Numerical Analysis Qualifying Exam
AMSC/CMSC 666, August 2010

(1) The idea of this problem is to give a direct proof of the uniqueness of the natural cubic spline interpolating data points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ with $x_0 < x_1 < \cdots < x_n$. The only analytic tool you will need is Rolle's theorem.

Suppose that $S_1(x), S_2(x)$ are natural cubic spline functions satisfying

$$S_1(x_i) = S_2(x_i) = y_i \quad \text{for } i = 0, 1, \ldots, n,$$
$$S_1''(x_0) = S_2''(x_0) = S_1''(x_n) = S_2''(x_n) = 0.$$

Let $S(x) = S_1(x) - S_2(x)$.

(a) Show that $S(x)$ is a natural cubic spline function.
(b) Show that $S''(x)$ has at least $n-1$ zeros in (x_0, x_n) and therefore at least $n+1$ zeros in $[x_0, x_n]$.
(c) Show that there is an $i \in \{0, 1, \ldots, n-1\}$ such that $S(x) \equiv 0$ on $[x_i, x_{i+1}]$.
(d) Show $S(x) \equiv 0$ on $[x_0, x_n]$ and therefore $S_1(x) \equiv S_2(x)$ on $[x_0, x_n]$.

(2) Derive the three-point Gaussian quadrature formula such that

$$\int_{-1}^{1} f(x)\, x^2 \, \mathrm{d}x \approx \sum_{j=1}^{3} f(x_j)\, w_j.$$

Give bounds on the error of this formula.

(3) Let $A \in \mathbb{R}^{N \times N}$ be symmetric and positive definite. Let $b \in \mathbb{R}^N$. Define $\langle y, z \rangle = y^T z$ for every $y, z \in \mathbb{R}^N$. Consider the following iterative method to solve $Ax = b$:

choose an initial iterate $x^{(0)} \in \mathbb{R}^N$;

compute the initial residual $r^{(0)} = b - Ax^{(0)}$;

$$\text{if } r^{(n)} \neq 0 \text{ then set} \quad \alpha_n = \frac{\langle r^{(n)}, r^{(n)} \rangle}{\langle Ar^{(n)}, r^{(n)} \rangle}, \quad \begin{aligned} x^{(n+1)} &= x^{(n)} + \alpha_n r^{(n)}, \\ r^{(n+1)} &= r^{(n)} - \alpha_n A r^{(n)}. \end{aligned}$$

(a) Show that if $r^{(n)} \neq 0$ then $\|y - x\|_A$ is minimized along the line $y = x^{(n)} + \alpha r^{(n)}$ parametrized by $\alpha \in \mathbb{R}$ when $\alpha = \alpha_n$ given by the above formula. Here $x \in \mathbb{R}^N$ is the solution of $Ax = b$ while $\|\cdot\|_A$ denotes the A-norm, which is defined by $\|z\|_A^2 = \langle Az, z \rangle$ for any $z \in \mathbb{R}^N$.
(b) Show that if $r^{(n)} \neq 0$ then $\langle r^{(n+1)}, r^{(n)} \rangle = 0$.
(c) Let $e^{(n)} = x^{(n)} - x$ be the error of the n^{th} iterate. Show that if $e^{(n)} \neq 0$ then

$$\frac{\|e^{(n+1)}\|_A^2}{\|e^{(n)}\|_A^2} = 1 - \frac{\langle r^{(n)}, r^{(n)} \rangle}{\langle Ar^{(n)}, r^{(n)} \rangle} \frac{\langle r^{(n)}, r^{(n)} \rangle}{\langle r^{(n)}, A^{-1} r^{(n)} \rangle}.$$

(d) Let κ denote the quotient of the largest over the smallest eigenvalues of A. Derive a bound on $\|e^{(n)}\|_A$ in terms of κ and $\|e^{(0)}\|_A$ by using the Kantorovich inequality

$$1 \leq \frac{\langle Ay, y \rangle \langle y, A^{-1} y \rangle}{\langle y, y \rangle^2} \leq \frac{(\kappa + 1)^2}{4\kappa} \quad \text{for every nonzero } y \in \mathbb{R}^N.$$

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4. Consider the initial value problem
$$y' = f(x,y)$$
for $0 \le x \le 1$, $y(0) = Y_0$ and f is a smooth function. Let $0 = x_0 < x_1 < \cdots < x_N = 1$ be a uniform partition of the interval $[0,1]$ and denote by h the step size. Consider a method of the form
$$y_{n+1} = \alpha y_n + \beta y_{n-1} + h\gamma f(x_{n-1}, y_{n-1}), \quad n \ge 1.$$
   (a) Choose the constants α, β, and γ so that the order of the local truncation error is as high as possible.

   (b) Study the stability of the method and determine whether the method is convergent.

5. Consider the nonlinear equation $e^x = \sin x$.

   (a) Show that there is a solution $x_* \in (-\frac{5}{4}\pi, -\pi)$.

   (b) Consider the following iterative methods: (i) $x_{k+1} = \ln(\sin x_k)$ and (ii) $x_{k+1} = \arcsin(e^{x_k})$. What can you say about the local convergence of each of these methods for x_* as in (a) and their convergence order? If you use a theorem give its precise statement.

   (c) For x_* as in (a) give a method which is quadratically convergent. Justify why the method is quadratically convergent.

6. Consider the following boundary value problem where $f \in C^0([0,1])$ is given:
$$-u''(x) + u(x) = f(x) \quad \text{for } x \in (0,1), \qquad u(0) = 0, \quad u(1) = 0.$$
We are interested in $u'(1)$.

   (a) The boundary value problem has a weak formulation: Find $u \in V$ such that
$$\text{for all } v \in V: \qquad a(u,v) = \ell(v)$$
   Identify V, $a(u,v)$, $\ell(v)$.

   (b) Assume from now on that we have $u \in C^2([0,1])$. Let $w(x) = x$. Is $w \in V$? Show that $u'(1) = a(u,w) - \int_0^1 f(x)w(x)dx$.

   (c) We now use a finite element method with piecewise linear functions on a uniform mesh with mesh size $h = 1/n$, n integer, yielding a finite element solution u_h. Explain why we have an error estimate $\|u - u_h\|_{H^1(\Omega)} \le Ch$ and justify that all needed assumptions are satisfied.

   (d) Let $\alpha_h = a(u_h, w) - \int_0^1 f(x)w(x)dx$. Using the previous parts show that $|u'(1) - \alpha_h| \le Ch$.

Numerical Analysis Qualifying Exam
AMSC/CMSC 666, January 2010

(1) Let $w(x)$ be a positive continuous weight function on $[a, b]$. Suppose that

$$\int_a^b w(x)\,dx = 1\,, \qquad \int_a^b x\,w(x)\,dx = 2\,,$$
$$\int_a^b x^2 w(x)\,dx = 5\,, \qquad \int_a^b x^3 w(x)\,dx = 10\,.$$

(a) Take $p_0(x) = 1$. Find a linear polynomial $p_1(x)$ and a quadratic polynomial $p_2(x)$ such that $\{p_0, p_1, p_2\}$ is an orthogonal set on $[a, b]$ with respect to the weight function $w(x)$.

(b) Show that $a < -\sqrt{5}$ and $b > \sqrt{5}$. (You may quote any appropriate theorem.)

(c) Derive the two-point Gaussian quadrature formula

$$\int_a^b f(x)\,w(x)\,dx \approx w_1 f(x_1) + w_2 f(x_2)\,,$$

which is exact for all polynomials of degree ≤ 3 (i.e. find the weights and nodes).

(2) Consider the (unshifted) QR-method for finding the eigenvalues of an invertible matrix $A \in \mathbb{R}^{N\times N}$.

(a) Give the algorithm.

(b) Show that each of the matrices A_n generated by this algorithm are orthogonally similar to A.

(c) Show that if A is upper Hessenberg then so are each of the matrices A_n generated by this algorithm.

(d) Perform one step of this algorithm on

$$A = \begin{pmatrix} 12 & 7 \\ 5 & 4 \end{pmatrix}.$$

(e) The sequence $\{A_n\}$ generated by this algorithm starting from the above A has a limit. What are the diagonal entries of this limit? Give your reasoning.

(3) Let $A \in \mathbb{R}^{N\times N}$ be symmetric and positive definite. Let $b \in \mathbb{R}^N$. Consider solving $Ax = b$ using the stationary iterative method given by

$$x^{(n+1)} = x^{(n)} + B^{-1}\big(b - Ax^{(n)}\big)\,,$$

where $B \in \mathbb{R}^{N\times N}$ has an easily computable inverse. Suppose that $B + B^T - A$ is positive definite.

(a) Show that this method will converge for any initial iterate $x^{(0)}$.

(b) Let $e^{(n)} = x^{(n)} - x$ be the error of the n^{th} iterate. Show that each step of this method reduces the A-norm of $e^{(n)}$ whenever $e^{(n)} \neq 0$, where the A-norm of any $y \in \mathbb{R}^N$ is defined by

$$\|y\|_A \equiv \sqrt{y^T A y}\,.$$

(c) Show how the SOR method fits into this framework.

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4. We consider a function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ with continuous 2nd order derivatives. Assume that f has a zero x_* with nonsingular Jacobian $Df(x_*)$. For a parameter $\alpha \in \mathbb{R}$ we consider the iteration

$$x_{k+1} := x_k - \alpha Df(x_k)^{-1} f(x_k)$$

   (a) Show that $x_{k+1} - x_* = (1-\alpha)(x_k - x_*) + Q$ with $\|Q\| \le C\,\|x_k - x_*\|^2$ for x_k sufficiently close to x_*. *Hint:* Let $d := Df(x_k)^{-1} f(x_k)$. Write down the Taylor approximation of order 1 for $f(x_*)$ about x_k with a 2nd order remainder term R. Use this to find an expression for d.

   (b) Show that this method is locally convergent for $\alpha \in (0,2)$

   (c) What can you say about the convergence rate depending on $\alpha \in (0,2)$? Can the iteration converge to x_* for $\alpha < 0$ or $\alpha > 2$?

5. For the initial value problem $y' = f(t,y)$, $y(t_0) = y_0$ we consider the method

$$y_{j+1} = y_j + h\left[\alpha f_j + (1-\alpha) f_{j-1}\right]$$

   where $f_k := f(t_k, y_k)$.

   (a) Find the consistency order of the method depending on the parameter $\alpha \in \mathbb{R}$

   (b) Show that the method is convergent and give the order of convergence, using an appropriate theorem

   (c) Consider $y' = \lambda y$ and find the values $\mu = \lambda h \in \mathbb{R}$ in the region of absolute stability in the case (i) $\alpha = 1$ and (ii) the value of α with the highest consistency order.

6. Let $\Omega_1 = (0, \frac{1}{2})$, $\Omega_2 = (\frac{1}{2}, 1)$, $\Omega = (0,1)$. Let $k(x) = \begin{cases} \frac{1}{10} & \text{if } x \in \Omega_1 \\ 1 & \text{if } x \in \Omega_2 \end{cases}$. Consider the following variational problem: Find $u \in H_0^1(\Omega)$ such that

$$\text{for all } v \in H_0^1(\Omega): \qquad \int_0^1 k(x) u'(x) v'(x)\,dx = \int_0^1 v(x)\,dx.$$

   (a) Let u be the solution of the variational problem Let $u_j = u|_{\Omega_j}$ for $j = 1,2$. Assume (only for (a)) that $u \in C^0(\overline{\Omega})$, $u_j \in C^2(\overline{\Omega}_j)$ for $j = 1,2$ and $f \in C^0([0,1])$. What differential equation does u_j satisfy? Show that there holds $u_1(\frac{1}{2}) = u_2(\frac{1}{2})$ and $u_1'(\frac{1}{2}) = 10 u_2'(\frac{1}{2})$

   (b) We use a finite element method with piecewise linear functions on a uniform mesh with mesh size $h = 1/N$, N even. Find the matrix and right hand side vector of the resulting linear system.

   (c) Show an estimate of the form $\|u_h - u\|_{H^1(\Omega)} \le C h^\alpha$ for the finite element solution u_h and state what assumption you need about the solution u (remember (a)). State any theorems you use in your argument.

Numerical Analysis

AMSC 666
COMPREHENSIVE EXAM
AUGUST 2009

1. Let A be a real symmetric matrix of order n with n distinct eigenvalues, and let $v_1 \in \mathbb{R}^n$ be such that $\|v_1\|_2 = 1$ and the inner product $(v_1, u) \neq 0$ for every eigenvector u of A.

(a) Let $\mathcal{P}_n$ denote the linear space of polynomials of degree at most $n-1$. Show that

$$\langle p, q \rangle \equiv (p(A)v_1, q(A)v_1)$$

defines an inner product on $\mathcal{P}_n$, where the expression on the right above is the Euclidean product in $\mathbb{R}^n$

(b) Consider the recurrence

$$\beta_{j+1}v_{j+1} = Av_j - \alpha_j v_j - \beta_j v_{j-1}$$

where α_j and β_j are scalars and $v_0 \equiv 0$ Show that $v_j = p_{j-1}(A)v_1$, where $p_{j-1}(t)$ is a polynomial of degree $j-1$

(c) Suppose the scalars above are such that $\alpha_j = (Av_j, v_j)$ and β_{j+1} is chosen so that $\|v_{j+1}\|_2 = 1$ Use this to show that that the polynomials from part (b) are orthogonal with respect to the inner product from part (a)

2. Consider the n-panel trapezoid rule $I_n(f)$ for calculating the integral $\int_0^1 f(x)dx$ of a continuous function $f \in C[0,1]$,

$$I_n(f) = \sum_{k=0}^{n-1} \left(\frac{t_{k+1} - t_k}{2}\right)(f(t_k) + f(t_{k+1}))$$

where $0 = t_0 < t_1 < \cdots < t_n = 1$

(a) Find a piecewise linear function G such that

$$I_n(f) - \int_0^1 f(t)dt = \int_0^1 G(t)f'(t)dt$$

for any continuous function f such that f' is integrable over $[0,1]$ Hint: Find G by applying the fundamental theorem of calculus to $\int_{t_k}^{t_{k+1}} (f(t)G(t))'dt$.

(b) Apply the previous result to $f(x) = x^\alpha$, $0 < \alpha < 1$, to obtain a rate of convergence.

3. Let $C[a,b]$ denote the set of all real-valued continuous functions defined on the closed interval $[a,b]$. Let $\rho \in C[a,b]$ be positive everywhere in $[a,b]$. Let $\{Q_n\}_{n=0}^{\infty}$ be a system of polynomials, with $deg\, Q_n = n$ for each n, orthogonal with respect to the inner product

$$< g, h >= \int_a^b \rho(x) g(x) h(x) dx \quad , \quad \forall g, h \in C[a,b]$$

For a fixed integer $n \geq 2$, let $x_1, \ldots, x_n$ be the n distinct roots of Q_n in (a,b). Let

$$r_k(x) = \frac{(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)} \quad , k = 1, 2, \ldots, n$$

be polynomials of degree $n-1$. Show that

$$\int_a^b \rho(x) r_j(x) r_k(x) dx = 0 \quad , \quad \forall j \neq k$$

and that

$$\sum_{k=1}^{n} \int_a^b \rho(x) \left(r_k(x)\right)^2 dx = \int_a^b \rho(x) dx.$$

Hint: Use orthogonality to simplify $\int_a^b \rho(x) \left(\sum_{k=1}^{n} r_k(x)\right)^2 dx$.

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4. Given the two-point boundary value problem

$$\begin{cases} -u'' + \alpha u = f(x), \quad 0 \le x \le 1, \ \alpha > 0, \\ u'(0) = A, \\ u'(1) = B \end{cases}$$

(a) Set up the finite element approximation for this problem, based on piecewise linear elements in equidistant points. Determine the convergence rate in an appropriate norm.

(b) Explain whether $\alpha > 0$ is necessary for the convergence in part (a).

5. Consider the multistep approximation of $\dot{x} = f(t, x)$ of the form

$$w_{n+1} = -\theta w_n + (1+\theta) w_{n-1} + \frac{1}{2} h \Big[-\theta f_{n+1} + (4 + 3\theta) f_n \Big], \qquad f_n := f(t_n, w_n)$$

(a) Determine the order of accuracy as a function of the free real parameter θ.

(b) For which θ's is the method stable?

6. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonlinear smooth function. To determine a local minimum of f one can use a descent method of the form

$$x_{k+1} = x_k + \alpha_k d_k, \qquad k \ge 0,$$

where $0 < \alpha_k \le 1$ is a suitable parameter obtained by backtracking and d_k is a direction that is chosen such that

$$f(x_{k+1}) < f(x_k). \tag{1}$$

(a) Write the Newton method including the backtracking algorithm and examine whether or not there exists α_k which yields (1). Establish conditions on the Hessian $H(f(x))$ of $f(x)$ which guarantee the existence of α_k.

(b) If we replace the Hessian by the matrix $H(f(x)) + \gamma_k I$, here $\gamma_k \ge 0$ and I is the identity matrix, we obtain a Newton-like method. Find a condition on γ_k which leads to (1).

Numerical Analysis

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COMPREHENSIVE EXAM
JANUARY 2009

1. Let A be a real symmetric positive-definite matrix. Given a linear system of equations $Ax = b$, consider an iterative solution strategy of the form

$$x_{k+1} = x_k + \alpha_k r_k \, ,$$

where x_0 is arbitrary, $r_k = b - Ax_k$ is the residual and α_k is a scalar parameter to be determined.

(a) Derive an expression for α_k such that the Euclidean norm $\|r_{k+1}\|_2$ is minimized as a function of α_k. Is this expression always well-defined and nonzero?

(b) Show that with this choice,

$$\frac{\|r_k\|_2}{\|r_0\|_2} \leq \left(1 - \frac{\lambda_{min}(A)}{\lambda_{max}(A)}\right)^{k/2} .$$

2. Define the following linear functional over the space of continuous functions $C[-1, 1]$:

$$I_n(f) = \sum_{k=1}^{n} w_{n,k} f(x_{n,k})$$

where $-1 \leq x_{n,k} \leq 1$, and $f \in C[-1, 1]$.

(a) Let T_n denote the n^{th} Chebyshev polynomial. Assume

$$(i) \qquad \lim_{n\to\infty} I_n(T_j) = \int_{-1}^{1} T_j(x)dx \ , \ j = 0, 1, 2, \ldots$$

Show that for all $j \geq 0$,

$$\lim_{n\to\infty} I_n(x^j) = \int_{-1}^{1} x^j \, dx \, .$$

Note: we are not really using any property of Chebyshev polynomials except that they have a nonvanishing leading coefficient of the term of order n.

(b) Assume further that there exists an $M > 0$ such that

$$(ii) \qquad \sum_{k=1}^{n} |w_{n,k}| \leq M \ , \ n = 1, 2, \ldots$$

Show that for all $f \in C[-1,1]$,

$$\lim_{n\to\infty} I_n(f) = \int_{-1}^{1} f(x)dx \, .$$

3. Let $A \in \mathbb{R}^{n\times m}$ with $n \geq m$. Assume $rank(A) = m$.

(a) It is known that the symmetric matrix $A^T A$ can be factored as

$$A^T A = V \Lambda V^T$$

where the columns of V are the orthonormal eigenvectors of $A^T A$ and Λ is the diagonal matrix containing the corresponding eigenvalues. Using this as a starting point, derive the singular value decomposition of A. That is, show that there is a real orthogonal matrix U and a matrix $\Sigma \in \mathbb{R}^{n\times m}$ which is zero except for its diagonal entries $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m > 0$ such that $A = U\Sigma V^T$.

(b) Let $b \in \mathbb{R}^n$. Show that the decomposition from part (a) can be used to compute $x \in \mathbb{R}^m$ such that $\|b - Ax\|_2$ is minimal.

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JANUARY 2009

4. One wants to solve the equation $x + \ln x = 0$, whose root is $r \sim 0.5$, using one or more of the following iterative methods:

$$\{i\}\ x_{k+1} = -\ln x_k \quad \{ii\}\ x_{k+1} = e^{-x_k} \quad \{iii\}\ x_{k+1} = \frac{x_k + e^{-x_k}}{2}$$

(a) Which of the three methods can be used?

(b) Which method should be used?

(c) Give an even better iterative formula.

5. Consider the boundary value problem

$$\begin{cases} -u'' + e^u = 0, & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

Discretize the problem with a finite element method using continuous, piecewise linear functions on an equidistant grid. Quadrature is to be done with the trapezoidal rule. Write the method in the form

$$AU_h + F_h(U_h) = 0,$$

where $U_h \in \mathbb{R}^m$ denoted the vector of unknown nodal values of the approximate solution, A is an $m \times m$ matrix whose elements are independent of the discretization parameter h, and $F_h : \mathbb{R}^m \to \mathbb{R}^m$ is a nonlinear vector-valued function.

6. Determine the local order of accuracy and the stability properties of the two-step scheme

$$x_{i+2} - 3x_{i+1} + 2x_i = \Delta t \cdot \left[\frac{13}{12} f(t_{i+2}, x_{i+2}) - \frac{5}{3} f(t_{i+1}, x_{i+1}) - \frac{5}{12} f(t_i, x_i)\right],$$

as an approximation for the ODE $\dot{x}(t) = f(t, x)$. What is its convergence rate?

Comprehensive Exam
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August 2008

1. Let $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$, and $p \in \mathbf{R}^n$, with $m > n$, and let δ be a scalar. Show how the the constrained least squares problem

$$\begin{array}{ll} \text{minimize} & \|b - Ax\|_2, \\ \text{subject to} & p^{\mathrm{T}}x = \delta. \end{array}$$

can be reduced to solving a related unconstrained least squares problem. The algorithm should start by finding a Householder transformation H such that

$$Hp = \|p\|_2 \mathbf{e}_1$$

and setting $y = Hx$.

2. Prove or disprove that the following interpolants exist for all values of y_1, y_2, y_3 and all distinct values of x_1, x_2, x_3.

a. $y_i = c_1 + c_2 x_i + c_3 x_i^2$

b. $y_i = c_1 + c_2 x_i^2 + c_3 x_i^4$

3. Consider the linear system of equations $Ax = b$ where A is a symmetric positive-definite matrix of order n. The *conjugate gradient method* (CG) for solving this system is

Choose x_0, compute $r_0 = b - Ax_0$
Set $p_0 = r_0$
for $k = 0$ until convergence
  $\alpha_k = (r_k, r_k)/(p_k, Ap_k)$
  $x_k = x_k + \alpha_k p_k$
  $r_k = r_k - \alpha_k Ap_k$
  <Test for convergence>
  $\beta_k = (r_{k+1}, r_{k+1})/(r_k, r_k)$
  $p_{k+1} = r_{k+1} + \beta_k p_k$
end

where $(v, w) = \sum_{i=1}^{n} v_i w_i$ is the Euclidean inner product.

Let Q be some other positive-definite matrix of order n. We know that the forms

$$\langle v, w\rangle_A \equiv (Av, w), \qquad \langle v, w\rangle_Q \equiv (Qv, w)$$

are inner products on $\mathbb{R}^n$. In addition, a matrix M is symmetric with respect to the Q-inner product $\langle\ ,\ \rangle_Q$ if $\langle Mv, w\rangle_Q = \langle v, Mw\rangle_Q$ for all v, w in $\mathbb{R}^n$, and M is positive-definite with respect to $\langle\ ,\ \rangle_Q$ if $\langle Mv, v\rangle_Q > 0$ for all nonzero v.

a. Show that $Q^{-1}A$ is symmetric and positive-definite with respect to the Q-inner product.

b. In light of this fact, CG can be used to solve the system $Q^{-1}Ax = Q^{-1}b$ in an appropriate manner. Specify this algorithm and identify any extra costs required that would not be present with $Q = I$.

c. Use any facts you know about the conjugate gradient method to identify properties of Q that would be desirable for computing the solution x efficiently.

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4. (10 pts.)

(a) Show that the two-step method

$$y_{n+1} = 2y_{n-1} - y_n + h[\frac{5}{2}f(x_n, y_n) + \frac{1}{2}f(x_{n-1}, y_{n-1})] \quad (1)$$

is of order 2 but does not satisfy the root condition.

(b) Give an example to show that the method (1) need not converge when solving $y' = f(x, y)$.

5. (12 pts.) Consider the boundary value problem

$$-u'' + (1 + x)u = x^2, \; u'(0) = 1, \; u(1) = 1. \quad (2)$$

(a) Prove that (2) has at most one solution.

(b) Discretize the problem. Take a uniform partition of $[0, 1]$:

$$x_i = ih, \; i = 0, 1, 2, \ldots, n, \; h = 1/n. \quad (3)$$

Use the three point difference formula for u'' and the simplest difference formula for the boundary condition at $x = 0$. Write the resulting system as a matrix-vector equation, $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = (u_1, u_2, \cdots, u_{n-1})^T$.

(c) Prove that the equation found in (b) has a unique solution.

(d) Transform the problem (2) into an equivalent problem with homogeneous boundary conditions.

(e) Obtain the variational formulation of the problem formulated in (d). Specify the Sobolev space H involved. Prove that this problem has a unique solution, which we denote by v.

(f) Consider the approximation of v by piecewise linear finite elements. Define precisely the piecewise linear finite element subspace (use the partition (3)). Show that the finite element problem has a unique solution v_h.

(g) Show that $\|v - v_h\|_H \leq Ch$ and indicate how the constant C depends on derivatives of v.

6. (8 pts.) Let $f : \mathbb{R} \to \mathbb{R}$ be the nonlinear function with zero x_*:

$$f(x,y) = \begin{bmatrix} x^2 - 2x + y \\ 2x - y^2 - 1 \end{bmatrix}, \quad \mathrm{x}_* = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Consider the iteration

$$\mathrm{x}_{n+1} = \mathrm{x}_n - Af(\mathrm{x}_n), \quad A = \begin{bmatrix} 1 & 1/2 \\ 1 & 0 \end{bmatrix}. \tag{4}$$

(a) Prove (4) is locally convergent.

(b) Show that the convergence is at least quadratic.

(c) Write the Newton iteration and compare with (4).

Numerical Analysis

Comprehensive Exam
AMSC 666
January 2008

1. Consider the system $Ax = b$. The GMRES method starts with a point x_0 and normalizes the residual $r_0 = b - Ax_0$, so that $v_1 = \nu^{-1} r_0$ has 2-norm one. It then constructs orthonormal Krylov bases $V_k = (v_1\ v_2\ \cdots\ v_m)$ satisfying

$$AV_k = V_{k+1} H_k,$$

where H_k is a $(k+1)\times k$ upper Hessenberg matrix. One then looks for an approximations to x of the form

$$x(c) = x_0 + V_k c,$$

choosing c_k so that $\|r(c)\| = \|b - Ax(c)\|$ is minimized, where $\|\cdot\|$ is the usual Euclidean norm.

a. Show that c_k minimizes $\|\nu e_1 - H_k c\|$.

b. Suppose we chose to solve the least squares problem in Part a for c_k by the method of orthogonal triangularization (QR). What is the order of the floating-point operation count for this method? Give reasons.

2. We wish to approximate the integral $I(f) \equiv \int_a^b f(x)dx$.

a. State the composite trapezoidal rule $Q_{T,n}$ for approximating $I(f)$ on a uniform partitioning of width $h = (b-a)/n$, and give a formula for the error $I(f) - Q_{T,n}(f)$ that is in a form suitable for extrapolation.

b. Use the error formula to derive a new quadrature rule obtained by performing one step of extrapolation on the composite trapezoidal rule. What is this rule, and how does its error depend on h? You may assume here that f is as smooth as you need it to be.

3. Consider the *shifted QR iteration* for computing the eigenvalues of a 2×2 matrix A: starting with $A_0 = A$, compute

$$A_k - \sigma_k I = Q_k R_k \quad A_{k+1} = R_k Q_k + \sigma_k I$$

where σ_k is a scalar shift.

a. If

$$A_k = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

specify the orthogonal matrix Q_k used to perform this step when Givens rotations are used. The matrix should be described in terms of the entries of the shifted matrix.

b. Suppose $a_{21} = \delta$, a small number, and $|a_{12}| \leq (a_{11} - a_{22})^2$. Demonstrate that with an appropriate shift σ_k, the $(2,1)$-entry of A_{k+1} is of magnitude $O(\delta^2)$. What does this suggest about the convergence rate of the QR iteration?

AMSC 667
Comprehensive Exam
January 2008

4. Consider the system

$$x = 1 + h\frac{e^{-x^2}}{1+y^2}, \qquad y = .5 + h\arctan(x^2+y^2).$$

(a) Show that if the parameter $h > 0$ is chosen sufficiently small, then this system has a unique solution (x^*, y^*) within some rectangular region.
(b) Define a fixed point iteration scheme for solving the system and show that it converges..

5.
(a) Outline the derivation of Adams-Bashforth methods for the numerical solution of the initial value problem $y' = f(x,y)$, $y(x_0) = y_0$.
(b) Derive the Adams-Bashforth formula

$$y_{i+1} = y_i + h[-\frac{1}{2}f(x_{i-1}, y_{i-1}) + \frac{3}{2}f(x_i, y_i)]. \tag{1}$$

(c) Analyze the method (1). To be specific, find the local truncation error, prove convergence and find the order of convergence.

6. Consider the problem

$$-u'' + u = f(x), \quad 0 \le x \le 1, \quad u(0) = u(1) = 0. \tag{2}$$

(a) Give a variational formulation of (2), i.e., express (2) as

$$u \in H, \quad B(u,v) = F(v) \quad \text{for all } v \in H. \tag{3}$$

Define the space H, the bilinear form B, and the linear functional F, and state the relation between (2) and (3).
(b) Let $0 = x_0 < x_1 < \cdots < x_n = 1$ be a mesh on $[0,\,1]$ with $h = \max_{j=0,\ldots,n-1}(x_{j+1} - x_j)$, and let

$$V_h = \{u : u \text{ continuous on } [0,\,1],\ u|_{[x_j,\,x_{j+1}]} \text{ is linear for each } j, u(0) = u(1) = 0\}.$$

Define the finite element approximation, u_h to u based on the approximation space V_h. What can be said about $\|u - u_h\|_1$, the error in the Sobolev norm on $H^1(0,1)$?
(c) Derive an estimate for $\|u - u_h\|_0$, the error in $L_2(0,1)$. Hint: Let w solve

$$\begin{cases} -w'' + w = u - u_h, \\ w(0) = w(1) = 0. \end{cases}$$

We characterize w variationally by

$$w \in H_0^1, \quad B(w,v) = \int (u - u_h) v \, dx, \text{ for all } v \in H_0^1.$$

Let $v = u - u_h$ to get

$$B(w, u - u_h) = \int (u - u_h)^2 \, dx = \|u - u_h\|_{L_2}^2. \tag{4}$$

Use the formula (4) to estimate $\|u - u_h\|_{L_2}$.

(d) Suppose $\{\phi_1^h, \cdots, \phi_{N_h}^h\}$ is a basis for V_h where $N_h = \dim V_h$, so $u_h = \Sigma_{j=1}^{N_h} c_j^h \phi_j^h$, for appropriate coefficients c_j^h. Show that

$$\|u - u_h\|_1^2 = \|u\|_1^2 - C^T AC,$$

where $C = [\, c_1^h, \cdots, c_{N_h}^h]^T$ and A is the stiffness matrix.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN QUALIFYING EXAM

NUMERICAL ANALYSIS (Ph.D. Version)

AUGUST 2007

Instructions

a. Answer all six questions.

b. Each question will be assigned a grade from 0 to 10. If some problems have multiple parts, be sure to go on to subsequent parts even if there is a part you cannot do.

c. Use a different set of sheets for each question. Write the problem number and your **code number** (not your name) on the outside sheets.

d. Keep scratch work on separate pages or on a separate set of sheets.

AMSC 666
Comprehensive Exam
August 2007

Problem 1.

A set of functions $\{g_1, \ldots, g_n\} \subset C[a, b]$ is a <u>Chebyshev system</u> if

(i) The set is linearly independent.

(ii) If g is a linear combination of $\{g_1, \ldots, g_n\}$ which is not identically zero g has at most $n-1$ distinct zeros in $[a, b]$.

(a) Show that $\{g_1, \ldots, g_n\}$ is a Chebyshev system if and only if for any n distinct points $x_1, \ldots, x_n \in [a, b]$, the matrix A with $a_{i,j} = g_j(x_i), 1 \leq i, j \leq n$ is nonsingular.

