $\varphi^{\mathfrak{A}}$ is either finite or cofinite (meaning its complement is finite). Prove that there is exactly one non-principal complete type $\Phi(x)$ in the single variable x consistent with T.

4. a) Let T be a consistent recursively axiomatized theory containing the axioms for Q. Prove that there is a formula $\varphi(x)$ such that

$$T \models \varphi(\bar{n}) \text{ for all } n \in \omega \text{ but } T \not\models \forall x \varphi(x).$$

b) Let $R \subseteq \omega \times \omega$ be r.e., and assume that $|\omega \setminus R_k| = 2$ for every $k \in \omega$. Prove that R is recursive.

5. a) Assume that $A \subseteq \omega$ is such that

$$\{e : W_e = \emptyset\} \subseteq A \text{ and } \{e : W_e = \omega\} \cap A = \emptyset.$$

Prove that A is not recursive.

b) Assume that $A \subseteq \omega$ is such that $K \leq_m A$. Prove that A contains an infinite r.e. subset.

[Recall that $K = \{e : e \in W_e\}$]

6. a) Let T be a consistent, decidable theory in a language L with just finitely many non-logical symbols. Prove that $T \subseteq T^*$ for some complete, decidable theory T^* of L.

[Hint: Let $\{\sigma_n : n \in \omega\}$ be a recursive list of all sentences of L ...]

b) Prove that $TOT \equiv_m INF$.

[Recall that $TOT = \{e : W_e = \omega\}$ and $INF = \{e : W_e \text{ is infinite}\}$]

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

January 2001

LOGIC (Ph.D./M.A. version)

1. a) Assume that $L \subseteq L'$, let T' be an L'-theory and let $\mathfrak{A}$ be an L-structure. Assume that there is no $\mathfrak{B}' \models T'$ such that $\mathfrak{A}$ is elementarily equivalent to the L-reduct of $\mathfrak{B}'$. Prove that there is some $\sigma \in Sn_L$ such that $\mathfrak{A} \models \sigma$ and $T' \models \neg\sigma$.

b) Let $L^{nl} = \{E\}$ where E is a binary relation symbol. Let K be the class of all L-structures $\mathfrak{A}$ for which $E^{\mathfrak{A}}$ is an equivalence relation on A with at least one finite $E^{\mathfrak{A}}$-class. Prove that there is no theory T of L such that $K = Mod(T)$.

[Hint: Assume that $K \subseteq Mod(T)$ and find $\mathfrak{A} \models T$ such that $\mathfrak{A} \notin K$.]

2. a) Let L contain at least the binary relation symbol E, and let $\mathfrak{A}$ be an infinite ω-saturated L-structure such that $E^{\mathfrak{A}}$ is an equivalence relation on A. Assume that whenever $\mathfrak{A} \prec \mathfrak{B}$ and $a \in A$ then

$$\{b \in B : E^{\mathfrak{B}}(a, b) \text{ holds}\} \subseteq A.$$

Prove that there is some $n_0 \in \omega$ such that every $E^{\mathfrak{A}}$-class has at most n_0 elements.

b) Let L be a countable language containing at least the unary relation symbols P_n for $n \in \omega$, and let T be a theory of L. Assume that T has a model $\mathfrak{A}$ such that for every $\varphi(x) \in Fm_L$ if $\varphi^{\mathfrak{A}} \neq \emptyset$ then there is some $k \in \omega$ such that $(\varphi^{\mathfrak{A}} \cap P_k{}^{\mathfrak{A}}) \neq \emptyset$. Prove that T has a model $\mathfrak{B}$ such that $B = \bigcup_{k\in\omega} P_k{}^{\mathfrak{B}}$.

3. Let T be a complete theory in a countable language L. Recall that a complete type $\Phi(x)$ consistent with T is said to be *non-principal* provided it does not contain a complete formula $\varphi(x)$.

a) Assume that $\Phi(x)$ is a non-principal complete type consistent with T. Prove that T has some model which contains infinitely many elements realizing $\Phi(x)$.

b) Assume that there are no non-principal complete types $\Phi(x)$ in the single free variable x consistent with T. Prove that there are only finitely many complete types in the single free variable x consistent with T.

4. a) Let A and B be r.e. subsets of ω. Assume that $(A \cup B)$ is recursive. Prove that there are recursive sets $A' \subseteq A$ and $B' \subseteq B$ such that $(A \cup B) = (A' \cup B')$.

b) Let A be an infinite r.e. subset of ω. Prove that there is an infinite recursive set B with $B \subseteq A$.

5. a) Give a theory T in a language L with just finitely many non-logical symbols which has an r.e. set of axioms but is such that

$$\{n \in \omega : T \text{ has a model } \mathfrak{A} \text{ with } |A| = n\}$$

is not recursive. Prove that it has these properties.

b) Assume that $R \subseteq \omega \times \omega$ is r.e. Let $A = \{k \in \omega : R_k \text{ is infinite}\}$. Prove that A is Π_2.

