DEPARTMENT OF MATHEMATICS UNIVERSITY OF MARYLAND
WRITTEN QUALIFYING EXAMINATION

ORDINARY DIFFERENTIAL EQUATIONS (PH. D. VERSION)

Instructions. Answer each the following six questions. Use a different answer sheet (or a different set of sheets) for each question. Write the problem number and your code number (not your name) on the top of each answer sheet. Keep scratch work on separate sheets.

Your work on each question will be assigned a grade from 0 to 10. Some problems have multiple parts or ask you to do more than one task. Be sure to go on to subsequent parts even if there is some part that you cannot do. Parts of a question need not have the same weight.

Carefully show all your steps, justify all your assertions, state precisely any definitions and theorems that you use, and explain your arguments in complete English sentences. Cross out any material that is not to be graded.

(1) Find all values of parameter A such that the equation

$$\dddot{x} + 3A\ddot{x} - 4A^2x = 0$$

admits a nontrivial periodic solution satisfying

$$x(0) = \dot{x}(0) = -\ddot{x}(0).$$

(2) Let f and g be smooth functions such that $g(t) \to 0$ as $t \to +\infty$, $f(0) = 0$, $f'(0) < 0$.

(a) Show that there exists $T, \varepsilon_0 > 0$ such that for $\tau > T, |\varepsilon| < \varepsilon_0$ the solution to the initial value problem

$$\dot{x} = f(x) + g(t), \quad x(\tau) = \varepsilon$$

is defined for all $t \geq T$.

(b) Show that there exists T such that for $\tau > T$ the solution to the initial value problem

$$\dot{x} = f(x) + g(t), \quad x(\tau) = 0$$

is Lyapunov asymptotically stable.

(3) Consider the system of equations

$$\dot{x} = a(t)x + b(t)y, \quad \dot{y} = c(t)x + d(t)y$$

where a, b, c, d are continuous functions with period 1. Suppose that $a(t) > 0$ and $d(t) > 0$. Show that for any natural number $k > 2$ the system cannot have a periodic solution $(x(t), y(t))$ with minimal period k.

(4) Let $x(t)$ be a bounded trajectory of a C^1 vector field on the plane. Assume that $x(0) \in \omega(x)$. Show that x is either fixed or periodic point.

(5) Let $A_1, A_2 \dots A_k$ be a finite set of points in $\mathbb{R}^3$. Consider the Hamiltonian system on $\mathbb{R}^3 \times (\mathbb{R}^3 - \cup_{j=1}^k A_j)$ with Hamiltonian

$$H(p,q) = \frac{1}{2}\sum_{j=1}^{3} p_j^2 + U(q)$$

where U is smooth on $\mathbb{R}^3 - \cup_{j=1}^k A_j$. Suppose that U is bounded at infinity that is there exists $R > 0$ such that for $|q| \geq R$ we have $|U(q)| \leq 1$.

(a) Show that solutions can not escape to infinity in finite time. That is if $0 < T < +\infty$ and $(p(t), q(t))$ is a solution defined on $[0, T)$ then

$$\sup_{t \in [0,T)} |q(t)| < +\infty.$$

(b) Let $[0, T)$ be the maximal forward interval of existence for some trajectory $(p(t), q(t))$ where $T < +\infty$. Show that $\lim_{t \uparrow T} q(t)$ exists and is equal to A_j for some $1 \leq j \leq k$.

(6) Consider the equation $\ddot{x} + x = f(x,t)$ where f is a smooth bounded function. Let $x(t)$ be a solution such that

$$x^2(t) + (\dot{x})^2(t) \to +\infty \text{ as } t \to +\infty$$

Let $N(T)$ be the number of isolated zeroes of x on the interval $[0, T]$. Show that $\lim_{T \to +\infty} \frac{N(T)}{T}$ exists and compute it.

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(1) Consider the initial value problem

$$\dot{x} = x^3 - e^{t^2}x^2, \quad x(0) = \xi$$

Show that there exists $\xi > 0$ such that the maximal solution to this initial value problem is not defined for all $t > 0$.