(b) Let $f \in C^{m+1}[a, b]$ be such that $f^{(m+1)}(x) \neq 0$ for all $x \in [a, b]$. Let $g_j(x) = x^{j-1}, j = 1, \ldots, m+1$. Show that $\{g_1, \ldots, g_{m+1}, f\}$ is a Chebyshev system. For this, you may use results from polynomial interpolation without proof.

Problem 2.

Let

$$I_n(f) = \sum_{k=1}^{n} w_{n,k} f(x_{n,k}), \qquad a \leq x_{n,k} \leq b$$

be a sequence of integration rules.

(a) Suppose

$$\lim_{n \to \infty} I_n(x^k) = \int_a^b x^k \, dx, \quad k = 0, 1, \cdots \tag{1}$$

and

$$\sum_{k=1}^{n} |w_{n,k}| \leq M, \quad n = 1, 2, \cdots \tag{2}$$

for some constant M. Show that

$$\lim_{n \to \infty} I_n(f) = \int_a^b f(x) \, dx \text{ for all } f \in C[a, b]$$

(b) Show that if all $w_{n,k} > 0$ then (1) implies (2).

Problem 3.

Consider the real system of linear equations

$$Ax = b \tag{1}$$

where A is nonsingular and satisfies

$$(v, Av) > 0$$

for all real v, where the Euclidean inner product is used here.

(a) Show that $(v, Av) = (v, Mv)$ for all real v, where $M = \frac{1}{2}(A + A^T)$ is the symmetric part of A.

(b) Prove that

$$\frac{(v, Av)}{(v, v)} \geq \lambda_{\min}(M) > 0,$$

where $\lambda_{\min}(M)$ is the minimum eigenvalue of M.

(c) Now consider the iteration for computing a series of approximate solutions to (1),

$$x_{k+1} = x_k + \alpha r_k,$$

where $r_k = b - Ax_k$ and α is chosen to minimize $\|r_{k+1}\|_2$ as a function of α. Prove that

$$\frac{\|r_{k+1}\|_2}{\|r_k\|_2} \leq \left(1 - \frac{\lambda_{\min}(M)^2}{\lambda_{\max}(A^TA)}\right)^{1/2}.$$

AMSC/CMSC 667 Written Exam: August 2007

4. Consider the problem of solving a nonlinear system of ODE

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}),$$

by an implicit method. The n-th step consists of solving for the unknown $\mathbf{y}$ a nonlinear algebraic system of the form

$$\mathbf{y} = \alpha h \mathbf{f}(t_n, \mathbf{y}) + \mathbf{g}_{n-1}, \tag{1}$$

where $\mathbf{g}_{n-1} \in \mathbb{R}^n$ is known, $\alpha > 0$ and h is the stepsize. Let $\mathbf{f} \in C^1$.

(a) Write (1) as a fixed point iteration and find conditions in h and $\mathbf{f}(t_n, \mathbf{y})$ that guarantee local convergence for this iteration.

(b) Write the Newton method for (1) and give conditions on h and $\mathbf{f}(t_n, \mathbf{y})$ that guarantee local convergence for this iteration. State precise additional assumptions on $\mathbf{f}$ that guarantee quadratic convergence.

5. This problem is about choosing between a specific single-step and a specific multistep methods for solving the ODE:

$$y' = f(t, y).$$

(a) Write the trapezoid method, define its local truncation error and estimate it.

(b) Show that the truncation error for the following multistep method is of the same order as in (a):

$$y_{n+1} = 2y_n - y_{n-1} - hf(t_{n-1}, y_{n-1}) + hf(t_n, y_n).$$

(c) What could be said about the global convergence rate for these two methods? Justify your conclusions for both methods.

6. Consider the boundary value problem

$$L(u) = -u'' + bu = f, \quad x \in I \equiv [0, 1], \qquad u(0) = u(1) = 0, \tag{2}$$

where $b \geq 0$ is a constant. Let $\{0 = x_0 < x_1 < \cdots < x_n = 1\}$ be a uniform mesh with meshsize h. Let

$$\mathbb{V}_h = \{v \in C[0, 1] : v|_{[x_{i-1}, x_i]} \text{ is linear for each } i, v(0) = v(1) = 0\}$$

be the corresponding finite element space, and let $u_h = R_h(u)$ be the corresponding finite element solution of (2). Note that R_h is a projection operator, the Ritz projector, onto the finite dimensional space $\mathbb{V}_h$ with respect to the *energy* scalar product $a(\cdot, \cdot)$ induced by problem (2).

(a) Let $|\cdot|_1$ be the H^1-seminorm, namely $|v|_1^2 = \int_I |v'|^2$ for all $v \in H_0^1(I)$. Find the constant Λ in terms of the parameter b such that

$$|u_h|_1 \leq \Lambda |u|_1.$$

Hint: recall the Poincaré inequality $\|v\|_0 \leq \frac{1}{2}|v|_1$ for all $v \in H_0^1(I)$, where $\|\cdot\|_0$ denotes the L^2-norm.

(b) If $I_h u \in \mathbb{V}_h$ is the Lagrange interpolant of u, then prove $R_h(I_h u) = I_h u$. Deduce

$$|u_h - I_h u|_1 \leq \Lambda |u - I_h u|_1.$$

(c) Use (b) to derive the error estimate

$$|u - u_h|_1 \leq (1 + \Lambda)|u - I_h u|_1,$$

and bound the right-hand side by a suitable power of h. Make explicit the required regularity of u.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN QUALIFYING EXAM

NUMERICAL ANALYSIS (Ph.D. Version)

JANUARY 2007

Instructions

a. Answer all six questions.

b. Each question will be assigned a grade from 0 to 10. If some problems have multiple parts, be sure to go on to subsequent parts even if there is a part you cannot do.

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d. Keep scratch work on separate pages or on a separate set of sheets.

Specify a numerical algorithm for finding v_j and give a convergence proof for this algorithm demonstrating that it is convergent under appropriate circumstances.

Problem 3.

Suppose A is an upper triangular, nonsingular matrix. Show that both Jacobi iterations and Gauss-Seidel iterations converge in finitely many steps when used to solve $A\mathbf{x} = \mathbf{b}$.

AMSC/CMSC 667 Written Exam: January 2007

4. Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be C^1, suppose $\mathbf{f}(\mathbf{x}^*) = 0$ and assume $\mathbf{f}'(\mathbf{x}^*)$ is nonsingular. Consider the following iteration

$$\mathbf{x}_{k+1} = \mathbf{x}_k - B_k^{-1}\mathbf{f}(\mathbf{x}_k), \qquad (k \geq 0).$$

(a) Derive the following error equation for $\mathbf{e}_{k+1} = \mathbf{x}_{k+1} - \mathbf{x}^*$:

$$\mathbf{e}_{k+1} = B_k^{-1}\Big(\big(B_k - \mathbf{f}'(\mathbf{x}^*)\big)\mathbf{e}_k - \int_0^1 \big(\mathbf{f}'(s\mathbf{x}_k + (1-s)\mathbf{x}^*) - \mathbf{f}'(\mathbf{x}^*)\big)ds\ \mathbf{e}_k\Big).$$

(b) Let $B_k = B \in \mathbb{R}^{n\times n}$ be a fixed matrix. Find conditions on B that guarantee local convergence. What rate of convergence do you expect and why?

(c) Find sufficient conditions on $\|B_k - \mathbf{f}'(\mathbf{x}_k)\|$ for the convergence to be superlinear. What choice of B_k corresponds to the Newton method and what rate of convergence do you expect?

5. Let $f(t,y)$ be uniformly Lipschitz with respect to y. Let $y(t)$ be the solution to the initial value problem: $y' = f(t,y), y(0) = y_0$. Consider the trapezoid method

$$(1) \qquad y_{i+1} = y_i + \frac{h}{2}\Big(f(t_i, y_i) + f(t_{i+1}, y_{i+1})\Big), \qquad (i \geq 0).$$

(a) Find a condition on the stepsize h that ensures (1) can be solved uniquely for y_{i+1}.

(b) Define local truncation error and estimate it. Examine the additional regularity of f needed for this estimate.

(c) Prove a global error estimate for (1).

6. Consider the 2-point boundary value problem

$$(2) \qquad -au'' + bu = f(x) \quad 0 < x < 1, \qquad u(0) = u_0, u(1) = u_1,$$

with $a, b > 0$ constants and $f \in C[0,1]$. Let $\mathcal{P} = \{x_i\}_{i=0}^{n+1}$ be a uniform partition of $[0,1]$ with meshsize h.

(a) Use centered finite differences to discretize (2). Write the system as

$$A\mathbf{U} = \mathbf{F},$$

and identify $A \in \mathbb{R}^{n\times n}$ and $\mathbf{U}, \mathbf{F} \in \mathbb{R}^n$. Prove that A is nonsingular.

(b) Define truncation error and derive a bound for this method in terms of h. State without proof an error estimate of the form

$$\max_{1\leq i\leq n} |u(x_i) - U_i| \leq Ch^s$$

and specify the order s.

(c) Prove the following discrete monotonicity property: If $\mathbf{U}_i$ is the solution corresponding to a forcing f_i, for $i = 1, 2$, and $f_1 \leq f_2$ then $\mathbf{U}_1 \leq \mathbf{U}_2$ componentwise.

Numerical Analysis Qualifying Exam
AMSC/CMSC 666, August 2006

(1) Consider the definite integral

$$I(f) = \int_a^b f(x)\,\mathrm{d}x\,.$$

Let $Q_n(f)$ denote the approximation of $I(f)$ by the composite midpoint rule with n uniform subintervals. For every $j \in \mathbb{Z}$ set

$$x_j = a + j\frac{b-a}{n}\,, \qquad x_{j+\frac{1}{2}} = \frac{x_j + x_{j+1}}{2}\,.$$

Let $K(x)$ be defined by

$$K(x) = -\tfrac{1}{2}(x - x_j)^2 \quad \text{for } x_{j-\frac{1}{2}} < x \leq x_{j+\frac{1}{2}}\,.$$

Assume that $f \in C^2([a,b])$.

(a) Show that the quadrature error $E_n(f)$ satisfies

$$E_n(f) \equiv Q_n(f) - I(f) = \int_a^b K(x) f''(x)\,\mathrm{d}x\,.$$

Hint: Use integration by parts over each subinterval $[x_{j-1}, x_j]$.

(b) Derive a sharp bound on the error of the form

$$|E_n(f)| \leq M_n \|f''\|_\infty \quad \text{for every } f \in C^2([a,b])\,.$$

Here $\|\cdot\|_\infty$ denotes the maximum norm over $[a,b]$. Recall that the above bound is sharp when the inequality is an equality for some nonzero f.

(2) Consider the (unshifted) QR-method for finding the eigenvalues of an invertible matrix $A \in \mathbb{R}^{N\times N}$.

(a) Give the algorithm.

(b) Show that each of the matrices A_n generated by this algorithm are orthogonally similar to A.

(c) Show that if A is upper Hessenberg then so are each of the matrices A_n generated by this algorithm.

(d) Let

$$A = \begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix}\,.$$

For this A the sequence $\{A_n\}$ has a limit. Find this limit. Give your reasoning.

(3) Let $A \in \mathbb{R}^{N\times N}$ be symmetric and positive definite. Let $b \in \mathbb{R}^N$. Consider solving $Ax = b$ using the conjugate gradient method. The n^{th} iterate $x^{(n)}$ then satifies

$$\big\|x^{(n)} - x\big\|_A \leq \|y - x\|_A \quad \text{for every } y \in x^{(0)} + \mathcal{K}_n\big(r^{(0)}, A\big)\,,$$

where $\|\cdot\|_A$ denotes the vector A-norm, $r^{(0)}$ is the initial residual, and

$$\mathcal{K}_n\big(r^{(0)}, A\big) = \operatorname{span}\big\{r^{(0)}, Ar^{(0)}, \cdots, A^{n-1}r^{(0)}\big\}\,.$$

(a) Prove that the error $x^{(n)} - x$ is bounded by

$$\big\|x^{(n)} - x\big\|_A \leq \big\|p_n(A)\big\|_A \big\|x^{(0)} - x\big\|_A\,,$$

where $p_n(z)$ is any real polynomial of degree n or less that satisfies $p_n(0) = 1$. Here $\big\|p_n(A)\big\|_A$ denotes the matrix norm of $p_n(A)$ that is induced by the vector A-norm.

(b) Let $T_n(z)$ denote the n^{th} Chebyshev polynomial. Let λ_{min} and λ_{max} denote respectively the smallest and largest eigenvalues of A. Apply the result of part (a) to

$$p_n(z) = \frac{T_n\left(\dfrac{\lambda_{max} + \lambda_{min} - 2z}{\lambda_{max} - \lambda_{min}}\right)}{T_n\left(\dfrac{\lambda_{max} + \lambda_{min}}{\lambda_{max} - \lambda_{min}}\right)}$$

to show that

$$\big\|x^{(n)} - x\big\|_A \leq \frac{1}{T_n\left(\dfrac{\lambda_{max} + \lambda_{min}}{\lambda_{max} - \lambda_{min}}\right)} \big\|x^{(0)} - x\big\|_A\,,$$

You can use without proof the fact that

$$\big\|p_n(A)\big\|_A = \max\big\{|p_n(z)| \,:\, z \in \operatorname{Sp}(A)\big\}\,,$$

where $\operatorname{Sp}(A)$ denotes the set of eigenvalues of A, and the facts that for every $n \in \mathbb{N}$ the polynomial $T_n(z)$ has degree n, is positive for $z > 1$, and satisfies

$$T_n(\cos(\theta)) = \cos(n\theta) \qquad \text{for every } \theta \in \mathbb{R}\,.$$

AMSC/CMSC 667 Written Exam, August 2006

(4) Suppose that $f:\mathbb{R}\to\mathbb{R}$ is smooth and that the boundary value problem

$$u'' = f(u) \text{ on } (0,1), \quad u(0)=u(1)=0$$

has a unique solution.

(a) For $N\in\mathbb{N}$ let $x_i = i/N$, $i=0,\ldots,N$. Write down a system of $N+1$ equations to obtain an approximation u_i for the solution u at x_i by replacing the second derivatives by a symmetric difference quotient.

(b) Write the system of equations in the form $F(U)=0$. Define domain and range of F and explain the meaning of the variable U.

(c) Formulate Newton's method for the solution of the system in (b) with $U^{(0)}=0$. Give explicit expressions for all objects involved (as far as this is reasonable). Determine a *sufficient* condition that ensures that the iterates $U^{(k)}$ in the Newton scheme are defined. Without doing any further calculations, can you decide whether the sequence $U^{(k)}$ converges? Why or why not?

(5) Consider the boundary value problem

$$-u'' = f, \quad 0<x<1,$$

with boundary conditions $u(0)=0$ and $u'(1)+\alpha u(1)=0$. Here α is a given positive number.

(a) Describe a Galerkin method to solve this problem using piecewise linear functions with respect to a uniform mesh.

(b) Derive the matrix equations for this Galerkin method. Write out explicitly that equation of the linear system which involes α.

(6) Consider the linear multistep method

$$y_{n+1} = \frac{4}{3}y_n - \frac{1}{3}y_{n-1} + \frac{2}{3}hf(x_{n+1},y_{n+1})$$

for the solution of the initial value problem $y'(x)=f\big(x,y(x)\big)$, $y(0)=y_0$.

(a) Show that the truncation error is of order 2.

(b) State the condition for consistency of a linear multistep method and verify it for the scheme in this problem.

(c) Does the scheme satisfy the root condition and/or the strong root conditions?

AMSC 666 Comprehensive Exam: Spring 2006

1. Let f be continuous on $[0, 1]$. A polynomial p of degree not greater than n is said to be a best (or Chebyshev) approximation to f if p minimizes the expression

$$E(p) = \max_{x \in [0,1]} |f(x) - p(x)|.$$

a. Show that a sufficient condition for p to be a best approximation is that there exist points $x_0 < x_1 < \cdots < x_{n+1}$ in $[0, 1]$ such that

$$f(x_i) - p(x_i) = -[f(x_{i+1}) - p(x_{i+1})] = \pm E(p), \qquad i = 0, 1, \ldots, n.$$

b. Compute the best linear approximation to x^2 in $[0, 1]$. [Hint: Drawing of a line through the parabola will allow you to conjecture where two of the points of oscillation must lie. Use the conditions from (a) to determine the third point and the coefficients of p.]