6. a) Recall that $\mathbf{K} = \{e \in \omega : e \in W_e\}$ and that $\mathbf{INF} = \{e \in \omega : W_e \text{ is infinite}\}$. Prove that $\mathbf{K} \leq_m \mathbf{INF}$.

b) Let $\mathcal{F}$ be a non-empty set of partial recursive functions of one argument and let $I = \{e \in \omega : \{e\} \in \mathcal{F}\}$. Prove that $I \not\leq_m (\omega \setminus I)$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

August 2000

LOGIC (Ph.D./M.A. version)

1. a) Let T be a theory of a language L containing (at least) the binary relation symbol E and so that for every $\mathfrak{A} \models T$, $E^{\mathfrak{A}}$ is an equivalence relation on A. Assume further that whenever $\mathfrak{A} \models T$, $\mathfrak{A} \prec \mathfrak{B}$, $a \in A$ and $b \in (B \setminus A)$ then $\mathfrak{B}_B \models \neg E(\bar{a}, \bar{b})$. Prove that there is some $n_0 \in \omega$ such that for every $\mathfrak{A} \models T$ all $E^{\mathfrak{A}}$-classes have $\leq n_0$ elements.

b) Let the only non-logical symbol of L be the binary relation symbol E. Let $\mathfrak{A}$ be the L-structure in which $E^{\mathfrak{A}}$ is an equivalence relation on A with infinitely many 2 element classes and infinitely many 3 element classes and no other classes. Let $\mathfrak{A} \subseteq \mathfrak{B}$ where $\mathfrak{B}$ adds exactly one more 2 element class and nothing else. Prove that $\mathfrak{A} \prec \mathfrak{B}$. [Hint: why are $\mathfrak{A}$ and $\mathfrak{B}$ elementarily equivalent?]

2. a) Is the structure $(\mathbf{R}, +, \cdot, 0, 1)$ ω-saturated? Explain.

b) Assume that the L-structure $\mathfrak{A}$ realizes exactly three different complete L-types in one free variable. Prive that the same is true of every model of $Th(\mathfrak{A})$.

3. a) Let T be a complete theory in a countable language L, and let $\Phi(x)$ be an L-type. Assume that in every model of T the type Φ is realized by at most 2 elements. Prove that there is a formula $\varphi(x)$ of L such that for every $\mathfrak{A} \models T$, $\Phi^{\mathfrak{A}} = \varphi^{\mathfrak{A}}$.

b) Let T be a complete theory in a countable language L which has no prime model. Let $\Phi(x)$ be an L-type omitted on some model of T. Prove that T has at least two nonisomorphic countable models omitting Φ.

4. a) Assume that $R \subseteq \omega \times \omega$ is r.e. and that R_k is infinite for all $k \in \omega$. Prove that there is a strictly increasing recursive function f on ω such that $f(k) \in R_k$ for all $k \in \omega$.

b) Prove that there is a function $g : \omega \to \omega$ such that for every *recursive* function f on ω there is some $n_0 \in \omega$ so that for all $n \geq n_0$ we have $f(n) < g(n)$.

5. a) Assume that $R \subseteq \omega \times \omega$ is r.e. but not recursive and that $\bigcup_{k\in\omega} R_k$ is recursive. Prove that $R_k \cap R_l \neq \emptyset$ for some $k \neq l$.

b) Let f_1 and f_2 be partial recursive functions and assume that $f_1 \neq f_2$. Let $B_1 = \{e : \{e\} = f_1\}$ and let $B_2 = \{e : \{e\} = f_2\}$. Prove that there is no recursive set A such that $B_1 \subseteq A$ and $B_2 \cap A = \emptyset$.

6. a) Prove that $\{e : 0 \in W_e\}$ is an m-complete r.e. set.

b) Let REC $= \{e : W_e$ is recursive $\}$. Use Post's Theorem to prove that REC is r.e. in $\emptyset''$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

January 2000

LOGIC (Ph.D./M.A. version)

1. a) Let a theory T and sentences σ_n for $n \in \omega$ be given. Assume that $T \models (\sigma_n \rightarrow \sigma_{n+1})$ and $T \not\models (\sigma_{n+1} \rightarrow \sigma_n)$ for all $n \in \omega$. Prove that T has a model $\mathfrak{A}$ such that $\mathfrak{A} \models \neg\sigma_n$ for all $n \in \omega$.

   b) Let L be a language containing at least the binary relation symbol E, and let $\mathfrak{A}$ be an L-structure so that $E^{\mathfrak{A}}$ is an equivalence relation on A. Assume that for every elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ and every $b \in B$ there is some $a \in A$ such that $E^{\mathfrak{A}}(a, b)$ holds. Prove that $E^{\mathfrak{A}}$ has just finitely many equivalence classes.

2. a) Prove that $(\mathrm{Q}, \leq)$ is isomorphically embeddable in some $\mathfrak{B} \equiv (\omega, \leq)$.

   b) Prove or disprove: $(\mathrm{Z}, +)$ has a proper elementary submodel.