(2) Let A be a constant two-by-two matrix with real coefficients such that $\mathrm{Tr} A = 0$. Show that the equation $\dot{x} = Ax$ has a continuous first integral.

(3) Consider a linear equation on $\mathbb{R}$

$$\dot{x} = a(t)x$$

where $a(t)$ is a continuous function. Suppose that for each $\varepsilon > 0$ there exists $T(\varepsilon)$ such that for each $n \in \mathbb{N}$, for each $m \in \mathbb{N}$ such that $m > T(\varepsilon)$ we have

$$|x(m+n)| \le \varepsilon |x(n)|.$$

Is it true that 0 is uniformly forward asymptotically stable? Prove it or give a counterexample.

(4) Show that for any $\varepsilon > 0$ there exist $0 < x_0 < \varepsilon, 0 < y_0 < \varepsilon$ such that the solution to the initial value problem

$$\dot{x} = (2x - 5y + x^3), \quad x(0) = x_0$$

$$\dot{y} = (x - 4y - y^3), \quad y(0) = y_0$$

stay in the first quadrant for all $t \ge 0$.

(5) Consider an area preserving vector field on the plane with isolated fixed points. Let $x(t)$ be bounded non-periodic orbit. Show that
$$\lim_{t\to+\infty} x(t)$$
exists.

Hint. Show that the ω-limit set of $x(t)$ consists of a single point.

(6) Let $x(t,\tau,\xi)$ is the solution to the initial value problem
$$\dot{x} = x\sin(x-t)+1, \quad x(\tau)=\xi.$$
Note that $x(t,0,0)=t$. Compute $\frac{\partial x}{\partial \tau}(t,0,0)$.

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Carefully show all your steps, justify all your assertions, state precisely any definitions and theorems that you use, and explain your arguments in complete English sentences Cross out any material that is not to be graded

(1) Consider a one degree of freedom Hamiltonian system with the Hamiltonian

$$H(p,q) = g(q)p^2$$

where $g(q)$ is a strictly positive C^∞ function. Find necessary and sufficient conditions on g so that all solutions are defined for all times.

(2) Find values of the parameter A such that the solution to the initial value problem

$$\ddot{x} = x + Ae^{-t}, \quad x(0) = 1, \quad \dot{x}(0) = 0$$

is bounded as $t \to +\infty$?

(3) Show that the origin is an uniformly asymptotically stable solution for the equation

$$\dot{x} = y, \quad \dot{y} = -x - y^3(1 - 5\sin t).$$

(4) Let $x(t,a)$ denote the solution to the initial value problem

$$\dot{x} = 10 - \frac{t^2-1}{t^2+1} - \sin t - x^2, \quad x(a) = 0.$$

(a) Show that $x(t,a)$ is defined for all $t \geq a$.

(b) Show that the following limit exists $\lim_{a\to-\infty} x(0,a)$.

(5) Prove that a divergence free C^∞ vector field ($\mathrm{div}(f) = 0$) in the plane does not have (nontrivial) isolated periodic orbits.

(6) Consider an equation

$$\begin{aligned} \dot{x} &= (x^2+y^2)(4y-x^2) \\ \dot{y} &= (x^2+y^2)(x+7y^3) \end{aligned}$$

(a) Show that there exists a nontrivial solution $(x(t), y(t))$ such that $\lim_{t\to+\infty}(x^2(t)+y^2(t)) = 0$.
(b) Let $(x(t), y(t))$ be any solution of part (a). Compute

$$\lim_{t\to+\infty} \frac{x(t)}{y(t)}.$$

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Carefully show all your steps, justify all your assertions, state precisely any definitions and theorems that you use, and explain your arguments in complete English sentences. Cross out any material that is not to be graded.

(1) For which values of $x(0), \dot{x}(0), \ddot{x}(0)$ and $\dddot{x}(0)$ is the solution of
$$\ddddot{x} + 2\ddot{x} + x = 0$$
bounded?