2. We will be concerned with the least squares problem of minimizing

$$\rho^2(x) = \|b - Ax\|^2.$$

Here A is an $m \times n$ matrix of rank n (which implies that $m \geq n$) and $\|\cdot\|$ is the Euclidean vector norm. Let

$$A = (Q_1 \;\; Q_2) \begin{pmatrix} R \\ 0 \end{pmatrix}$$

be the QR decomposition of A. Here Q_1, Q_2, and R are respectively $m \times n$, $m \times (m - n)$, and $n \times n$.

a. Show that the solution of the least squares problem satisfies the QR equation $Rx = Q_1^{\mathrm{T}} b$ and that the solution is unique. Further show that $\rho(x) = \|Q_2^{\mathrm{T}} b\|$.

b. Use the QR equation to show that the least squares solution satisfies the normal equations $(A^{\mathrm{T}} A)x = A^{\mathrm{T}} b$.

3. Let A be real symmetric and let $u_0 \neq 0$ be given. For $k = 1, 2, \ldots$ define u_{k+1} as the linear combination of the vectors Au_k, $u_k, \ldots, u_0$ with the coefficient of Au_k equal to one that is orthogonal to the vectors $u_k, \ldots, u_0$; i.e.,

$$u_{k+1} = Au_k - \alpha_k u_k - \beta_k u_{k-1} - \gamma_k u_{k-2} - \cdots .$$

a. Find formulas for α_k and β_k.

b. Show that $\gamma_k = \delta_k = \cdots = 0$. Where do you use the symmetry of A?

c For which nonzero vectors u_0 does $u_1 = 0$ hold.

AMSC/CMSC 667 Written Exam, January 2006

4. (a) Describe Newton's method for finding a root of a smooth fucntion $f : \mathbf{R} \to \mathbf{R}$.

(b) Assume that $f : \mathbf{R} \to \mathbf{R}$ is a smooth function, satisfies

$$f'(x) > 0, \quad f''(x) > 0, \qquad \text{for all } x \in \mathbf{R},$$

and has a root x^*. Draw a geometric picture illustrating the convergence of the method and give an analytic prove that Newton's method converges to x^* for any initial guess $x_0 \in \mathbf{R}$.

5. The goal of this problem is to solve the boundary value problem

$$-u'' = f \text{ on } (0,1), \quad u(0) = 0,\, u(1) = 0$$

in a suitable finite element space.

(a) For $N \in \mathbb{N}$ let $x_i = i/N$, $i = 0, \dots, N$. Define a suitable $N-1$ dimensional subspace V_N in H_0^1 associated with the points x_i. Let $\phi_1, \dots, \phi_{N-1}$ be *any* basis in V_N. Explain how you can determine the coefficients u_i in the representation of the finite element solution

$$u_N = \sum_{i=1}^{N-1} u_i \phi_i$$

by solving a linear system. Prove that there exists a unique solution.

(b) Show that

$$a(u,v) = \int_0^1 u'v' \, dx$$

defines an inner product on H_0^1 and thus a notion of orthogonality in H_0^1.

(c) Let $\phi_1 = 1 - 2|x - 1/2|$ be the basis of the one-dimensional space V_2. Find an *orthogonal* basis in V_4 that contains the basis function ϕ_1. Sketch the basis functions. Indicate how you would construct a basis of V_{2^n} that contains the basis of $V_{2^{n-1}}$.

(d) What is the structure of the linear system in (a) for this special basis?

6. For solving the equation $y' = f(x,y)$, consider the scheme

$$y_{n+1} = y_n + \frac{h}{2}(y_n' + y_{n+1}') + \frac{h^2}{12}(y_n'' - y_{n+1}'')$$

where $y'_n = f(x_n, y_n)$ and $y''_n = f_x(x_n, y_n) + f(x_n, y_n)f_y(x_n, y_n)$.

(a) Show that this scheme is fourth-order accurate.

(b) For stability analysis, one takes $f(x, y) = \lambda y$. State what it means for $\bar{\lambda} = h\lambda$ to belong to the region of absolute stability for this scheme, and show that the region of absolute stability contains the entire negative real axis.

Numerical Analysis Qualifying Exam
AMSC/CMSC 666, August 2005

(1) Derive the one-, two-, and three-point Gaussian quadrature formulas such that

$$\int_{-1}^{1} f(x)\, x^4 \,\mathrm{d}x \approx \sum_{j=1}^{n} f(x_j)\, w_j \,.$$

Give bounds on the error of these formulas.

(2) We wish to solve $Ax = b$ iteratively where

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} .$$

Show that for this A the Jacobi method and the Gauss-Seidel method both converge. Explain why for this A one of these methods is better than the other.

(3) Consider the three-term polynomial recurrence

$$p_{k+1}(x) = (x - \mu_k)p_k(x) - \nu_k^2 p_{k-1}(x) \quad \text{for } k = 1, 2, \cdots ,$$

initialized by $p_0(x) = 1$ and $p_1(x) = x - \mu_0$, where each μ_k and ν_k is real and each ν_k is nonzero.

(a) Prove that each $p_k(x)$ is a monic polynomial of degree k, and that for every $n = 0, 1, \cdots$ one has

$$\operatorname{span}\{p_0(x), p_1(x), \cdots , p_n(x)\} = \operatorname{span}\{1, x, \cdots , x^n\} .$$

(b) Show that for every $k = 0, 1, \cdots$ the polynomial $p_k(x)$ has k simple real roots that interlace with the roots of $p_{k-1}(x)$.

(c) Show that for every $n = 0, 1, \cdots$ the roots of $p_{n+1}(x)$ are the eigenvalues of the symmetric tridiagonal matrix

$$T = \begin{pmatrix} \mu_0 & \nu_1 & 0 & \cdots & 0 \\ \nu_1 & \mu_1 & \nu_2 & \ddots & \vdots \\ 0 & \nu_2 & \mu_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \nu_n \\ 0 & \cdots & 0 & \nu_n & \mu_n \end{pmatrix} .$$

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4. (a) Given a smooth function $f(x)$, state the secant method for the approximate solution of nonlinear equation in $\mathbb{R}$

$$f(x) = 0.$$

(b) State the order of convergence for this method, and explain how to derive it.

(c) Are there situations in which the order of convergence is higher? Explain your answers.

5. Consider the initial value problem

$$y' = f(t, y), \qquad y(0) = y_0.$$

(a) Write the ODE in integral form and explain how to use the trapezoidal quadrature rule to derive the *trapezoidal method* with uniform time-step $h = T/N$:

$$y_{n+1} = y_n + \frac{h}{2}\Big(f(t_{n+1}, y_{n+1}) + f(t_n, y_n)\Big).$$

(b) Define the concept of absolute stability. That is, consider applying the method to the case $f(t, y) = \lambda y$ with real λ. Show that the region of absolute stability contains the entire negative real axis of the complex $h\lambda$ plane.

(c) Suppose that $f(t, y) = Ay$ where $A \in \mathbb{R}^{n \times n}$ is a symmetric matrix and $y \in \mathbb{R}^n$. Examine the properties of A which guarantee that the method is absolutely stable (Hint: study the eigenvalues of A).

6. Consider the following two-point boundary value problem in $(0, 1)$:

$$(1) \qquad -u_{xx} + u = 1, \qquad u'(0) = u'(1) = 0.$$

(a) Give a variational formulation of (1), i.e., express it as

$$(2) \qquad u \in H: \quad a(u, v) = F(v), \qquad \text{for all } v \in H.$$

Define the function space H, the bilinear form a, and the linear functional F, and state the relation between (1) and (2). Show that the solution u is unique.

(b) Write the finite element method with piecewise linear elements over a uniform partition $P = \{x_i = (i-1)h\}_{i=1}^N$ with meshsize $h = 1/(N-1)$. If $\mathbf{U} = (u_i)_{i=1}^N$ is the vector of nodal values of the finite element solution, find the (stiffness) matrix A and right-hand side $\mathbf{F}$ such that $A\mathbf{U} = \mathbf{F}$. Show that A is symmetric and positive definite. Show that the solution U is unique.

(c) Consider two partitions P_1 and P_2 of $[0, 1]$, with P_2 a refinement of P_1. Let V_1 and V_2 be the corresponding piecewise linear finite element spaces. Show that V_1 is a subspace of V_2.

(d) Let $u_1 \in V_1$ and $u_2 \in V_2$ be the finite element solutions. Show the orthogonality equality

$$\|u - u_1\|_{H^1}^2 = \|u - u_2\|_{H^1}^2 + \|u_1 - u_2\|_{H^1}^2.$$

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January 2005

1. Given $f \in C[a,b]$, let p_n^* denote the best uniform approximation to f among polynomials of degree $\leq n$, i.e.
$$\max_{x\in[a,b]} |f(x) - p_n(x)|$$
is minimized by the choice $p_n = p_n^*$. Let $e(x) = f(x) - p_n^*(x)$. Prove that there are at least two points $x_1, x_2 \in [a,b]$ such that
$$|e(x_1)| = |e(x_2)| = \max_{x\in[a,b]} |f(x) - p_n^*(x)|$$
and $e(x_1) = -e(x_2)$.

2.
   (a) Find the node x_1 and the weight w_1 so that the integration rule
$$I = \int_0^1 x^{-1/2} f(x)\,dx \approx w_1 f(x_1)$$
is exact for linear functions. (No knowledge of orthogonal polynomials is required.)
   (b) Show that no one-point rule for approximating I can be exact for quadratics.
   (c) In fact
$$I - w_1 f(x_1) = cf''(\xi) \text{ for some } \xi \in [0,1].$$
Find c.
   (d) Let $f(x)$ and $g(x)$ be two polynomials of degree ≤ 3. Suppose f and g agree at $x = a$, $x = b$ and $x = (a+b)/2$. Show
$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx.$$

3. Let $A \in R^{n\times n}$ be nonsingular and $\mathbf{b} \in \mathbf{R^n}$. Consider the following iteration for the solution of $A\mathbf{x} = \mathbf{b}$.
$$\mathbf{x_{k+1}} = \mathbf{x_k} + \alpha(\mathbf{b} - A\mathbf{x_k}).$$
   (a) Show that if all the eigenvalues of A have positive real part then there will be some real α such that the iterates converge for any starting vector $\mathbf{x_0}$. Discuss how to choose α optimally in case A is symmetric and determine the rate of convergence.
   (b) Show that if some eigenvalues of A have negative real part and some have positive real part, then there is no real α for which the iterations converge.

(c) Let $\rho = \|I - \alpha A\| < 1$ for a matrix norm associated to a vector norm. Show that the error can be expressed in terms of the difference between consecutive iterates, namely

$$\|\mathbf{x} - \mathbf{x_{k+1}}\| \leq \frac{\rho}{1-\rho}\|\mathbf{x_k} - \mathbf{x_{k+1}}\|$$

(The proof of this is short but a little tricky.)

(d) Let A be the tridiagonal matrix

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 3 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 3 & 1 \\ 0 & \cdots & \cdots & \cdots & -1 & 3 \end{pmatrix}$$

Find a value of α that guarantees convergence.

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4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonlinear smooth function. To determine a (local) minimum of f one can use a descent method of the form

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k, \qquad (k \geq 0), \tag{1}$$

where $0 < \alpha_k \leq 1$ is a suitable parameter obtained by backtracking and $\mathbf{d}_k$ is a descent direction, i.e. it satisfies

$$f(\mathbf{x}_{k+1}) < f(\mathbf{x}_k). \tag{2}$$

(a) Write the *steepest descent* (or gradient) method and show that there exist $0 < \alpha_k \leq 1$ such that the resulting method satisfies (2).

(b) Write the *Newton method* and examine whether or not there exist α_k which yield (2). Establish conditions on the Hessian $H(f(\mathbf{x}))$ of $f(\mathbf{x})$ which guarantee the existence of α_k.

(c) If we replace the Hessian by the matrix $H(f(\mathbf{x})) + \gamma_k I$, where $\gamma_k \geq 0$ and I is the identity matrix, we obtain a *quasi-Newton method*. Find a condition on γ_k which leads to (2).

5. Consider the following nonlinear autonomous initial value problem in $\mathbb{R}$ with $f \in C^2$:

$$y' = f(y), \qquad y(0) = y_0.$$

(a) Write the ODE in integral form and use the mid-point quadrature rule to derive the *mid-point method* with uniform time-step $h = T/N$:

$$y_{n+1} = y_n + hf\Big(\frac{y_{n+1} + y_n}{2}\Big).$$

(b) Define truncation error T_n. Assuming that $T_n = O(h^3)$, show an estimate for the error $|y(t_N) - y_N|$. What is the order of the method? (Hint: use that f is Lipschitz continuous).

(c) Prove that $T_n = O(h^3)$ for this method. (Hint: expand first $y(t_{n+1})$ and $y(t_n)$ around $\tau_n = t_n + \frac{h}{2}$ and next expand $f(\frac{1}{2}(y(t_{n+1}) + y(t_n)))$. Also expand $y'(t)$ around τ_n).

6. Consider the following two-point boundary value problem in $(0,1)$ with $\varepsilon \ll 1 \leq b$:

$$-\varepsilon u_{xx} + b u_x = 1, \qquad u(0) = u(1) = 0.$$

(a) Write the finite element method with piecewise linear elements over a uniform partition of $(0,1)$ with meshsize h. If $\mathbf{U} = (U_i)_{i=0}^{I+1}$ is the vector of nodal values of the finite element solution, find the (stiffness) matrix A and right-hand side $\mathbf{F}$ such that $A\mathbf{U} = \mathbf{F}$. Is A symmetric? Is A positive definite?

(b) Find a relation between the three parameters h, ε, b for A to be an M-matrix, i.e. to have $a_{ii} > 0, a_{ij} \leq 0$ if $i \neq j$, and $a_{ii} \geq -\sum_{i \neq j} a_{ij}$.

(c) Consider the *upwind* modification of the ODE

$$-\Big(\varepsilon + \frac{b}{2}h\Big) u_{xx} + b u_x = 1.$$

Show that the resulting matrix A is an M-matrix without restrictions on h, ε and b.

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Numerical Analysis I
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AMSC666 Comprehensive exam, 2004

Answer questions #1, #4 and one other question from #2, #3.

1. To compute $\sqrt{2}$ we consider the following Eudoxos iterations: starting with $x_0 = y_0 = 1$ we set $x_{n+1} = x_n + y_n$ followed by $y_{n+1} = x_{n+1} + x_n$. Then $y_n/x_n \longrightarrow \sqrt{2}$.

   (a) Explain the Eudoxos method in terms of the power method.

   (b) How many iterations are required for an error $|y_n/x_n - \sqrt{2}| \leq 10^{-6}$?

2. Let $\{p_n(x)\}$ be a sequence of monic polynomials orthogonal on $[a, b]$ with respect to the positive weight function $w(x)$ (p_n has degree n). Show that the p_n satisfy a three term recursion formula of the form

$$p_n(x) = (x - a_n)p_{n-1}(x) - b_n p_{n-2}(x).$$

   Give expressions for the coefficients a_n and b_n.

3. (a) Find $\{p_0, p_1, p_2\}$ such that p_i is a polynomial of degree i and this set is orthogonal on $[0, \infty)$ with respect to the weight function $w(x) = e^{-x}$.

   (b) Find the weights and nodes of the 2 point Gaussian formula

$$\int_0^\infty f(x)e^{-x}dx \approx w_1 f(x_1) + w_2 f(x_2).$$

   Note: $\int_0^\infty x^n e^{-x}dx = n!$, $0! = 1$.

4. We are interested to compute an approximate value $w(x) \sim e^x$ for all real x, with *relative* error

$$\frac{|w(x) - e^x|}{e^x} \leq 0.5 \times 10^{-2}.$$

   To this end we express x as $x \log_2 e = m + \theta$, i.e., $e^x = 2^m 2^\theta$, where m is an integer and $|\theta| \leq 1/2$ and we shall use the near min-max polynomial of a minimal degree, $p(\theta)$ to compute 2^θ with 'sufficient' accuracy.

   What is the 'sufficient' accuracy required for computing 2^θ so that the requirement of relative error $\leq 0.5 \times 10^{-2}$ is met, what is the near min-max polynomial $p(\theta)$ in this case, and what is the final representation of the approximate value $w(x) = 2^m p(\theta)$?