3. a) Let L be a countable language containing at least the binary relation symbol E, and let T be a theory of L such that for every model $\mathfrak{A}$ of T, $E^{\mathfrak{A}}$ is an equivalence relation on A. Assume that for every model $\mathfrak{A}$ of T some $E^{\mathfrak{A}}$ class is infinite. Prove that there is some formula $\varphi(x)$ of L consistent with T so that whenever $\mathfrak{A}$ is a model of T, $a \in A$ and $\mathfrak{A}_A \models \varphi(\bar{a})$ then the $E^{\mathfrak{A}}$-class of a is infinite.

   b) Let T be a complete theory in a countable language L, let $\Phi(x)$ and $\Psi(x)$ be L-types, and let $\mathfrak{A}$ be an ω- saturated model of T. Assume that $\Phi^{\mathfrak{A}} = (A \setminus \Psi^{\mathfrak{A}})$. Prove that there is some formula $\varphi(x)$ of L such that for every model $\mathfrak{B}$ of T, $\Phi^{\mathfrak{B}} = \varphi^{\mathfrak{B}}$.

4. a) Let $R \subseteq \omega \times \omega$ be r.e. and assume that $R_k \neq \emptyset$ for all $k \in \omega$ and that $R_k \cap R_l = \emptyset$ for all $k \neq l$. Prove that there is some r.e. $C \subseteq \omega$ such that $|R_k \cap C| = 1$ for all $k \in \omega$.

   b) Let $X \subseteq \omega$ and a formula $\varphi(x)$ of the language of arithmetic be given. Assume that φ weakly represents X in every consistent theory T containing Q. Prove that X is recursive.

5. a) Let T be a recursively axiomatizable theory and assume that T has just finitely many complete extensions (in the same language). Prove that T is decidable.

   b) Define $f : \omega \to \omega$ by $f(e) =$ the least d such that $\{d\} = \{e\}$. Prove that f is not recursive.

6. a) Give an example (with proof) of a set $X \subseteq \omega$ which is Π_1 but not Σ_1.

   b) Prove or disprove: $\{\ulcorner\sigma\urcorner : \mathfrak{N} \models \sigma\}$ is arithmetic.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

August, 1999

LOGIC (Ph.D./M.A. version)

1. a) Let T and T' be theories of L such that for every L- structure $\mathfrak{A}$, $\mathfrak{A} \models T$ iff $\mathfrak{A} \not\models T'$. Prove that T is finitely axiomatizable.

   b) Prove that every countable linear order can be isomorphically embedded in $(\mathbf{Q}, \leq)$.

2. a) Prove or disprove: $(\mathbf{R}\setminus\{0\}, \leq)$ is an elementary substructure of $(\mathbf{R}, \leq)$.

   b) Let T be a complete ω-categorical theory in a countable language L. Prove that there is an integer k such that for every model $\mathfrak{A}$ of T and every formula $\varphi(x)$ of L with just one free variable, if $\varphi^{\mathfrak{A}}$ has more than k elements then $\varphi^{\mathfrak{A}}$ is infinite.

3. Let T be a complete theory in a countable language L, let $\mathfrak{A}$ be an ω-saturated model of T, and let $\Phi(x)$ be a type in one free variable consistent with T. Assume that Φ is realized in $\mathfrak{A}$ by *exactly* two elements of A. Prove that Φ is realized by exactly two elements in every model of T.

4. a) Assume that $R \subseteq \omega \times \omega$ is r.e. and $\bigcup_{k\in\omega} R_k = \omega$. Prove that there is some recursive $S \subseteq R$ such that $\bigcup_{k\in\omega} S_k = \omega$ and $S_k \cap S_l = \emptyset$ whenever $k \neq l$.

b) Let T be a consistent recursively axiomatizable extension of the theory Q. Find a formula $\varphi(x)$ such that $T \models \varphi(\bar{n})$ for all $n \in \omega$ but $T \not\models \forall x \varphi(x)$. (Be sure to show that the formula you define has this property.)

5. a) Let L be a language with just finitely many non-logical symbols and let $L' = L \cup \{c\}$ where c is a constant symbol not in L. Assume that T' is a finitely axiomatizable essentially undecidable theory of L' and let $T = T' \cap Sn_L$. Prove that T is essentially undecidable.

b) Prove that $A = \{e \in \omega : \{e\}(e) = e\}$ is not recursive.

6. An r.e. set $A \subseteq \omega$ is said to be *simple* if $(\omega \setminus A)$ is infinite but does not contain an infinite r.e. subset.

a) Prove that the intersection of two simple r.e. sets is simple.

b) Show that $K = \{e : e \in W_e\}$ is not simple.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

January, 1999

LOGIC (Ph.D./M.A. version)

1. a) Let L be a language containing at least the binary relation symbol E and let T be a theory of L so that in every model $\mathfrak{A}$ of T, $E^{\mathfrak{A}}$ is an equivalence relation on A. Assume that in every model $\mathfrak{A}$ of T, every $E^{\mathfrak{A}}$-class is finite. Prove that there is some $n \in \omega$ so that in every model $\mathfrak{A}$ of T, every $E^{\mathfrak{A}}$-class contains at most n elements.

   b) Let Σ_1 and Σ_2 be sets of sentences of L such that there is no sentence θ of L so that $\Sigma_1 \models \theta$ and $\Sigma_2 \models \neg\theta$. Prove that $(\Sigma_1 \cup \Sigma_2)$ has a model.