(2) Prove or disprove the following statement. All solutions to the equation $\dot{x} = x^2 \cos t - e^t x$ are defined for all times.

(3) Investigate the forward stability (that is, decide if the point is unstable, positively stable or asymptotically positively stable) of the fixed point $(1, 1)$ for the following systems of equations.
$$(a) \quad \begin{aligned} \dot{x} &= x(y^2 - y), \\ \dot{y} &= y(x - x^2) \end{aligned}$$
$$(b) \quad \begin{aligned} \dot{x} &= x(y^2 - y) - (x^{10} + y^{10} - 1)^2(x - 1), \\ \dot{y} &= y(x - x^2) - (x^{10} + y^{10} - 1)^2(y - 1). \end{aligned}$$

(4) (a) Let A be a 2×2 matrix with strictly positive elements. Prove that if $\det(A) = 1$ and if $x \in \mathbb{R}^2$ is a vector with positive components then $||A^n x|| \to \infty$.

(b) Prove that the system of equations
$$\begin{aligned} \dot{x} &= (2 + \sin t)y \\ \dot{y} &= (4 - \cos^2 t)x \end{aligned}$$
has no nontrivial periodic solutions.

(5) Let $f(x) = (f_1(x), f_2(x))$ and $g(x) = (g_1(x), g_2(x))$ be perpendicular C^1 vector fields in the plane (that is,
$$f_1(x)g_1(x) + f_2(x)g_2(x) \equiv 0).$$

Prove that if f has a (nontrivial) periodic orbit, then g has a fixed (singular) point.

(6) Let N(a,b, T) be the number of zeroes of the solution of the initial value problem

$$\ddot{x} + x\cos\left(\frac{t^4+2t^3+3t^2+4t+5}{5t^4+4t^3+3t^2+4t+1}\right) = 0 \quad x(0) = a, \dot{x}(0) = b$$

on the segment [0, T]. For $a^2+b^2 \neq 0$ compute

$$\lim_{T\to\infty} N(T)/T.$$

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
AUGUST 2008

Ordinary Differential Equations.

Instructions to the Student.

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number **(NOT YOUR NAME)** on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a well known theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

e. It is your responsibility to make sure that you understand correctly what a particular question asks you to do. If you have any questions during the exam, ask the faculty consultant.

(1) Let

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

Let $\dot{x} = Ax$ for $x \in \mathbb{R}^3$ and $x(t)$ be a solution. We say that $x(t)$ grows linearly if $\lim_{t\to+\infty} |x(t)|/t = c > 0$ and superlinearly if $\lim_{t\to+\infty} |x(t)|/t = +\infty$. Find all initial conditions $x(0)$ such that $x(t)$ are

(a) bounded;
(b) grow linearly;
(c) grow superlinearly.

(2) Consider the nonautonomous equation $\dot{x} = X(t,x)$ on $\mathbb{R}^n$. Assume that $X(t,x)$ is continuous and that there exists a continuous function $K(t)$ on $\mathbb{R}$ such that for all x_1 and $x_2 \in \mathbb{R}^n$

$$\|X(t,x_1) - X(t,x_2)\| \le K(t)\|x_1 - x_2\|.$$

Show that all solutions are defined for all times.

(3) Let $(x(\varepsilon, t), y(\varepsilon, t))$ denote the solution to the equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \tan t & \varepsilon \cos t \\ \varepsilon \sin t & \varepsilon \ln(1+t^2) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

on the time interval $[0,1]$ with initial condition

$$(x(\varepsilon, 0), y(\varepsilon, 0)) = (1, 0).$$

Compute

$$\lim_{\varepsilon \to 0} \frac{y(\varepsilon, 1)}{\varepsilon}.$$

(4) Consider the equation

$$\dot{x} = -x^3 + \frac{t}{t^4 + t^2 + 1}.$$

Show that for every solution there exists the limit

$$\lim_{t \to +\infty} x(t).$$

(You can use the fact that every solution is defined for all times without the proof).