   Note. Given $\ln 2 = 0.6931$.

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5. (a) State Newton's Method for the approximate solution of

$$f(x) = 0,$$

where $f(x)$ is a real valued function of the real variable x.

(b) State and prove a convergence result for the method.

(c) What is the typical order of convergence? Are there situations in which the order of convergence is higher? Explain your answers to these questions.

6. Consider the boundary value problem

$$\begin{cases} -u''(x) + u(x) = f(x),\ 0 \le x \le 1 \\ u'(0) = 2,\ u(1) = 0. \end{cases} \tag{1}$$

(a) Derive a variational formulation for (1).

(b) What do we mean by the Finite Element Approximation u_h to u?

(c) State and prove an estimate for

$$\|u-u_h\|_1 \equiv \left(\int_0^1 |u(x) - u_h(x)|^2\,dx + \int_0^1 |u'(x) - u_h'(x)|^2\,dx\right)^{1/2}.$$

(d) Prove the formula

$$\|u - u_h\|_1^2 = \|u\|_1^2 - \|u_h\|_1^2.$$

7. Consider the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0,$$

where f satisfies the Lipschitz condition

$$|f(t, y) - f(t, z)| \le L\,|y - z|$$

for all t, y, z. A numerical method called *the midpoint rule* for solving this problem is defined by

$$y_{n+1} = y_{n-1} + 2hf(t_n, y_n),$$

where h is a time step and $y_n \approx y(t_n)$ for $t_n = t_0 + nh$. Here y_0 is given and y_1 is presumed to be computed by some other method.

(a) Suppose the problem is posed on a finite interval $t_0 \leq t \leq b$, where $h = (b - t_0)/M$. Show directly, i.e., without citing any major results, that the midpoint rule is stable. That is, show that if $\hat{y}_0$ and $\hat{y}_1$ satisfy

$$|y_0 - \hat{y}_0| \leq \epsilon, \quad |y_1 - \hat{y}_1| \leq \epsilon,$$

then there exists a constant C independent of h such that

$$|y_n - \hat{y}_n| \leq C\epsilon$$

for $0 \leq n \leq M$.

(b) Suppose instead that we are interested in the long time behavior of the midpoint rule applied to the particular example $f(t, y) = \lambda y$. That is, let h be fixed and let $n \to \infty$, so that the rule is applied over a long time interval. Show that in this case the midpoint rule does not produce an accurate approximation to the solution of the initial value problem.

CMSC/MAPL 666
Comprehensive Exam
January 2004

1. Let $a = x_0 < x_1 < \cdots < x_n = b$ be an arbitrary fixed partition of the interval $[a, b]$. A function $q(x)$ is a quadratic spline function if
   (i.) $q \in C^1[a, b]$.
   (ii.) On each subinterval $[x_i, x_{i+1}], q$ is a quadratic polynomial

   The problem is to construct a quadratic spline $q(x)$ interpolating data points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$. The construction is similar to the construction of the cubic spline but much easier.
   (a) Show that if we know $z_i := q'(x_i), i = 0, \ldots, n$ we can construct q.
   (b) Find equations which enable us to determine the z_i. You should find that one of the z_i can be prescribed arbitrarily, for instance, $z_0 = 0$.

2.
   (a) Give the definition of the QR algorithm for finding the eigenvalues of a matrix A. Define both the unshifted version and the version with shifts $\chi_0, \chi_1, \chi_2, \cdots$.
   (b) Show that in each case the matrices $A_1, A_2, \cdots$ generated by the QR algorithm are unitarily equivalent to A (i.e. $A_i = U_i A U_i^H$, U_i unitary).
   (c) Let

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$$

   Use plane rotations (Givens rotations) to carry out one step of the QR algorithm on A, first without shifting and then using the shift $\chi_0 = 1$. Which seems to do better ? The eigenvalues of A are $\lambda_1 = 1.382$, $\lambda_2 = 3.618$. (Recall that a plane rotation is a matrix of the form

$$Q = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

   with $c^2 + s^2 = 1$.)

3. Let A be an $N \times N$ symmetric, positive definite matrix. Then we know that solving $A\mathbf{x} = \mathbf{b}$ is equivalent to minimizing the functional $F(\mathbf{x}) := \frac{1}{2}\langle \mathbf{x}, A\mathbf{x}\rangle - \langle \mathbf{b}, \mathbf{x}\rangle$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbf{R}^N$. To solve the problem by minimization of F we consider the general iterative method $\mathbf{x}_{\nu+1} = \mathbf{x}_\nu + \alpha_\nu \mathbf{d}_\nu$.
   (a) When $\mathbf{x}_\nu$ and $\mathbf{d}_\nu$ are given, show that the value of α_ν which minimizes $F(\mathbf{x}_\nu + \alpha \mathbf{d}_\nu)$ as a function of α is given in terms of the residual $\mathbf{r}_\nu$:

$$\alpha_\nu = \frac{\langle \mathbf{r}_\nu, \mathbf{d}_\nu \rangle}{\langle \mathbf{d}_\nu, A\mathbf{d}_\nu \rangle}, \qquad \mathbf{r}_\nu := \mathbf{b} - A\mathbf{x}_\nu.$$

(b) Let $\{\mathbf{d_j}\}_{j=1}^{N}$ be an A-orthogonal basis of $\mathbf{R}^N$, $\langle \mathbf{d_j}, A\mathbf{d_k}\rangle = 0,\ j \neq k$. Consider the expansion of the solution $\mathbf{x}$ in this basis:

$$\mathbf{x} = \sum_{j=1}^{N} \hat{x}_j \mathbf{d_j}.$$

Use that A-orthogonality of the $\mathbf{d_j}$ to compute the $\hat{x}_j$ in terms of the solution $\mathbf{x}$ and the $\mathbf{d_j}$'s.

(c) Let $\mathbf{x}_\nu$ denote the partial sum

$$\mathbf{x}_\nu := \sum_{j=1}^{\nu-1} \hat{x}_j \mathbf{d_j}, \quad \nu = 2, 3, \ldots,$$

so that $\mathbf{x}_{\nu+1} = \mathbf{x}_\nu + \hat{x}_\nu \mathbf{d}_\nu$ where the $\hat{x}_\nu$'s are the coefficients found in (b). Use that fact that $x_\nu \in \text{span}\{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_{\nu-1}\}$ and the A-orthogonality of the $\mathbf{d_j}$'s to conclude that the coefficient $\hat{x}_\nu$ is given by the optimal α_ν, i.e.,

$$\hat{x}_\nu = \frac{\langle \mathbf{r}_\nu, \mathbf{d}_\nu\rangle}{\langle \mathbf{d}_\nu, A\mathbf{d}_\nu\rangle} = \alpha_\nu.$$

4. (a) Consider the boundary value problem

$$\begin{cases} -u''(x) + b(x)u(x) = f(x), 0 \le x \le 1 \\ u(0) = u_l,\ u(1) = u_r, \end{cases} \tag{1}$$

where $b(x), f(x) \in C[0,1]$, and $b(x) \ge 0$. Formulate a difference method for the approximate solution of (1) on a uniform mesh of size h. Explain how $u''(x)$ is approximated by a difference quotient.

(b) Suppose $b(x) = 0$ and $u_l = u_r = 0$ in (1). Formulate a finite element method for the approximate solution of (1) in this special case, again on a uniform mesh. Using the standard "hat functions" basis for the finite element space, write out the finite element equations explicitly. Show that if an appropriate quadrature formula is used on the right-hand side of the finite element equations, they (the finite element equations) are the same as the finite difference equations.

(c) Show that the matrix in (b) is nonsingular.

5. Consider the following dissipative initial value problem,

$$\begin{cases} y' + f(y) = 0,\ 0 \le x \le 1 \\ y(0) = y_0, \end{cases} \tag{2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is smooth and satisfies $0 \le f'(y)$.

(a) Write the Backward Euler Method for (1). This gives rise to an algebraic equation. Explain how you would solve this equation.

(b) Derive an error estimate of the form

$$|y(x_n) - y_n| \le \frac{M}{2}h,\ n = 1, 2, \ldots,$$

where $M = \max_{0\le x\le 1} |y''(x)|$. Do this directly, not as an application of a standard theorem. (Note that there is no exponential on the right hand side.)

6. This problem is concerned with *Broyden's method* for solving a nonlinear system of equations $F(x) = 0$:

$$x_{n+1} = x_n + \delta_n, \qquad \delta_n = -B_n^{-1} F(x_n),$$

where x_0 and B_0 are given and the matrices $\{B_n\}$ are defined by the recurrence

$$B_n = B_{n-1} + \frac{F(x_n)\, \delta_{n-1}^T}{\delta_{n-1}^T \delta_{n-1}}.$$

(a) Show that if B is a nonsingular matrix, u and v are arbitrary vectors, and $1 + v^T B^{-1} u \neq 0$, then

$$(B + uv^T)^{-1} = \left(I - \frac{B^{-1} u v^T}{1 + v^T B^{-1} u} \right) B^{-1}.$$

This is known as the *Sherman-Morrison* formula.

(b) Use the result of part (a) to derive a representation for B_n^{-1} of the form

$$B_n^{-1} = (I - w_{n-1} v_{n-1}^T) \cdots (I - w_1 v_1^T)(I - w_0 v_0^T) B_0^{-1}.$$

(c) Explain how this representation can be used to derive an implementation of Broyden's method. You may assume here that $B_0 = I$. Outline the computational costs and storage requirements of this implementation. Why would you want to use this implementation instead of some other one?

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August 2003 Comprehensive Exam

INSTRUCTIONS: Results used, but not proved, must be stated clearly, indicating the assumptions under which they are valid.

1. Let $a = x_0 < x_1 < \cdots < x_n = b$ be a mesh on $[a, b]$. Let $\ell_i(x)$ be defined on $[a, b]$ as follows:
    1. ℓ_i is piecewise linear on $[a, b]$.
    2. $\ell_i(x_i) = 1$.
    3. $\ell_i(x_j) = 0$ $(j \neq i)$.

   (a) Let $f(x)$ be continuous on $[a, b]$. Determine the values of coefficients c_i such that the function
   $$\ell(x) = \sum_{i=0}^{n} c_i \ell_i(x)$$
   satisfies $\ell(x_i) = f(x_i)$.

   (b) Let $h = \max_i\{x_{i+1} - x_i\}$. Let f be twice continuously differentiable on $[a, b]$. By deriving an upper bound on $|f(x) - \ell(x)|$, where ℓ is the function defined in part a, show that $\lim_{h\to 0} |f(x) - \ell(x)| = 0$ and estimate the rate of convergence. You may use any results from polynomial interpolation theory.

2. The following improper integral $I(f)$ is to be approximated using the following 2-point quadrature $Q(f)$:
$$I(f) = \int_0^\infty f(x)e^{-x}dx \approx \omega_1 f(x_1) + \omega_2 f(x_2) = Q(f).$$

   (a) Determine the *nodes* x_1 and x_2 that maximize the polynomial degree of $Q(f)$.

   (b) Having x_1 and x_2, compute the *weights* ω_1 and ω_2.

   (c) Give an integral representation formula for the error $E(f) = I(f) - Q(f)$ (Hint: use Hermite polynomial interpolation).

3. Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric and positive definite matrices, and let $\mathbf{b} \in \mathbb{R}^n$. Consider the quadratic function $Q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T\mathbf{b}$ for $\mathbf{x} \in \mathbb{R}^n$ and a *descent* method to approximate the solution of $A\mathbf{x} = \mathbf{b}$:
$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k.$$

   (a) Define the concept of descent direction $\mathbf{d}_k$ and show how to compute the *optimal* stepsize α_k.

   (b) Formulate the *steepest descent* (or gradient) method and write a pseudocode which implements it.

   (c) Let B^{-1} be a preconditioner of A. Show how to modify the steepest descent method to work for $B^{-1}A\mathbf{x} = B^{-1}\mathbf{b}$, and write a pseudocode. Note that $B^{-1}A$ may not be symmetric (Hint: proceed as with the conjugate gradient method).

4 . The solution (π, π) of the system of equations $\mathrm{f}(\mathrm{x}) = 0$, where

$$\mathrm{f}(\mathrm{x}) = \begin{bmatrix} \sin x + \cos 2y - x - 1 + \pi \\ \sin 3x - \sin y + 2y - 2\pi \end{bmatrix}, \quad \mathrm{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

is to be computed by the iterative method

$$\mathrm{x}_{i+1} = \mathrm{x}_i - A^{-1}\mathrm{f}(\mathrm{x}_i), \tag{1}$$

where

$$A = \begin{bmatrix} -2 & 0 \\ -3 & 3 \end{bmatrix}.$$

(a) Show that there is a $\delta > 0$ such that (1) converges provided $|x_0 - \pi| < \delta, |y_0 - \pi| < \delta$. You may use a standard theorem to prove this.

(b) Prove the convergence is quadratic. For this part give direct proof of quadratic convergence (do not just quote a general theorem). Hint: Note that the Jacobian $\mathrm{f}'(\mathrm{x})$ is Lipschitz continuous.

5 . (a) Define the backward Euler method and trapezoidal rule for computing the numerical solution to an initial value problem $y' = f(t, y)$, $y(t_0)$ given.

(b) Consider the model problem $y' = \lambda y$, $y(0) = 1$, for $t \in [0, T]$, with $\lambda < 0$. For both the methods from part (a), derive expressions for the numerical solutions obtained for this model problem, and show that both numerical methods are convergent when applied to it.

(c) Another desirable property for the numerical solution is for it to display behavior qualitatively similar to that of the exact solution for very large T, using as small a number of time steps as possible (so that $|h\lambda|$ may be large). Which of the two methods under consideration here does a better job with respect to this criterion, for the problem of part (b)?

6 . Consider the boundary value problem

$$-u'' + u = f(x),\ 0 \le x \le 1, \quad u(0) = u(1) = 0. \tag{1}$$

(a) Give a variational formulation of (1), *i.e.*, express (1) as

$$u \in H, \quad B(u, v) = F(v), \text{for all } v \in H. \tag{2}$$

Define the space H, the bilinear form B, and the linear functional F, and state the relation between (1) and (2).

(b) What is meant by the finite element approximation, u_h, to the solution, u (h is the mesh parameter)? Is it guaranteed that u_h exists and is unique? What can be said about $\|u - u_h\|_1$ = the error in the Sobolev space, $H^1(0,1)$?

(c) Letting $e = u - u_h$, show that

$$B(e, e) = B(u, u) - B(u_h, u_h),$$

and hence that

$$B(u_h, u_h) \le B(u, u).$$

AMSC/CMSC 666
January 2003 Comprehensive Exam

INSTRUCTIONS: Results used, but not proved, must be stated clearly, indicating the assumptions under which they are valid.

1. This problem is about approximating mixed second derivatives by finite differences, which turns out to be useful in approximating Hessians. Let u be a continuous function in a neighborhood of the origin $(0,0) \in \mathbb{R}^2$. For $h > 0$ small, consider the four points in $\mathbb{R}^2$

$$\mathbf{x}_1 = (h,h), \quad \mathbf{x}_2 = (-h,h), \quad \mathbf{x}_3 = (-h,-h), \quad \mathbf{x}_4 = (h,-h).$$

(a) Use $\mathbf{x}_1$ and $\mathbf{x}_2$ and polynomial interpolation to derive a finite difference approximation $\partial_x^h u(0,h)$ of $\partial_x u(0,h)$, along with an error estimate. Indicate clearly the regularity of u required for this estimate to hold.

(b) Derive a similar approximation $\partial_x^h u(0,-h)$ for $\partial_x u(0,-h)$ using $\mathbf{x}_3$ and $\mathbf{x}_4$.

(c) Combine $\partial_x^h u(0,h)$ with $\partial_x^h u(0,-h)$ appropriately to obtain a finite difference approximation $\partial_{xy}^h u(0,0)$ of $\partial_{xy} u(0,0)$. Prove an error estimate and indicate clearly the regularity of u required.

2(a). State the Weierstrass approximation theorem for functions defined on $[0,1]$.
2(b). Let

$$Q_n(f) \equiv \sum_{i=0}^{n} \omega_i^{(n)} f(x_i^{(n)}) \cong \int_0^1 f(x)\,dx \equiv I(f)$$

be a sequence of integration formulas with the following properties:

1. The weights $\omega_i^{(n)}$ are nonnegative.
2. $x_i^{(n)} \in [0,1]$.
3. $Q_n(p) = I(p)$ for all polynomials p of degree not greater than n.