2. a) Let $\mathfrak{A}$ be an L-structure and let $\varphi(x)$ be a formula of L. Prove that $\varphi^{\mathfrak{A}}$ is finite iff there is no $\mathfrak{B}$ so that $\mathfrak{A} \prec \mathfrak{B}$ and $\varphi^{\mathfrak{A}} \neq \varphi^{\mathfrak{B}}$.

   b) Let $\{\varphi_i(x) : i \in \omega\}$ be an infinite set of L-formulas and let $\mathfrak{A}$ be an ω-saturated L-structure. Assume that for every $a \in A$ there is some $i \in \omega$ such that $\mathfrak{A}_A \models \varphi_i(\bar{a})$. Prove that for every L-structure $\mathfrak{B}$ elementarily equivalent to $\mathfrak{A}$, for every $b \in B$ there is an $i \in \omega$ such that $\mathfrak{B}_B \models \varphi_i(\bar{b})$.

3. a) Let T be a complete theory in a countable language L that has an infinite model. Prove that T is ω-categorical iff all models of T realize precisely the same n-types for each $n \in \omega$.

   b) Let L be a countable language and let $\mathfrak{A}$ be an infinite, countable, saturated L-structure. Prove that there is a proper elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ that is isomorphic to $\mathfrak{A}$.

4. a) Let T be a theory in a language $L \supseteq \{S, \bar{0}\}$ that contains only finitely many non-logical symbols. Assume that every recursive relation is representable in T. Prove that T is undecidable.

   b) Let L be a countable language and let $L' = L \cup \{c\}$, where c is a constant symbol not in L. Let Σ be a set of sentences of L, let $T = Cn_L(\Sigma)$ and let $T' = Cn_{L'}(\Sigma)$. Prove that T is undecidable iff T' is undecidable.

5. a) Let $E \subseteq \omega \times \omega$ be r.e. Assume that E is an equivalence relation on ω and assume that $C \subseteq \omega$ is an r.e. set that contains exactly one element from each E-class. Prove that E is recursive.

   b) Let $A \subseteq \omega$ be non-empty. Carefully prove that A is the domain of some partial recursive function iff A is the range of some total recursive function.

6. a) Let A be a non-empty, proper subset of ω. Assume that A is recursive. Prove that there are numbers $a \in A$ and $b \in (\omega \backslash A)$ such that $W_a = W_b$.

   b) Let X be a non-empty subset of ω. Assume that X is r.e. Let $I = \{e \in \omega : W_e = X\}$. Prove that every r.e. subset A of ω is many-one reducible to I.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

August, 1998

LOGIC (Ph.D./M.A. version)

1. a) Let T be a theory of L and let σ be a sentence of L. Assume that for every model $\mathfrak{A}$ of T, $\mathfrak{A} \models \sigma$ iff A is finite. Prove that there is some $n \in \omega$ such that every model of T with at least n elements is infinite.

   b) Let $\mathfrak{A}$ be a proper elementary extension of $(\omega, <)$. Prove that there is an infinite sequence $\{a_n\}_{n\in\omega}$ of elements of A such that $a_{n+1} <^{\mathfrak{A}} a_n$ holds for all $n \in \omega$.

2. a) Let $\mathfrak{A}$ be an infinite L-structure. Assume that for every formula $\varphi(x)$ of L, either $\varphi^{\mathfrak{A}}$ is finite or $(\neg\varphi)^{\mathfrak{A}}$ is finite. Prove that there is exactly one complete 1-type $\Gamma(x)$ consistent with T that can be realized by infinitely many elements in some model of T.

   b) Let T be a complete theory in a countable language L and let $\Phi(x)$ be a type consistent with T. Assume that Φ is omitted in some model of T. Prove that there is another model of T in which Φ is realized by infinitely many elements.

3. a) Let T be a complete theory in the language $L = \{+, \cdot, <, S, \bar{0}\}$ such that $Q \subseteq T$ but $(\omega, +, \cdot, <, S, 0) \not\models T$. Prove that there is some formula $\varphi(x)$ of L such that $T \models \exists x\varphi(x)$ but $T \models \neg\varphi(\bar{n})$ for every $n \in \omega$.

   b) Let $\mathfrak{A}$ be the countable model of an ω-categorical theory in a countable language L. Prove that $\mathfrak{A}$ has a non-trivial automorphism.

4. a) Prove that every infinite r.e. $A \subseteq \omega$ contains an infinite recursive subset.

b) Let $R \subseteq \omega \times \omega$ be r.e. and satisfy the following conditions:

$$\bigcup_{k \in \omega} R_k = \omega \quad \text{and} \quad R_k \cap R_l = \emptyset \text{ whenever } k \neq l.$$

Prove that R is recursive. (Recall that $R_k = \{l : R(k,l) \text{ holds}\}$).

5. a) Let $X \subseteq \omega$ be r.e. but not recursive. Let $\varphi(x)$ be a Σ-formula in the language $L = \{+, \cdot, <, S, \bar{0}\}$ that defines X in $(\omega, +, \cdot, <, S, 0)$. Prove that there is some consistent theory $T \supseteq Q$ such that $T \vdash \varphi(\bar{n})$ for some $n \notin X$.

b) Prove that there is a partial recursive function f that cannot be extended to a total recursive function (i.e., there is no total recursive function g such that $g(k) = f(k)$ whenever $f(k)$ is defined).