(5) Consider an equation $\dot{x} = X(x)$ on $\mathbb{R}^n$. Which of the statements (a)–(c) below are true? If a statement is true give a proof, if a statement is false give a counter example.

(a) If $n = 1$ and $\operatorname{div}(X)(x) < 0$ for all $x \in \mathbb{R}^1$, then X has a fixed point.

(b) If $n = 2$ and $\operatorname{div}(X)(x) < 0$ for all $x \in \mathbb{R}^2$, then every periodic solution is a fixed point.

(c) If $n = 3$ and $\operatorname{div}(X)(x) < 0$ for all $x \in \mathbb{R}^3$, then every periodic solution is a fixed point.

(6) Show that for some $\mu \neq 0$.

$$\begin{cases} \dot{x} = \mu x - y + xy - xy^2 - x^3 \\ \dot{y} = x + \mu y - x^2 - y^3 \end{cases}$$

has a nontrivial periodic solution. (A solution is called non-trivial if it is not a fixed point).

ODE QUALIFYING EXAM.

Wednesday January 2008, 9 am–1pm, room 3206.

Instructions. Answer all six questions. Your answer on each question will be assigned a grade from 0 to 10. Your grade will be based on your work that is shown as well as on your answer. Carefully show all your steps, state precisely any definitions and theorems that you use, and explain your argument in complete English sentences. Cross out any material that is not to be graded. Some problems may have multiple parts or ask you to do more than one thing. Be sure to go on to subsequent parts even if there is some part you can not do. If you are asked to prove a result and then apply it to a given situation you may receive partial credit for the correct application.

(1) Consider the nonlinear equation $\dot{x} = a(t)x^3$, where $t \in [0, \infty)$, $x \in \mathbb{R}$ and $a : [0, \infty) \to \mathbb{R}$ is continuous.

(a) (3 pts) Find all solutions.

Let φ denote the zero solution.

(b) (2 pts) Show that φ is uniformly stable if $a(t) < 0$ for all t.

(c) (2 pts) Show that φ is asymptotically stable if there is a number $b > 0$ such that $a(t) < -b$ for all t.

Call the zero solution UNIFORMLY ASYMPTOTICALLY STABLE if there exists $\delta > 0$ such that for any solution $x(t)$ satisfying $x(t_0) = x_0$ where $||x_0|| \leq \delta$

$$x(t_0 + T) \to 0 \text{ as } T \to \infty$$

uniformly in $||x_0|| \leq \delta$ and $t_0 \geq 0$.

(d) (3 pts) Show that if $a(t) = -\frac{1}{t+1}$, then φ is asymptotically stable but not uniformly asymptotically stable.

(2) Determine the stability of the critical point 0 of the van der Pol equation

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0$$

for all values of $\mu \in \mathbb{R}$.

(3) Consider the equation
$$x' = f(x) + g(x),$$
where $x \in \mathbb{R}^n$, $f, g : \mathbb{R}^n \to \mathbb{R}^n$. Assume that for all $x \in \mathbb{R}^n$ $||f(x)|| \leq 1$ and for all $x_1, x_2 \in \mathbb{R}^n$
$$||f(x_1) - f(x_2)|| \leq L||x_1 - x_2||.$$
Assume also that g is continuous and for all $x \in \mathbb{R}^n$ and some $\varepsilon > 0$ $||g(x)|| \leq \varepsilon$. Show that if $x_1(t)$ and $x_2(t)$ are two solutions with $x_1(0) = x_2(0)$, then
$$||x_1(t) - x_2(t)|| \leq \frac{2\varepsilon}{L}\left(e^{Lt} - 1\right).$$

(4) Consider the system
$$x' = -x(e^x - \cos x), \quad y' = x\frac{y^2}{y+1} + e^{-t}.$$
Show that if $x(0) > 0$ and $y(0) > 0$ then $\lim_{t\to+\infty} y(t) = +\infty$.