Prove that for any function f that is continuous in [0,1] we have

$$\lim_{n\to\infty} Q_n(f) = I(f).$$

[Hint: First show that $\omega_i^{(n)} \le 1$.]

3. A rotation in the (i,j)-plane $(i < j)$is an identity matrix whose (i,i)-, (i,j)-, (j,i)-, and (j,j)-elements have been replaced by c, s, $-s$, c, where $c^2 + s^2 = 1$.

a. Show that for any $x, y \in \mathbb{R}$ there is a 2×2 plane rotation R_{12} such that

$$R_{12}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{x^2+y^2} \\ 0 \end{pmatrix}.$$

Give pseudocode for a program `rotgen(x, y, c, s, nu)` that generates the appropriate c and s from x and y and also returns $\nu = \sqrt{x^2+y^2}$.

b. Let A be an $(n+1)\times n$ upper Hessenberg matrix (i.e., A is zero below its first subdiagonal). Give pseudocode for an algorithm that reduces A to upper triangular form by premultiplication by n plane rotations.

c. Give the number of additions and multiplications required for the algorithm in part b. Give only the highest order term in n.

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4. Suppose $F : \mathcal{R}^n \to \mathcal{R}^n$ has a zero, x^*: $F(x^*) = 0$. Suppose the Jacobian $F'(x^*)$ is nonsingular and $F'(x)$ is Lipschitz continuous. Consider the variant of Newton's method for computing x^*, namely

$$x_{k+1} = x_k - B(x_k)F(x_k),$$

where $B(x_k)$ is a matrix that represents an approximation to the inverse of the Jacobian, $F'(x_k)^{-1}$. Assume $B(x)$ satisfies

$$\|I - B(x)F'(x^*)\| \leq \alpha,$$

where α is a positive constant less than one. Prove that the errors $e_k = x^* - x_k$ satisfy

$$\|e_{k+1}\| \leq \alpha\|e_k\| + c\,\|e_k\|^2,$$

where c is a constant.

5. Consider the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0,$$

where we assume $f(t,y), f_t(t,y) = \frac{\partial f(t,y)}{\partial t}$, and $f_y(t,y) = \frac{\partial f(t,y)}{\partial y}$ are continuous on the strip $S = [t_0, b] \times R$, and $|f_y(t,y)| \leq K$ on S. Suppose we approximate the solution $y(t)$ by the Euler Method: $y(t_i) \approx y_i$, where y_i is the Euler approximation at $t_i = t_0 + ih, i = 0, 1, \ldots, n$, where $h = \frac{b-t_0}{n}$.

(a) Establish the relation

$$y(t_{i+1}) - y_{i+1} = [1 + hf_y(t_i, \eta_i)][y(t_i) - y_i] + \frac{y''(\xi_i)h^2}{2}, \quad (1)$$

where ξ_i is between t_i and t_{i+1} and η_i is between $y(t_i)$ an d y_i. Note that (1) can be written

$$\begin{aligned} \text{Global Error at } t_{i+1} &= [1 + hf_y(t_i, \eta_i)] \times \text{Global Error at } t_i \\ &+ (\text{Local Truncation Error at } t_i); \end{aligned}$$

$1 + hf_y(t_i, \eta_i)$ is called the *amplification factor.*

(b) Use the relation in (1) to prove that

$$|y(t_i) - y_i| \le \frac{\overbrace{Mh}^{(b-t_0)}}{2}, \text{ for } i = 1, \ldots, n,$$

where M is an upper bound for $|y''(t)|$, for $t_0 \le t \le b$, provided

$$f_y(t, y) \le 0, \text{ for } (t, y) \in S, \text{ and } 0 < h < \frac{2}{K}.$$

Recall that K is a bound on $|f_y(t, y)|$.

6. Let $u(x)$ denote a function on the interval (a, b) such that $u(a) = 0$ and $u'(x)^2$ is integrable.

(a) Prove that

$$\int_a^b u(x)^2 dx \le \frac{[b-a]^2}{2} \int_a^b u'(x)^2 dx.$$

(b) Derive a matrix analogue of this relation when it is applied to a function

$$u_h(x) = \sum_{i=1}^{n} u_i \phi_i(x)$$

determined from a finite element discretization with piecewise linear basis functions (the usual hat functions) defined on a uniform mesh. That is, derive matrices A and M such that the result from part (a) leads to

$$(Mu, u) \le \frac{(b-a)^2}{2} (Au, u)$$

where u is the vector of coefficients defining u_h. Show the precise structure of A and M.

(c) Use Gerschgorin's Theorem to show that

$$\frac{(Au, u)}{(Mu, u)} \le ch^{-2},$$

where c is independent of the discretization mesh size h.

Note: For parts (b) and (c), you may assume for simplicity that a Dirichlet condition $u(b) = 0$ holds at the right endpoint.

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Comprehensive Exam
August 2002

1.

a. Let X be an $n \times p$ matrix of rank p. Given an n-vector y show that the 2-norm of the residual $r = y - Xb$ is minimized when r is orthogonal to the columns of X. Deduce that the solution b satisfies the normal equations $(X^{\mathrm{T}}X)b = X^{\mathrm{T}}y$. You may assume there is an orthogonal matrix whose first p columns form a basis for the column space of X.

b. Let x be an approximate eigenvector of A. Determine μ such that the residual $r = Ax - \mu x$ is minimized.

2. Suppose there is a quadrature formula

$$\int_a^b f(x)\,dx \approx w_a f(a) + w_b f(b) + \sum_{j=1}^{n} w_j f(x_j)$$

which produces the exact integral whenever f is a polynomial of degree $2n+1$. Here the nodes $\{x_j\}_{j=1}^n$ are all distinct. Prove that the nodes lie in the open interval (a, b), and the weights w_a, w_b and $\{w_j\}_{j=1}^n$ are positive.

3. Let A be a symmetric matrix of order n.

a. Let v_1 be an arbitrary vector in $\mathbb{R}^n$ with $\|v_1\|_2 = 1$, and let $v_{-1} = 0 \in \mathbb{R}^n$. Consider the three-term recurrence

$$\gamma_{j+1} v_{j+1} = Av_j - \delta_j v_j - \gamma_j v_{j-1}, \quad 1 \le j \le k,$$

where γ_{j+1} is chosen so that $\|v_{j+1}\|_2 = 1$ You may assume that $k < n$ and the process does not "break down," i.e., $v_j \neq 0$ for $j \le k$. Show that δ_j can be chosen so that the generated vectors $\{v_j\}$ satisfy $(v_j, v_i) = 1$ for $j = i$ and 0 for $j \neq i$.

b. Now consider the linear system of equations $Ax = b$, and let $v_1 = b/\|b\|_2$ in the algorithm above. Show that an approximate solution

$$x_k = \sum_{j=1}^{k} a_j v_j \tag{1}$$

whose residual $r_k = b - Ax_k$ is orthogonal to $span\{v_1, v_2, \ldots, v_k\}$ can be constructed by solving a system of equations

$$T_k a = \|b\|_2 e_1$$

for the vector of coefficients a in (1), where T_k is a tridiagonal matrix of order k and e_1 is the unit vector $(1, 0, \ldots, 0)^T$.

AMSC 667

August '02 Comprehensive Exam

INSTRUCTIONS: Results used, but not proved, must be stated clearly, indicating the assumptions under which they are valid.

4. The Newton method is used to approximate the zero of the function $f(x) = \tan(x/5)$, with starting value $x_0 = 5$. The results are reported in the following table

x_k
$5.0000000e+0$
$2.7267564e+0$
$5.0937737e-1$
$3.5171112e-3$
$0.1160185e-9$.

(a) Write the Newton method for this function and explain why the iterates are monotonically decreasing towards the zero $x_* = 0$.
(b) Define the concept of quadratic rate of convergence, and check whether the above method converges quadratically or not.
(c) Use Taylor expansions to find the actual rate of convergence (do not just quote a theorem). Explain the Newton method in geometric terms, and use this construction to explain the rate of convergence.

5. Let $u : [0,1]^2 \to \mathbb{R}$ be the solution of the *heat equation*:

$$\begin{aligned} u_t - u_{xx} &= f(t,x), && t, x \in (0,1) \\ u(t,0) = u(t,1) &= 0, && t \in (0,1) \\ u(0,x) &= u_0(x), && x \in [0,1]. \end{aligned}$$

(a) Write this equation in operator form as $u_t + Au = f$, where $Au = -u_{xx}$. Write the (backward) implicit Euler method for the ODE $u_t + Au = f$ with constant stepsize τ.
(b) Discretize the term Au of (a) via centered differences on a uniform partition of $[0,1]$ with meshsize h.
(c) Let $\mathbf{U}^n = (U_i^n)_{i=1}^I$ be the nodal values of the discrete solution at time step n for $1 \le n \le N$. Show that $\mathbf{U}^n$ solves a system $\mathbf{M}\mathbf{U}^n = \mathbf{b}^{n-1}$ with matrix $\mathbf{M}$ satisfying

$$m_{ii} > \sum_{j \neq i} |m_{ij}|, \qquad m_{ij} \le 0 \quad i \neq j.$$

(d) Show that $u_0, f \ge 0$ imply $U_i^n \ge 0$ for all n, i.

6. Let $y(t)$ be the solution of the scalar initial value problem $y' = f(t,y)$ with $y(0) = y_0$. Let $\{y_n\}_{n=0}^N$ be the solution of the following Runge-Kutta method

$$\begin{aligned} k_1 &= f(t_n, y_n), \\ k_2 &= f(t_n + h, y_n + hk_1), \\ y_{n+1} &= y_n + \frac{h}{2}(k_1 + k_2). \end{aligned}$$

(a) Define truncation error T_n on the interval $[t_n, t_{n+1}]$.
(b) Show that $T_n = O(h^3)$ provided $f \in C^2$. What would be the order of T_n if $f \in C^k$ with $k < 2$?
(c) Derive a global error estimate and establish the global rate of convergence. Sketch a proof.

Numerical Analysis

AMSC 666
Comprehensive Exam
January 2002

1. Let A be a nonsingular matrix, and let $A = Q - R$ be a splitting such that the eigenvalue of $Q^{-1}R$ with maximum modulus, denoted $\rho(Q^{-1}R)$, is less than 1.
a. What is the stationary iterative method based on this splitting for solving $Ax = b$?
b. Let $x^{(m)}$ denote the mth iterate. Suppose that $Q^{-1}R = SDS^{-1}$ where D is diagonal. Prove that

$$\frac{\log(\kappa(S)/\epsilon)}{-\log(\rho(Q^{-1}R))}$$

iterations suffice to reduce the relative error $\|x - x^{(m)}\|_2/\|x - x^{(0)}\|_2$ to ϵ. Here $\kappa(S) = \|S\|_2\,\|S^{-1}\|_2$.

2. Consider the polynomial recurrence

$$p_{k+1}(x) = (x - \alpha_{k+1})p_k(x) - \beta_{k+1}^2 p_{k-1}(x), \quad k = 0, 1, 2, \ldots$$

where $p_0 = 1$, $p_{-1} = 0$, and α_k and β_k are scalars.
a. Show that the roots of $p_k(x)$ are the eigenvalues of the tridiagonal matrix

$$J_k = \begin{pmatrix} \alpha_1 & \beta_2 & & & \\ \beta_2 & \alpha_2 & \beta_3 & & \\ & & \ddots & & \\ & & \beta_{k-1} & \alpha_{k-1} & \beta_k \\ & & & \beta_k & \alpha_k \end{pmatrix}.$$

b. Derive conditions on α_k and β_k such that the polynomials satisfy

$$\int_{-1}^{1} p_i(x)p_j(x)\,dx = 0, \quad i \neq j.$$

Of what use are the roots in this case?

3. Determine a and b such that $ax + b$ is the best ∞-norm approximation to x^2 on [0,1].

Numerical Analysis

AMSC 667

January '02 Comprehensive Exam

INSTRUCTIONS: Results used, but not proved, must be stated clearly, indicating the assumptions under which they are valid.

4. Suppose $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is of class C^2, suppose $f(x^*) = 0$, and suppose the Jacobian $Df(x^*)$ is positive definite. For λ a real parameter, consider the iteration:

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \lambda \mathbf{f}(\mathbf{x}_k). \tag{1}$$

(a) Write sufficient conditions for this iteration to converge locally; specifically, determine the range of λ for which (1) converges.
(b) Is it possible to choose λ in such a way that convergence is superlinear?
(c) Suppose λ is replaced by a matrix $\mathbf{A}_k \in \mathbb{R}^{n\times n}$. How would you choose $\mathbf{A}_k$ to achieve quadratic convergence.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be smooth, and $u : [0,1] \to \mathbb{R}$ be the solution of the *periodic* 2-point boundary value problem

$$-u'' = f(x) \quad 0 < x < 1, \qquad u(0) = u(1), \quad u'(0) = u'(1). \tag{2}$$

(a) Show that for this problem to have a solution, f must satisfy the *compatibility condition*

$$\int_0^1 f(x)dx = 0.$$

Show that in this case, the kernel of this problem, *i.e.*, the space of solutions corresponding to $f(x) = 0$, is the space of constant functions.
(b) Discretize (2) using finite differences on a uniform partition $x_0 = 0 < x_1 < \cdots < x_{N-1} < x_N = 1$ with meshsize h. Write the resulting system in terms of the unknown nodal values $\mathbf{U} = \{U_n\}_{n=1}^N$. Note that this requires eliminating the unknown U_0.
(c) Show that the system can be written so that the resulting matrix $\mathbf{A}$ is symmetric; display $\mathbf{A}$, showing the pattern of zero and nonzero entries. Show that $\mathbf{A}$ is singular, and find its kernel (nullspace).
(d) Find a discrete compatibility counterpart of (a) for the discrete problem $\mathbf{AU} = \mathbf{F}$ to have a solution.

6. Consider the following *implicit* method for the initial value problem $y' = f(t,y)$ with $y(0) = y_0$:

$$y_n = y_{n-1} + \frac{h}{2}\Big(f(t_{n-1}, y_{n-1}) + f(t_n, y_n)\Big), \qquad 1 \le n \le N,$$

where $h > 0$ is the constant stepsize. Assume that the ODE is *dissipative*, *i.e.*, $\frac{\partial f}{\partial y} \le 0$.
(a) Show that the nonlinear algebraic problem involving y_n always admits one and only one solution. Note that h is not assumed to be small here.
(b) Define the local truncation error T_n on the interval $[t_{n-1}, t_n]$, and find its order, provided $f \in C^2$
(c) Show that the error $e_n = y(t_n) - y_n$ between the exact solution $y(t_n)$ and the corresponding approximate solution y_n obeys the following relation:

$$|e_n| \le |e_{n-1}| + |T_n|, \text{ for all } 1 \le n \le N.$$

(d) Deduce the error estimate

$$|e_N| \le \sum_{n=1}^{N} |T_n|$$

(with stability constant = 1), and derive the global rate of convergence, provided $f \in C^2$.

MAPL/CMSC 666: Numerical Analysis (I)
Qualifying Examination Problems
January 2001

1. Let $C[a,b]$ denote the set of all real-valued functions defined and continuous on a closed interval $[a,b]$. Let $\rho \in C[a,b]$ be positive everywhere in $[a,b]$.

1a. (10%) Let $f \in C[a,b]$. For each integer $n \geq 0$, let $p_n \in \mathcal{P}_n$ be the best uniform approximation of f in $\mathcal{P}_n$ on $[a,b]$:

$$\max_{a \leq x \leq b} |f(x) - p_n(x)| \leq \max_{a \leq x \leq b} |f(x) - q_n(x)| \qquad \forall q_n \in \mathcal{P}_n,$$

where $\mathcal{P}_n$ is the set of all real polynomials of degree $\leq n$. Show that

$$\lim_{n \to \infty} \int_a^b \rho(x)[f(x) - p_n(x)]^2 dx = 0.$$

1b. (60%) Let $\{Q_n\}_{n=0}^{\infty}$ be a system of polynomials, with $\deg Q_n = n$ for each n, orthogonal with respect to the inner product

$$\langle g, h \rangle = \int_a^b \rho(x) g(x) h(x)\, dx \qquad \forall g, h \in C[a,b].$$

For a fixed integer $n \geq 2$, let $x_1, \cdots, x_n$ be the n distinct roots of Q_n in (a,b). Let

$$l_k(x) = \frac{(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}, \qquad k = 1, \cdots, n.$$

Show that

$$\int_a^b \rho(x) l_j(x) l_k(x)\, dx = 0, \qquad 1 \leq j < k \leq n,$$

and that

$$\sum_{k=1}^n \int_a^b \rho(x) \left[l_k(x)\right]^2 dx = \int_a^b \rho(x) dx.$$

2. The trapezoidal numerical integration rule on a closed interval $[a, b]$ is given by

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)\,[f(a)+f(b)]\,.$$

Let $f : [a, b] \to \mathcal{R}$ be twice continuously differentiable.