6. a) Prove that there is some $e \in \omega$ such that $\{e\}(2e) = 3e + 1$.

b) Let $A = \{\ulcorner \sigma \urcorner : \sigma \text{ is a sentence of } L = \{+, \cdot, <, S, \bar{0}\} \text{ and } \models \sigma\}$. Prove that A is a complete r.e. set.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
January 1998

LOGIC

1. a) Let L be a countable language containing at least the binary relation symbol E, and let T be a theory of L so that $E^{\underline{A}}$ is an equivalence relation on A for every model $\underline{A}$ of T. Assume that whenever $\underline{A}$ is a model of T and $\underline{B}$ is an elementary extension of $\underline{A}$ then every element of (B — A) has Its $E^{\underline{B}}$ -class contained in (B — A). Prove that there is some integer n such that in every model $\underline{A}$ of T every $E^{\underline{A}}$ -class has size < n.

b) Let T be a consistent theory in the countable language L and let $\Phi(x)$ and $\Psi(x)$ be types consistent with T. Assume that for every model $\underline{A}$ of T we have $\Psi^{\underline{A}} = A - \Phi^{\underline{A}}$. Prove that there is some formula $\varphi(x)$ such that $\Phi^{\underline{A}} = \varphi^{\underline{A}}$ for every model $\underline{A}$ of T.

2. Let T be a complete theory in a countable language L and let $\Phi(x)$ be a complete type of T. Assume that T has models $\underline{A}$ and $\underline{B}$ so that $|\Phi^{\underline{A}}| = 1$ and $|\Phi^{\underline{B}}| = 2$.

a) Prove that T has a model omitting Φ.

b) Prove that T has a model $\underline{C}$ so that $\Phi^{\underline{C}}$ is infinite.

3. a) Prove that $(\omega, +)$ has no proper elementary substructures.

b) Let T be a complete ω-categorical theory in a countable language.

Prove that there is an integer n such that for every formula $\varphi(x)$ and every model $\underline{A}$ of T, if $\varphi^{\underline{A}}$ is finite than $|\varphi^{\underline{A}}| < n$.

4. a) For any $R \subseteq \omega \times \omega$ we define $R_k = \{l : R(k,l) \text{ holds}\}$. Assume that R is r.e. and $\bigcup_{k \in \omega} R_k = \omega$. Prove that there is some recursive $S \subseteq R$ such that $\bigcup_{k \in \omega} S_k = \omega$ and further $S_k \cap S_l = \emptyset$ whenever $k \neq l$.

b) Let $A, B \subseteq \omega$ and assume that B is r.e. but not recursive and that $B \leq_m A$. Prove that A contains an infinite r.e. subset.

5. a) Prove that $\{e : W_e \neq \omega\} \leq_m \{e : W_e \text{ is finite}\}$.

b) Let A_n be arbitrary subsets of ω for every n in ω. Prove that there is some $B \subseteq \omega$ such that $A_n \leq_T B$ for every n.

6. a) Prove that $REC = \{e : W_e \text{ is recursive}\}$ is Σ^0_3.

b) Prove that $A \leq_T \{\ulcorner\sigma\urcorner : \underline{N} \models \sigma\}$ for every arithmetic $A \subseteq \omega$, where $\underline{N}$ is the standard model of arithmetic on the natural numbers.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
August 1997

LOGIC

1. a) Let L be a language containing at least the binary relation symbol E. Let $\underline{A}$ be an L-structure in which E is interpreted as an equivalence relation on the universe. Assume that every element of every elementary extension of $\underline{A}$ belongs to the E-class of some element of A. Prove that there are just finitely many E-classes in $\underline{A}$.

b) Let L and L' be languages with $L \subseteq L'$. Let T' be an L'-theory, and let $\underline{A}$ be an L-structure. Assume that there is no model of T' whose L-reduct is elementarily equivalent to $\underline{A}$. Prove that there is some L-sentence σ such that $A \models \sigma$ and $T' \models \neg\sigma$.

2. a) Let T be a complete theory of a language L and let $\Phi(x)$ be an L-type. Assume that Φ is realized by at most one element in every model of T. Prove that there is some formula $\varphi(x)$ such that $\Phi^{\underline{A}} = \varphi^{\underline{A}}$ for every model $\underline{A}$ of T.

b) Let $\underline{A}$ be the countable model of an ω-categorical theory in a countable language L. Let X be a subset of A fixed by all automorphisms of $\underline{A}$ (that is, if $a \in X$ then $h(a) \in X$ for every automorphism h of $\underline{A}$). Prove that X is definable in $\underline{A}$ by some L-formula. (You may assume that if $(\underline{A},a) \equiv (\underline{A},b)$ then $(\underline{A},a) \cong (\underline{A},b)$, and also the Ryll-Nardzewski characterization of ω-categorical theories).