(5) Let $x(t)$ be a non trivial solution of the equation
$$x'' = -x^3 + \frac{x^2 - 1}{x^4 + 1}x'.$$
Show that between any two zeroes of x there is a zero of x' and vice versa between any two zeroes of x' there is a zero of x.

(6) Let X be a smooth bounded vector field on the plane.

(a) (3 pts) Prove that if X has a periodic solution then it has a fixed point.

(b) (7 pts) Suppose that X has no fixed points. Prove that for any solution $x(t)$
$$\lim_{t\to+\infty} ||x(t)|| = +\infty.$$

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF MARYLAND

WRITTEN GRADUATE QUALIFYING EXAM

ORDINARY DIFFERENTIAL EQUATIONS (PH.D. VERSION)

August 2007

Instructions. Answer all six questions. Your work on each question will be assigned a grade from 0 to 10. Your grade will be based on the work that is shown as well as your answer. Carefully show all your steps, justify all your assertions, state precisely any definitions and theorems that you use, and explain your arguments in complete English sentences. Cross out any material that is not to be graded. Some problems may have multiple parts or ask you to do more than one thing. Be sure to go on to subsequent parts even if there is some part you cannot do. If you are asked to prove a result and then apply it to a given situation you may receive partial credit for the correct application.

1. Consider the system $\dot{x} = Ax$ where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

Let $x(t)$ be a solution of the above system. We are saying that $x(t)$ is *growing linearly* if $\lim_{t\to\infty} \frac{|x(t)|}{t} = c > 0$, and $x(t)$ is *growing faster than linearly* if $\lim_{t\to\infty} \frac{|x(t)|}{t} = \infty$.
Find all initial conditions $x(0)$ such that the respective solutions $x(t)$ are

(a) bounded

(b) growing linearly

(c) growing faster than linearly

2. (a) Consider a system

$$\dot{x} = f(t, x) \tag{1}$$

where $f(t,x)$ is a locally Lipschitz function defined in $(0,\infty) \times \mathbf{R}^n$.
Let us denote $x(t,\tau,\xi)$ a solution of (1) satisfying initial condition $x(\tau) = \xi$.
Let $\phi(t)$ be a solution of (1) defined for $0 < t < \infty$, and let β be a positive constant.
Recall that $\phi(t)$ is uniformly stable on $[\beta,\infty)$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any $\tau \geq \beta$ if ξ satisfies $|\xi - \phi(\tau)| < \delta$ then $x(t,\tau,\xi)$ is defined on $[\tau,\infty)$ and $|x(t,\tau,\xi) - \phi(t)| < \epsilon$ for all $t \geq \tau$.
Let α be another positive constant, $\alpha \neq \beta$. Prove that $\phi(t)$ is uniformly stable on $[\alpha,\infty)$ if and only if $\phi(t)$ is uniformly stable on $[\beta,\infty)$.

(b) Consider $\dot{x} = A(t)x$ where $A(t)$ is an $n\times n$ matrix which continuously depends on $t \in (0,\infty)$ and satisfies $\int_1^\infty |A(t)| < \infty$.
Prove that the zero solution is uniformly stable.

3. Suppose the IVP $\dot{x} = f(t,x), \quad x(t_0) = x_0$, where $f(t,x)$ is a continuous function defined in $\mathbf{R}^2$, has two different solutions $\phi(t)$ and $\psi(t)$ such that $\phi(t_1) > \psi(t_1)$.
   Prove that for any c satisfying $\phi(t_1) > c > \psi(t_1)$ the IVP has a solution $x(t)$ satisfying $x(t_1) = c$.

4. Let $A = \{(x,y), \quad 1/2 \le x^2 + y^2 \le 2\}$, and let

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned} \tag{2}$$

   be a C^1 system defined in some open domain $D \subset \mathbf{R}^2$ containing A. Suppose that for initial conditions $(x,y) \in A$ respective solutions $(x(t), y(t))$ remain in A for $t \ge 0$ and there exists $c > 0$ such that $x\dot{y} - y\dot{x} \ge c$.
   Let $P \subset A$ be the set of points $(x,y) \in A$ which belong to periodic orbits of the system (2).