2a. (50%) Prove that there exists $\xi \in (a, b)$ such that

$$\int_a^b f(x)\,dx = \frac{1}{2}(b-a)\,[f(a)+f(b)] - \frac{1}{12}(b-a)^3 f''(\xi).$$

2b. (50%) Let $N \geq 1$ be an integer, $h = (b-a)/N$, and $x_k = a + kh$, $k = 0, \cdots, N$. Prove the following error formula for the composite trapezoidal numerical integration rule

$$\int_a^b f(x)\,dx = \left\{ \frac{h}{2}[f(a)+f(b)] + h\sum_{k=1}^{N-1} f(x_k) \right\} - \frac{(b-a)f''(\eta)}{12}h^2,$$

where $\eta \in (a, b)$ depends on f.

3. Suppose that $A \in \mathcal{R}^{m\times n}$, $x \in \mathcal{R}^n$, and $b \in \mathcal{R}^m$, with $m \geq n$. Consider the problem

$$\min_x \|Ax - b\|^2 + \lambda\|x\|^2$$

where λ is a real scalar.

3a. (50%) Develop an algorithm to solve the problem if $\lambda > 0$.

3b. (20%) What is the highest order term in the operations count for your algorithm? (You do not need to compute the constant in front of the powers of m and n.)

3c. (30%) In what way is the problem changed if $\lambda < 0$?

January MAPL 667 Exam

Clearly specify all the theorems you use.

1. Let A be the $n \times n$ tridiagonal matrix with values 4 on the diagonal, and values -1 above and below the diagonal. Let $x \in \mathbb{R}^n$ and consider the nonlinear system

$$Ax + \begin{pmatrix} x_1^3 \\ \vdots \\ x_n^3 \end{pmatrix} = b \tag{1}$$

with $b \in \mathbb{R}^n$. We consider the iteration method $x^{(k+1)} = T(x^{(k)})$ where the iteration function $y = T(x)$ is defined implicitly by

$$4y + \begin{pmatrix} y_1^3 \\ \vdots \\ y_n^3 \end{pmatrix} = (4I - A)x + b \tag{2}$$

   (a) Show that the system (2) always has a unique solution $y \in \mathbb{R}^n$ for any given $x, b \in \mathbb{R}^n$.

   (b) Show that T satisfies $\|T(x) - T(y)\|_\infty \leq \frac{1}{2}\|x - y\|_\infty$ for $x, y \in \mathbb{R}^n$. (Hint: Express $T(x)$ in terms of the inverse function of $\psi(t) = 4t + t^3$.)

   (c) Use (b) to show that the system (1) has a unique solution $x_* \in \mathbb{R}^n$. Let $x^{(0)} \in \mathbb{R}^n$ and $x^{(k+1)} = T(x^{(k)})$. Prove an estimate which shows how $\|x^{(k)} - x_*\|_\infty$ decreases for $k \to \infty$.

2. Consider the initial value problem $y' = f(x, y)$, $y(0) = a$ and the multistep method with constant stepsize h

$$3y_{n+1} - 4y_n + y_{n-1} = 2hf(x_{n+1}, y_{n+1})$$

where $x_n = nh$.

   (a) Assume that $y_{n-1} = y(x_{n-1})$, $y_n = y(x_n)$ and show that $|y_{n+1} - y(x_{n+1})| \leq Ch^3$ for some constant $C > 0$ independent of h, provided that $f(x, y)$ is sufficiently smooth. Assuming the method is stable, what is its order of convergence?

   (b) Let $f(x, y) = 4y$. Determine whether we have $\sup_{n \geq 0} |y_n| < \infty$ for all initial values (i) in the case $h = 1$, (ii) in the case $h = \frac{1}{2}$.

3. Consider the boundary value problem

$$-u''(x) + u(x) = f(x), \qquad x \in (0, 1) \tag{3}$$

with periodic boundary conditions

$$u(0) = u(1), \quad u'(0) = u'(1) \tag{4}$$

Consider the function space $V = \{ v \in H^1(0,1) \colon v(0) = v(1) \}$ and the variational formulation

$$\text{Find } u \in V \text{ such that for all } v \in V\text{:} \quad \int_0^1 \Big(u'(x)v'(x) + u(x)v(x) \Big) dx = \int_0^1 f(x)v(x)dx \tag{5}$$

   (a) Let $u \in C^2[0, 1]$ be a solution of (5). Show that u is a solution of (3), (4).

   (b) Let $h = 1/N$ with $N \in \mathbb{N}$. We partition $[0, 1]$ into N subintervals of length h. Let V_h be the *subspace* of the space V consisting of piecewise linear functions on this partition. What is the dimension of V_h? Give a basis of V_h consisting of functions of smallest support (nodal basis functions).

   (c) Give the finite element formulation corresponding to (5). Let $\vec{U}$ be the coefficient vector of u with respect to the basis. Determine the stiffness matrix A and the right hand side vector $\vec{F}$ so that the finite element solution satisfies $A\vec{U} = \vec{F}$. Give explicit values (in terms of N) for all entries of the stiffness matrix A.

January MAPL 667 Exam

Clearly specify all the theorems you use.

4. Let A be the $n \times n$ tridiagonal matrix with values 4 on the diagonal, and values -1 above and below the diagonal. Let $x \in \mathbb{R}^n$ and consider the nonlinear system

$$Ax + \begin{pmatrix} x_1^3 \\ \vdots \\ x_n^3 \end{pmatrix} = b \tag{1}$$

with $b \in \mathbb{R}^n$. We consider the iteration method $x^{(k+1)} = T(x^{(k)})$ where the iteration function $y = T(x)$ is defined implicitly by

$$4y + \begin{pmatrix} y_1^3 \\ \vdots \\ y_n^3 \end{pmatrix} = (4I - A)x + b \tag{2}$$

(a) Show that the system (2) always has a unique solution $y \in \mathbb{R}^n$ for any given $x, b \in \mathbb{R}^n$.

(b) Show that T satisfies $\|T(x) - T(y)\|_\infty \leq \frac{1}{2}\|x - y\|_\infty$ for $x, y \in \mathbb{R}^n$. (Hint: Express $T(x)$ in terms of the inverse function of $\psi(t) = 4t + t^3$.)

(c) Use (b) to show that the system (1) has a unique solution $x_* \in \mathbb{R}^n$. Let $x^{(0)} \in \mathbb{R}^n$ and $x^{(k+1)} = T(x^{(k)})$. Prove an estimate which shows how $\|x^{(k)} - x_*\|_\infty$ decreases for $k \to \infty$.

5. Consider the initial value problem $y' = f(x, y)$, $y(0) = a$ and the multistep method with constant stepsize h

$$3y_{n+1} - 4y_n + y_{n-1} = 2hf(x_{n+1}, y_{n+1})$$

where $x_n = nh$.

(a) Assume that $y_{n-1} = y(x_{n-1})$, $y_n = y(x_n)$ and show that $|y_{n+1} - y(x_{n+1})| \leq Ch^3$ for some constant $C > 0$ independent of h, provided that $f(x, y)$ is sufficiently smooth. Assuming the method is stable, what is its order of convergence?

(b) Let $f(x, y) = 4y$. Determine whether we have $\sup_{n \geq 0} |y_n| < \infty$ for all initial values (i) in the case $h = 1$, (ii) in the case $h = \frac{1}{2}$.

6. Consider the boundary value problem

$$-u''(x) + u(x) = f(x), \qquad x \in (0, 1) \tag{3}$$

with periodic boundary conditions

$$u(0) = u(1), \quad u'(0) = u'(1) \tag{4}$$

Consider the function space $V = \{ v \in H^1(0, 1) \colon v(0) = v(1) \}$ and the variational formulation

$$\text{Find } u \in V \text{ such that for all } v \in V\colon \quad \int_0^1 \Big(u'(x)v'(x) + u(x)v(x)\Big)dx = \int_0^1 f(x)v(x)dx \tag{5}$$

(a) Let $u \in C^2[0, 1]$ be a solution of (5). Show that u is a solution of (3), (4).

(b) Let $h = 1/N$ with $N \in \mathbb{N}$. We partition $[0, 1]$ into N subintervals of length h. Let V_h be the *subspace* of the space V consisting of piecewise linear functions on this partition. What is the dimension of V_h? Give a basis of V_h consisting of functions of smallest support (nodal basis functions).

(c) Give the finite element formulation corresponding to (5). Let $\vec{U}$ be the coefficient vector of u with respect to the basis. Determine the stiffness matrix A and the right hand side vector $\vec{F}$ so that the finite element solution satisfies $A\vec{U} = \vec{F}$. Give explicit values (in terms of N) for all entries of the stiffness matrix A.

MAPL 666

Comprehensive Exam

August 2000

1. Let $A \in \mathcal{R}^{m\times n}$ be a matrix of rank n. Let $C = (A^T A)^{-1}$ be the *variance-covariance matrix*. Suppose that we have a QR factorization ($Q \in \mathcal{R}^{m\times n}$ with orthonormal columns, $R \in \mathcal{R}^{n\times n}$ upper triangular) of A available.

1a. (30%) Show that $C = R^{-1}(R^T)^{-1}$.

1b. (40%) Give an efficient algorithm for computing the main diagonal of C, given R.

1c. (30%) What is the relation of C to the normal equations for the least squares problem

$$\min_x \|Ax - b\| ?$$

2a. (50%) Let $X = [X_1, X_2]$ be a nonsingular $n \times n$ complex matrix. Let A be an $n \times n$ complex matrix, and suppose $AX_1 = X_1 M$. Show that

$$X^{-1}AX = \begin{bmatrix} M & B \\ 0 & C \end{bmatrix}$$

for some matrices B and C.

2b. (50%) Prove that each eigenvalue of M and each eigenvalue of C is an eigenvalue of A. Choose an eigenvector of M and express an eigenvector of A in terms of it. Repeat for an eigenvector of C.

3a. (20%) Write down the compound trapezoidal rule for approximating the integral

$$I(f) = \int_a^b f(x)dx$$

Use $n + 1$ function evaluations.

3b. (30%) Write down the error formula for your formula from part a.

3c. (50%) Sketch the details of a computer program that uses the compound trapezoidal rule to produce an approximation $Q(f)$ for $I(f)$ such that $|I(f) - Q(f)| < \epsilon$ for some parameter ϵ provided by the user. Make sure that you make efficient use of function values.

CMSC/MAPL 667
Comprehensive Exam
August 2000

Problem 1

Let $G\colon R^n - R^n$ be continuously differentiable and have a fixed point x_*. For some starting point x_0 consider the iteration

$$x_{k+1} = G(x_k), \qquad k = 0, 1, 2 \ldots$$

(a) Define the concepts of global and local convergence for the iteration.
(b) Show that if in the matrix 1-norm, $\|\cdot\|_1$, we have $\|G'(x_*)\|_1 < 1$, then the iteration converges locally.
(c) The mapping

$$G(x, y) = \begin{pmatrix} \alpha(3x^2 + 3x + y - 1) \\ 1 + \alpha(x^3 - y + 1) \end{pmatrix}$$

has the fixed point

$$x_* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

for any value of the parameter α. For which values of α does the local convergence result apply?

Problem 2

Consider the boundary value problem

$$\begin{cases} -u''(x) + u(x) = f(x) \\ u(0) = 0,\ u'(1) = 2, \end{cases} \tag{1}$$

where $f \in L_2(0, 1)$.
(a) Derive the variational formulation of (1), defining precisely the bilinear form, the linear functional, and the function spaces.
(b) Given a partition $\Delta_h = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$, with $h = \max_{i=1,\ldots,n}(x_i - x_{i-1})$, define the finite element approximation u_h to u using continuous, piecewise linear approximating functions.
(c) State the standard estimate, in terms of a power of h, for the finite element error, $e = u - u_h$, in the H^1-norm. Include a statement of any regularity assumptions the solution u should satisfy.

(d) Letting B be the bilinear form in the variational formulation of (1), show that

$$B(\epsilon, e) = B(u, u) - B(u_h, u_h).$$

Hint: Recall the basic orthogonality property of u_h.

Problem 3

Consider the initial value problem

$$\begin{cases} y' = f(x, y) \\ y(a) = A, \end{cases} \tag{1}$$

where $f(x, y)$ is defined on $[a, b] \times R$. For approximating of the solution $y(x)$ of (1), consider the Taylor Series Method,

$$\begin{cases} x_0 = a, y_0 = A \\ x_{i+1} = x_i + h \\ y_{i+1} = y_i + h\Phi(x_i, y_i, h, f), \ i = 0, 1, \cdots, \end{cases} \tag{2}$$

where

$$\Phi(x, y, h, f) \equiv f(x, y) + \frac{h}{2} f^{(1)}(x, y), \tag{3}$$

where

$$f^{(1)}(x, y) = f_x(x, y) + f_y(x, y) f(x, y). \tag{4}$$

Φ is an approximate slope. We also consider the exact slope, defined by

$$\Delta(x, y(x), h, f) = \frac{y(x + h) - y(x)}{h}. \tag{5}$$

(a) Show that the method is of order 2 (for the solution $y(x)$), *i.e.*, that there is a positive constant C such that

$$|\Phi(x, y(x), h, f) - \Delta(x, y(x), h, f)| \leq Ch^2, \text{ for all } a \leq x \leq b - h. \tag{6}$$

C will depend on $y(x)$; you will need to assume $f(x, y)$ has bounded partial derivatives of orders ≤ 2. Hint: Show that $y''(x) = f^{(1)}(x, y(x))$.

(b) In what follows you may assume that $\Phi(x, y, h, f)$ is Lipschitz continuous in y, *i.e.*, that

$$|\Phi(x, \overline{y}, h, f) - \Phi(x, y, h, f)| \leq K |\overline{y} - y|, \forall y, \overline{y} \in R. \tag{7}$$

Here is an outline of the proof that

$$|y(x_i) - y_i| \le C\frac{e^{K(b-a)} - 1}{K}h^2. \tag{8}$$

Complete the proof: specifically, answer questions (i), (ii), and (iii).

(i) Show that

$$y(x_{i+1}) = y(x_i) + h\Delta(x_i, y(x_i), h, f). \tag{9}$$

(ii) Letting $e_i = y(x_i) - y_i$, use (2),(9), (6), and (7) to show that

$$|e_{i+1}| \le |e_i|[1 + hK] + Ch^3. \tag{10}$$

(iii) Show that (8) follows from (10) and the

Lemma. Suppose $d_0, d_1, \cdots$ is a nonnegative sequence satisfying

$$d_{i+1} \le (1+\delta)d_i + M,\ i = 0, 1, \cdots,$$

for some $\delta, M \ge 0$. Then

$$d_i \le M\frac{e^{i\delta} - 1}{\delta},\ i = 0, 1, \cdots.$$

(You do not need to prove the Lemma.)

Numerical Analysis

CMSC/MAPL 666
Comprehensive Exam
January 2000

1. Let (50%) f be a function with 3 continuous derivatives. Let q be the quadratic polynomial that interpolates f at $x_0 < x_1 < x_2$. Let $h = \max(x_1 - x_0, x_2 - x_1)$ and
$$\max_{x_0 \le x \le x_2} |f'''(x)| = K.$$
   (a) Show that
$$\max_{x_0 \le x \le x_2} |f''(x) - q''(x)| \le Ch^\alpha,$$
   where $C > 0$ depends *only* on K and determine α_i. (Hint: the key to this is to prove that $f''(x) - q''(x)$ vanishes at some point in $[x_0, x_2]$. The result can then be obtained by integration.)