3. a) Prove that Th((Z,+)) does not have a countable ω-saturated model.

b) Let L be a countable language containing at least a binary relation symbol E. Let T be an L-theory stating (among other things) that E is an equivalence relation on the universe. Assume that T has a model $\underline{A}$ with the property that every L-formula $\varphi(x)$ satisfiable on $\underline{A}$ is satisfiable by some element of A from a finite E-class. Prove that T has a model in which all E-classes are finite.

4. a) Let R be a binary relation on ω which is r.e. but not recursive. Assume that $R_k \cap R_\ell = \emptyset$ for all $k \neq l$ (where $R_k = \{n: R(k,n) \text{ holds}\}$). Prove that $\bigcup_{k \in \omega} R_k$ is not recursive.

b) Let $A = \{\ulcorner \sigma \urcorner : Q \vdash \sigma\}$ where Q is the theory of the language of arithemetic used in undecidability results. Prove that every r.e. set of natural numbers is many-one reducible to A.

5. a) Assume $X \subseteq \omega$ is such that $\{e: W_e = \omega\} \subseteq X$ and $\{e: W_e = \emptyset\} \cap X = \emptyset$. Prove that X is not recursive.

b) Prove that $B = \{e: \{e\}(2e) = 3\}$ is a complete r.e. set.

6. a) Assume that $B \subsetneq \omega$ is infinite but contains no infinite r.e. subset. Assume that A is r.e. and $A \leq_m B$. Prove that A is recursive.

b) Recall that $COF = \{e: (\omega - W_e) \text{ is finite}\}$. Prove that COF is r.e. in $\emptyset''$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
January 1997

LOGIC

1. a) Prove that $(Z,<)$ has no proper elementary submodels.

   b) Let T be a complete theory in a countable language L containing (at least) a binary relation symbol E such that in every model of T, E is interpreted as an equivalence relation on the universe. Assume that in every ω-saturated model of T there is exactly one infinite E-class. Prove that there is some integer n such that that in every model of T every E-class with > n elements is infinite.

2. a) Let T be a consistent theory in a countable language L. Assume that for all formulas $\varphi(x)$ of L we have

$$T \models \forall x\ \varphi(x) \quad \text{iff} \quad T \models \varphi(c) \text{ for all constants } c \text{ of } L.$$

   Prove that T has a model $\underline{A}$ such that $A = \{c^{\underline{A}} : c \in L\}$.

   b) Let $\underline{A}$ be any L-structure and assume that $\underline{A}$ realizes exactly three different complete types. Show that the same is true for every L-structure $\underline{B}$ elementarily equivalent to $\underline{A}$.

3. a) Let T be a complete theory in a countable language L and let $\underline{A}$ be a countable atomic model of T. Assume that a and b are elements of A with the same complete type. Prove that $\underline{A}$ has an automorphism f such that $f(a) = b$.

   b) Let T be a complete theory in a countable language L. Assume there are only finitely many complete types $\Phi(x)$ in a single variable x consistent with T. Prove that there are only finitely many formulas $\varphi(x)$ of L up to equivalence with respect to T.

4. a) Let A and B be disjoint r.e. sets of natural numbers, and assume neither of them is recursive. Prove that $(A \cup B)$ is not recursive.

   b) Prove that any theory T with an r.e. set of axioms also has a recursive set of axioms.

5. a) Let T be a theory in a countable language L and assume that $\{n \in \omega :$ T has a model of cardinality $n\}$ is not recursive. Prove that T is undecidable.

   b) Let T be a consistent recursively axiomatizable theory in the usual language for arithmetic on the natural numbers. Assume that X is weakly representable in T by $\varphi(x)$ and that X is not recursive. Prove that there is some consistent recursively axiomatizable theory T′ containing T such that X is not weakly representable in T′ by $\varphi(x)$.

6. a) Prove that there are r.e. subsets A and B of ω which are disjoint but there is no recursive set C with $A \subseteq C$ and $(B \cap C) = \emptyset$

   b) Prove that $\{e : W_e \text{ is infinite}\} \leq_m \{e : W_e = \omega\}$.
   [Hint: first define a partial recursive function g(e,x) which converges iff $\{e\}(y)$ converges for some $y > x$]

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
August 1996

LOGIC

1. a) Let $\underline{A}$ be an L-structure and let $\varphi(x)$ be a formula of L. Prove that $\varphi^{\underline{A}}$ is finite iff $\varphi^{\underline{A}} = \varphi^{\underline{B}}$ for every elementary extension $\underline{B}$ of $\underline{A}$.

   b) Let T be a complete theory in a countable language L, let $\underline{A}$ be an ω-saturated model of T, and let $\Phi(x)$ and $\Psi(x)$ be L types. Assume that $\Psi^{\underline{A}} = A - \Phi^{\underline{A}}$. Prove that there is some formula $\varphi(x)$ of L such that $\Phi^{\underline{A}} = \varphi^{\underline{A}}$.

2. a) Let T be a countable language whose non-logical symbols include the binary relation $<$. Let T be a consistent theory of L such that $<^{\underline{A}}$ is a linear order of A for every model $\underline{A}$ of T. Assume that whenever $\underline{A}$ is a model of T there are a,b in A such that the $<^{\underline{A}}$-interval between a and b is infinite. Prove that there is some formula $\varphi(x,y)$ of L consistent with T such that whenever $\underline{A}$ is a model of T and $\underline{A}_A \models \varphi(\bar{a},\bar{b})$ then the $<^{\underline{A}}$-interval between a and b is infinte.

   b) Let L and L′ be languages with $L \subseteq L'$, let T_1' and T_2' be theories of L′ which contain precisely the same sentences of L, and let T be a theory of L. Prove that some model of T can be expanded to a model of T_1' iff some model of T can be expanded to a model of T_2'.

3. a) Let $\underline{A}$ be any L-structure, let $L' = L(A)$ and let $T' = Th(\underline{A}_A)$. Let $\underline{B}'$ be an L′-structure which is a model of T′. Assume that $\underline{B}'$ is an atomic model of T′. Prove that $\underline{B}$, the L-reduct of $\underline{B}'$, is isomorphic to $\underline{A}$.

   b) Let T be a complete ω-categorical theory in a countable language L. Prove that there is some integer k such that for every formula $\varphi(x)$ of L and every model $\underline{A}$ of T, if $|\varphi^{\underline{A}}| > k$ then $\varphi^{\underline{A}}$ is infinite.

4. a) Assume that $R \subseteq \omega \times \omega$ is r.e. and defines a strict linear order on ω with no last element (so $R(k,k)$ fails for all k). Prove that there is a strictly increasing recursive function f such that $R(f(k), f(k+1))$ holds for all k.

b) Let the non-logical symbols of L be $\{+, \cdot, <, s, \bar{0}\}$ and let $\underline{N}$ be the standard L-structure for arithmetic on the natural numbers. Prove that there is <u>no</u> listing $\{\varphi_n(x) : n \in \omega\}$ of all the formulas of L with x free such that $X = \{n : \underline{N} \models \varphi_n(\bar{n})\}$ is recursive.

5. a) Let L have as its only non-logical symbol the binary relation E and let T_0 be the L-theory asserting that E is an equivalence relation on the universe with infinitely many classes. Prove that there is a complete L-theory T which extends T_0 and is undecidable.

b) Let A be a non-empty r.e. subset of ω and define $I = \{e : A = W_e\}$. Prove that every r.e. set B is many-one reducible to I.

6. a) Let L be a language with finitely many non-logical symbols and let $L' = L \cup \{c\}$ where c is an individual constant symbol not in L. Let $\underline{A}'$ be a strongly undecidable L'-structure and let $\underline{A}$ be its reduct to L. Prove that $Th(\underline{A})$ is an undecidable L-theory.

b) Let $REC = \{e : W_e \text{ is recursive}\}$. Prove that REC is r.e. in $\emptyset''$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION

January 1996

LOGIC

1. a) Let L be a language whose non-logical symbols include the binary relation E. Let T be a theory of L such that $E^{\underline{A}}$ is an equivalence relation on A for every model $\underline{A}$ of T. Assume that in every model $\underline{A}$ of T there is exactly one infinite $E^{\underline{A}}$-class. Prove that there is some n in ω such that in every model $\underline{A}$ of T all finite $E^{\underline{A}}$-classes have at most n elements.

   b) Let T be a complete theory of some language L and let $\Phi(x)$ be an L-type consistent with T. Assume that Φ is omitted on some model of T. Prove that Φ is realized in some model of T by at least two different elements.

2. a) Let T be a complete theory in a countable language L and let $\underline{A}$ be the prime model of T. Let $\Phi(x)$ be any L-type. Prove that there is some L-type $\Psi(x)$ such that $\Psi^{\underline{A}} = A - \Phi^{\underline{A}}$.

   b) Let L be a countable language and let $L' = L \cup \{c_1, \ldots, c_k\}$ where $c_1, \ldots, c_k$ are individual constants not in L. Let T and T′ be complete theories of L and L′ respectively and assume $T \subseteq T'$.

   Prove that T has a countable universal model iff T′ has a countable universal model.

3. a) Let L be a countable language. An L-structure $\underline{A}$ is said to be <u>locally finite</u> iff every element of A belongs to a finite L-definable subset of A. Let T be a complete L-theory and assume that no model of T is locally finite. Prove that there is some L-formula $\varphi(x)$ consistent with T such that for every L-formula $\psi(x)$ and every model $\underline{A}$ of T, $\varphi^{\underline{A}} \cap \psi^{\underline{A}}$ is infinite provided it is not empty.

b) Let T be a complete theory in a countable language L. Let $\underline{A}$ be a countable model of T which is not prime and let $\Phi(x)$ be a type omitted on $\underline{A}$. Prove that there is some countable model of T which also omits Φ but is not isomorphic to $\underline{A}$.
[Warning: You cannot assume that T has a prime model.]