   (b) (50%) Now suppose $h = x_2 - x_1 = x_1 - x_0$ and f has 4 continuous derivatives. In this case show
$$|f''(x_1) - q''(x_1)| \le C'h^\beta$$
   where $\beta > \alpha$. What is C' in terms of the derivatives of f ?

2.
   (a) (40%) Find $\{p_0, p_1, p_2\}$ such that p_i is a polynomial of degree i and this set is orthogonal on $[0, \infty)$ with respect to the weight function $w(x) = e^{-x}$. (Note: $\int_0^\infty x^n e^{-x}\,dx = n!,\ 0! = 1$.)

   (b) (60%) Derive the 2 point Gaussian formula
$$\int_0^\infty f(x)e^{-x}\,dx \approx w_1 f(x_1) + w_2 f(x_2)$$
   i.e. find the weights and nodes.

3. Let A be an $n \times n$ nonsingular matrix, and consider the linear system $Ax = b$.
   (a) (30%) Write down the Jacobi iteration for solving $Ax = b$, in the way that it would be programmed on a computer.

   (b) (30%) Suppose A has m nonzero elements with $m << n^2$. How many operations per iteration does the Jacobi iteration take?

   (c) (40%) Assume that A is strictly diagonally dominant: for $i = 1 \ldots, n$,
$$|a_{ii}| > \sum_{j=1, j\ne i}^{n} |a_{ij}|.$$
   Show that the Jacobi iteration converges for any initial guess $x^{(0)}$. (Hint: You may use Gerschgorin's theorem without proving it.)

CMSC/MAPL 667
Comprehensive Exam
January 2000

1.

(a) (30%) Verify that the calculation of $1/a$, where a is a non-zero scalar, by Newton's method corresponds to the iterative method

$$x_{k+1} = x_k(2 - ax_k), \quad k \geq 0. \tag{1}$$

(b) (30%) By considering the successive residuals $r_k = 1 - ax_k$ show that (1) is convergent if and only if $0 < ax_0 < 2$.

(c) (40%) In analogy to (1) construct an iterative method for finding the inverse of a nonsingular matrix A and give necessary and sufficient conditions for convergence in terms of the initial guess X_0.

2. The *modified Euler method* for the initial value problem

$$y' = f(x, y), \; y(x_0) = y_0$$

is given by

$$y_{n+1} = y_n + hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n)).$$

Give a direct proof that this method is convergent. Do not quote any theorems. What is the order of the method ?
Hint: For this you will need to compute (or estimate) the local truncation error.

3. Set $I = (0,1)$. Let $\{0 = x_0 < x_1 < \cdots < x_n = 1\}$ be a partition of I. Set $h_n = x_n - x_{n-1}$. Let V_h be the finite element space

$$V_h = \{v \in C^0(I) : v|_{(x_{n-1}, x_n)} \in \mathcal{P}_1, \; v(0) = 0\},$$

where $\mathcal{P}_1$ is the space of first degree polynomials. Let $\{\phi_n\}_{n=1}^N$ be the canonical basis of V_h. Consider the functional $J : V_h \to \mathrm{R}$ defined by

$$v \mapsto J(v) = \int_0^1 (\frac{1}{2}(v')^2 - fv)\, dx,$$

where $f \in L^2(I)$.

(a) (20%) If $v = \sum_{n=1}^N v_n\phi_n \in V_h$ and $\tilde{v} = \{v_n\}_{n=1}^N$, then instead of J we can consider $F : \mathrm{R}^n \to \mathrm{R}$ where

$$\tilde{v} \mapsto F(\tilde{v}) = J(v).$$

Compute the gradient of F.

(b) (20%) Let $u_h = \sum_{n=1}^{N} u_n \phi_n$ be a minimizer of J:

$$u_h \in V_h : J(u_h) = \min_{v \in V_h} J(v).$$

Determine an equation satisfied by u_h or equivalently $\tilde{u}$, and relate to the piecewise linear finite element approximation of a two-point boundary value problem. Find the corresponding boundary conditions.

(c) (30%) Show that u_h, or equivalently $\tilde{u}$, exists and is unique.

(d) (30%) If u is the solution of the two-point boundary value problem, derive the error estimate

$$\|(u - u_h)'\|_{L^2(I)} \leq C \Big(\sum_{n=1}^{N} h_n^2 \|f\|_{L^2(x_{n-1}, x_n)}^2 \Big)^{\frac{1}{2}}.$$

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
IN
NUMERICAL ANALYSIS

AUGUST 1999

INSTRUCTIONS

1. Answer the three questions from 666 and the three questions from 667. This is for both the MAPL Masters and PhD exam.

2. Your work on each question will be assigned a grade from 0 to 10. If some problems have multiple parts, be sure to go on to subsequent parts even if there is a part you cannot do.

3. Use a different sheet (or different set of sheets) for each question. Write the problem number and your **code number** (not your name)on the top of every sheet.

4. Keep scratch work on separate sheets.

5. There is a four hour time limit.

CMSC/MAPL 666
Comprehensive Exam, August 1999

1. The idea of this problem is to give a direct proof of the uniqueness of the natural cubic spline interpolating data points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ with $x_0 < x_1 < \cdots < x_n$. The only analytic tool you will need is Rolle's theorem.
So suppose $S_1(x), S_2(x)$ are natural cubic spline functions satisfying

$$S_1(x_i) = S_2(x_i) = y_i, \quad i = 0, 1, \ldots, n, \quad S_j''(x_0) = S_j''(x_n) = 0, \quad j = 1, 2.$$

Let $S(x) = S_1(x) - S_2(x)$.
(a) Show that $S(x)$ is a natural cubic spline function.
(b) Show that $S''(x)$ has at least $n-1$ zeros on (x_0, x_n) and therefore at least $n+1$ zeros on $[x_0, x_n]$.
(c) Show that there is an $i, 0 \le i \le n-1$ such that $S(x) \equiv 0$ on $[x_i, x_{i+1}]$.
(d) Show $S(x) \equiv 0$ on $[x_0, x_n]$ and therefore $S_1(x) \equiv S_2(x)$ on $[x_0, x_n]$.

2. Let

$$I_n(f) = \sum_{k=1}^{n} w_{n,k} f(x_{n,k}), \qquad a \le x_{n,k} \le b$$

be a sequence of integration rules.
(a) suppose

$$\lim_{n \to \infty} I_n(x^k) = \int_a^b x^k \, dx, \quad k = 0, 1, \cdots \tag{1}$$

and

$$\sum_{k=1}^{n} |w_{n,k}| \le M, \quad n = 1, 2, \cdots \tag{2}$$

for some constant M. Show that

$$\lim_{n \to \infty} I_n(f) = \int_a^b f(x) \, dx \text{ for all } f \in C[a, b]$$

(b) Show that if all $w_{n,k} > 0$ then (1) implies (2).

3. Suppose $A \in R^{n \times n}$ is symmetric and positive definite. Consider the following iteration:

$$\begin{aligned} A_0 &= A \\ &\text{for } k = 0, 1, 2, \cdots \\ A_k &= L_k L_k^T \text{ (Cholesky)} \\ A_{k+1} &= L_k^T L_k \end{aligned}$$

Here L_k is lower triangular with positive diagonal elements.

(a) Show that A_k is symmetric and positive definite (so that the iteration is well defined) and similar to A.

(b) Show that if

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad a \geq c$$

has eigenvalues $\lambda_1 \geq \lambda_2 > 0$ then the A_k converge to $\mathrm{diag}(\lambda_1, \lambda_2)$. Hint: Carry out one step of the iteration and compare the elements of A_1 to those of A_0. The result follows from basic facts of analysis and matrix theory.

CMSC/MAPL 667
Comprehensive Exam
Fall 1999

INSTRUCTIONS: Results used, but not proved, must be stated clearly, indicating the assumptions under which they are valid.

1. Let $g : \mathrm{R} \to \mathrm{R}$ be a C^1 function with $g(\alpha) = \alpha$.
(a) Suppose $|g'(\alpha)| < 1$. Prove that if x_0 is chosen sufficiently close to α the iterations

$$x_{n+1} = g(x_n), \qquad n = 0, 1, \dots \tag{1}$$

converge to α
(b) Suppose $g \in C^2$ with $g(\alpha) = \alpha, g'(\alpha) = 0$. Show that the convergence of (1) is (at least) locally quadratic.
(c) By consideration of the function $g(x) = x - x^3, \alpha = 0$ show that the condition $|g'(\alpha)| < 1$ is not neccessary for the convergence of (1).

2. Given the ordinary differential equation $y' = f(x, y)$, consider the two-step method

$$y_{n+1} = 4y_n - 3y_{n-1} - 2hf(x_{n-1}, y_{n-1}), \qquad n \geq 1 \tag{2}$$

(a) Define truncation error and compute its order for (2).
(b) State the root condition and prove that (2) is unstable. Explain why the global error diverges even though the local error tends to 0 when $h \downarrow 0$.
(c) Construct an explicit example of divergence.

3. Consider the following boundary value problem in $(-1, 1)$:

$$-(a(x)u')' = f(x), \qquad u(-1) = 1, \quad u'(1) = -1, \tag{3}$$

with piecewise constant coefficient a

$$a(x) = \begin{cases} 1, & -1 < x < 0 \\ 2, & 0 < x < 1. \end{cases}$$

Let $x_0 = -1 < x_1 < \cdots < x_N = 1$ be a (non-uniform) partition of $(-1, 1)$ containing the origin. Let $h_n = x_n - x_{n-1}$ and $h = \max_{1 \leq n \leq N} h_n$ be the meshsize.
(a) Derive a variational formulation for (3) in the original variable u, indicating the bilinear and linear forms as well as the function spaces (do not change variables to make the boundary conditions zero). State and verify the assumptions of the Lax-Milgram Lemma.
(b) Let $\mathbf{V}_h(\alpha)$ be the finite element space of continuous piecewise linear functions $v(x)$ over the above partition such that $v(-1) = \alpha$. Write the corresponding discrete problem and find the matrix equation. Indicate clearly the spaces where the discrete solution u_h and test functions belong.
(c) Show that the discrete problem admits a unique solution u_h.
(d) Show that u_h is the piecewise linear interpolant of u (Hint: integrate by parts).

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
IN
NUMERICAL ANALYSIS

JANUARY 1999

INSTRUCTIONS

1. Answer the three questions from 666 and the three questions from 667. This is for both the MAPL Masters and PhD exam.

2. Your work on each question will be assigned a grade from 0 to 10. If some problems have multiple parts, be sure to go on to subsequent parts even if there is a part you cannot do.

3. Use a different sheet (or different set of sheets) for each question. Write the problem number and your **code number** (not your name)on the top of every sheet.

4. Keep scratch work on separate sheets.

5. There is a four hour time limit.

CMSC/MAPL 666
Comprehensive Exam
January 1999

1. Suppose $\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\}$ is a set of functions with the property that given any set of $n+1$ distinct numbers $x_0, x_1, \ldots, x_n$ and numbers $f_0, f_1, \ldots, f_n$, there is a unique function
$$\Phi(x) = \sum_{i=0}^{n} \alpha_i \phi_i(x)$$
such that $\Phi(x_k) = f_k$, $k = 0, 1, \ldots, n$. Suppose $\{\psi_0(y), \psi_1(y), \ldots, \psi_m(y)\}$ is a set of functions with the same property (with n replaced by m). Show that if we are given
$$\begin{array}{ll} x_0, x_1, \ldots, x_n, & \text{with } x_i \neq x_j \text{ for } i \neq j, \\ y_0, y_1, \ldots, y_m, & \text{with } y_i \neq y_j \text{ for } i \neq j, \end{array}$$
and numbers f_{ij}, $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, m$ there exists a unique function of the form
$$\Phi(x, y) = \sum_{\nu=0}^{n} \sum_{\mu=0}^{m} \alpha_{\nu\mu} \phi_\nu(x) \psi_\mu(y)$$
with $\Phi(x_i, y_j) = f_{ij}$, $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots m$.

2.
   (a) Assume that $f(x)$ is continuous and $f'(x)$ is integrable on $[0, 1]$. Show that the error E_n in the n-panel trapezoid rule for calculating $\int_0^1 f(x)\, dx$ can be written as
$$E_n(f) = \int_0^1 G(t) f'(t)\, dt$$
   where
$$G(t) = \frac{t_{j-1} + t_j}{2} - t \quad \text{for} \quad t_{j-1} \leq t \leq t_j, \; j = 1, \ldots, n.$$
   (b) Apply the result of (a) to $f(x) = x^\alpha$ for $0 < \alpha < 1$ to obtain a rate of convergence.

3.
   (a) Define the explicitly shifted QR algorithm for finding the eigenvalues of a matrix A.
   (b) Show that the explicitly shifted QR algorithm produces a sequence of matrices A_k which are similar to A.
   (c) Assume A is upper Hessenberg. Show how to impliment the explicitly shifted QR algorithm by using plane rotations.

(d) Consider one step of the explicitly shifted QR algorithm applied to A_k with shift α_{nn} (the (n,n) element of A_k). After premultiplication by the first $n-2$ rotations we arrive at a matrix of the form

$$\begin{pmatrix} * & * & * & \cdots & * & * & * \\ 0 & * & * & \cdots & * & * & * \\ 0 & 0 & * & \cdots & * & * & * \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & * & * & * \\ 0 & 0 & 0 & \cdots & 0 & a & b \\ 0 & 0 & 0 & \cdots & 0 & \epsilon & 0 \end{pmatrix}$$

where ϵ will be small if the algorithm is converging. Suppose $a >> \epsilon$. Show that after the complete QR step, the $(n, n-1)$ element of A_{k+1} will be given by

$$\frac{-\epsilon^2 b}{a^2 + \epsilon^2}.$$

Conclude that the explicitly shifted QR algorithm will tend to converge quadratically.

MAPL 667 QUALIFYING EXAM (SPRING 1999)

INSTRUCTIONS: Results used, but not proved, must be stated clearly, indicating the assumptions under which they are valid.

1. Consider the minimization problem $\min_{(x,y)\in\mathbf{R}^2} f(x,y)$ where

$$f(x,y) = \sin^2(x-y) + \frac{\alpha}{2}x^2.$$

(a) Write the algorithm for one step of a Newton method with backtracking for $\alpha > 0$.
(b) Find the rate of convergence to the absolute minimizer $(0,0)$ for $\alpha > 0$.
(c) Would backtracking be sufficient to make the method of (a) converge locally to $(0,0)$ if $\alpha = 0$? Explain.

2. Let $\mathbf{A} \in \mathbf{R}^{2\times 2}$ be a diagonalizable matrix with real eigenvalues $\lambda_1, \lambda_2 < 0$. Let $\mathbf{x}(t) \in \mathbf{R}^2$ be a solution to the linear 1st order system of ODEs

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

(a) Write the equations for one step of the Forward Euler Method, the Backward Euler Method and the Trapezoid Method for this system. Indicate which methods are explicit and which are implicit, and why.
(b) Find suitable restrictions on the time-step h in terms of λ_1, λ_2 for each method to be absolutely stable. To this end, introduce a change of variables $\mathbf{x} = \mathbf{P}\mathbf{y}$, where $\mathbf{P}$ is the matrix that diagonalizes $\mathbf{A}$, and obtain an ODE system for $\mathbf{y}$.
(c) Consider the following 2nd order linear ODE with (damping) parameter $\alpha \geq 1$:

$$y'' + 2\alpha y' + y = 0.$$

Convert this ODE into a 1st order system for $\mathbf{x} = (y, y')^T$, and explain why the implicit methods of (a) are preferable for large α.

3. Consider the following boundary value problem on $(0,1)$

$$-u'' + u' = 1, \qquad u(0) = 1, \quad u'(1) = -1.$$

(a) Write a variational formulation. Indicate clearly the bilinear form, forcing term, and functional spaces involved.
(b) Write a C^0 piecewise linear finite element approximation over a uniform partition of size $h = 1/N$. If $\mathbf{U} = (U_n)_{n=0}^N$ are the nodal values of the discrete solution, derive the n-th equation of the resulting linear system for $1 \leq n < N$ and $n = N$.
(c) Define and compute the truncation error for $1 \leq n < N$ for the finite element approximation of (b). Explain what the error in L^∞ between u and the finite element solution would be if the order of the truncation error for $n = N$ were the same as for $n < N$ (Note: you are not supposed to deal explicitly with $n = N$). Hint: proceed as you would with the n-th equation of the finite difference method.