4. a) Assume that A and B are r.e. subsets of ω such that $A \cup B$ is recursive. Prove that there are recursive sets $A' \subseteq A$ and $B' \subseteq B$ such that $A \cup B = A' \cup B'$.

b) Recall that if $\varphi(x)$ is a Σ-formula (in the language for arithmetic on the natural munbers) and if $Q \vdash \exists x \varphi(x)$ then $Q \vdash \varphi(\bar{n})$ for some n in ω. Prove that there is no total recursive function f such that whenever $\varphi(x)$ is a Σ-formula and $Q \vdash \exists x \varphi(x)$ then $Q \vdash \varphi(\overline{f(k)})$ where $k = \ulcorner\varphi\urcorner$.
[Hint: Let $\psi(x,y)$ be a Σ_1-formula representing in Q the relation "x is the Godel number of a proof from Q of the sentence whose Godel number is y" and consider the formulas $\varphi_1(x) = \psi(x,\bar{1})$.]

5. a) Given a language L_1 let $L_2 = L_1 \cup \{c\}$ where c is an individual constant not in L_1. Let T_2 be a finitely axiomatizable essentially undecidable theory of L_2 and let $T_1 = T_2 \cap Sn_{L_1}$. Prove that T_1 is also essentially undecidable.

b) Prove that $\{e : W_e \neq \omega\} \leq_m \{e : W_e \text{ is finite}\}$.
[Hint: First define a partial recursive function $f(e,x)$ which converges iff $\{e\}(y)$ converges for all $y < x$.]

6. a) Let A and B be subsets of ω. Prove that B is A-r.e. iff $B \leq_m A'$ where A' is the jump of A.

b) Let $C = \{\ulcorner\sigma\urcorner : \underline{N} \models \sigma\}$ where $\underline{N}$ is the standard model of arithmetic on the natural numbers. Prove that $A \leq_T C$ for all arithmetic sets A, and use this to conclude that C is not arithmetic.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
August 1995

LOGIC

1. a) Given a theory T and a sentence θ of L, assume that for every model $\underline{A}$ of T, $A \models \theta$ iff A is finite. Prove that there is some $n \in \omega$ such that for every model $\underline{A}$ of T, $A \models \theta$ iff A has at most n elements.

   b) Let $\underline{A}$ and $\underline{B}$ be L-structures and assume that $\underline{B}$ is a proper elementary extension of $\underline{A}$. Assume further that there is an L-formula $\varphi(x,y)$ such that $A = \{ b \in B : \underline{B}_B \models \varphi(\bar{b}, \bar{b}_0) \}$ for some b_0 in B. Prove that $b_0 \notin A$.

2. a) Let $T = Th((\mathbb{Q},+,\cdot,<,0,1))$. Prove that T does not have a countable saturated model.

   b) Let T be a complete L-theory, let L′ be a language containing L and let T′ be an L′-theory containing T. Assume that $\underline{A}$ is a model of T which has an elementary extension which can be expanded to an L′-structure which is a model of T′. Prove that every model $\underline{B}$ of T has an elementary extension which can be expanded to a model of T′.

3. Let L be a countable language containing (at least) a binary relation symbol $\leq$ and individual constants c_n for all $n \in \omega$. Let T be a complete theory of L containing (at least) the axioms that $\leq$ is a linear order of the universe and $c_n \leq c_{n+1}$ for all $n \in \omega$. Call a model $\underline{A}$ of T standard if for every $a \in A$ there is some $n \in \omega$ such that $\underline{A}_A \models \bar{a} \leq c_n$. Let $\underline{A}^*$ be an ω-saturated model of T.

   a) Prove that if $\underline{A}^*$ is standard then there is some $n \in \omega$ such that

   $$\underline{A}^* \models \forall x \, (x \leq c_n).$$

   b) Assume that for every L-formula $\varphi(x)$ such that $\underline{A}^* \models \exists x \, \varphi(x)$ there is some $n \in \omega$ such that $\underline{A}^* \models \exists x \, [\varphi(x) \wedge x \leq c_n]$. Prove that T has a standard model.

4. Let T be a recursively axiomatized extension of the theory Q which is true on $\underline{N} = (\omega, +, \cdot, <, 0, s)$. Let $R \subseteq \omega \times \omega$ be representable in T by the Σ_1-formula $\varphi(x,y)$. Let $X = \{k : \exists l\ R(k,l) \text{ holds}\}$.

a) Show X is weakly representable in T by $\exists y\ \varphi(x,y)$.

b) Assume X is not recursive. Prove that there is some $k \in \omega$ such that $T \models \neg \varphi(\bar{k}, \bar{l})$ for all $l \in \omega$ but $T \not\models \forall y \neg \varphi(\bar{k}, y)$.

5. a) Let $\mathcal{F}$ be a set of partial recursive functions of one argument, and let $I = \{ e : \{e\} \in \mathcal{F} \}$. Prove that $I \not\leq_m (\omega - I)$.

b) Let A and B be subsets of ω. Assume B is r.e. but not recursive and that $B \leq_m A$. Prove that A contains an infinite r.e. subset.

6. a) Let L_0 be the language with no non-logical symbols.

i) Show that there is a theory T_0 of L_0 which is undecidable.

ii) Can there be an undecidable L_0-theory T_0 which has only finite models? Explain.

b) Let X be an r.e. subset of ω. Let $I = \{ e : W_e = X \}$. Prove that I is Π_2.