   (a) Prove that P is a nonempty closed subset of A.
   (b) Provide examples such that
      i. $P = A$
      ii. P consists of a single orbit
      iii. P contains infinitely many orbits, $P \neq A$

5. Consider the two-dimensional system:

$$\begin{aligned} \dot{x} &= px + y - 3x + x^2 + y^2 \\ \dot{y} &= py + x - 3y + 2xy \end{aligned}$$

   (a) Find the real values of p such that the origin is a hyperbolic fixed point. For each such p determine the type and stability of the origin.
   (b) For those values of p such that the origin is *not* a hyperbolic fixed point, sketch the global phase portrait of the system.

6. Consider the boundary value problem

$$\ddot{u} + (\lambda + 2t^2(1-t^2))u = 0, \quad u(0) = u(1) = 0$$

   (a) Prove that there is $\lambda_0 > 0$ such that for $\lambda < \lambda_0$ there are only trivial solutions.
   (b) For how many values of $\lambda \le 101$ does this boundary value problem have a nontrivial solution ?

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF MARYLAND

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January 2007

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1. Consider a C^1 system $\dot{x} = f(x)$ in $\mathbf{R}^n$. Let $x(t)$ and $y(t)$ be solutions defined for $t \in \mathbf{R}$, and let c be a positive constant. Assume that $x(t_k) \to y(0)$ for an increasing sequence $t_k \to \infty$ such that $t_{k+1} - t_k < c$ for all k.
   Prove that $y(t)$ is a constant or a periodic solution.

2. Let $A(t)$ be a continuous $n \times n$ matrix with period T. Suppose the system
   $$\dot{y} = A(t)y \tag{1}$$
   has no periodic solutions of period T and that there are no non-zero constant solutions.
   Let $g(t) \in \mathbf{R}^n$ be any continuous vector with period T. Prove that the system
   $$\dot{y} = A(t)y + g(t) \tag{2}$$
   has a periodic (possibly constant) solution of period T.

3. Consider the differential equation
   $$\ddot{x} + \frac{1}{x^2} - \frac{1}{x^3} = 0$$
   for $x > 0$.
   (a) Sketch the phase portrait.
   (b) Identify all equilibrium solutions and all periodic solutions and determine their stability.
   (c) Let $(x(t), y(t))$ be a periodic solution. Suppose $y(t_0) = y(t_1) = 0$, $x_0 = x(t_0) < x_1 = x(t_1)$ and $y(t) \neq 0$ for $t_0 < t < t_1$.
   For $t_0 \leq t \leq t_1$ express t as a function of x. Then express the period of $(x(t), y(t))$ as a function of x_0 and x_1.
   (d) For nonperiodic solutions describe their asymptotic behav-

4. Consider the two-dimensional system:

$$\begin{aligned}\dot{x} &= -y + x(1 - \sqrt{x^2+y^2}) \\ \dot{y} &= x + y(1 - \sqrt{x^2+y^2})\end{aligned}$$

   (a) Show that there is a periodic orbit, which intersects the x axis at $(1, 0)$.

   (b) Find explicitly the Poincare return map on the positive x axis .

   (c) Determine the stability of the periodic orbit.

5. Prove the following particular case of the Sturm Comparison Theorem.
   Let $\phi(x)$ be a solution of

$$\ddot{y} + g_1(x)y = 0$$

   and let $\psi(x)$ be a solution of

$$\ddot{y} + g_2(x)y = 0$$

   where $g_1(x),\ \ g_2(x)$ are continuous on (a, b). Assume $g_2(x) > g_1(x)$ on (a, b). If $x_1 \in (a, b),\ \ x_2 \in (a, b)$ are two consecutive zeros of $\phi(x)$, then $\psi(x_3) = 0$ at some point $x_3 \in (x_1, x_2)$.

6. Consider a differential equation $\dot{x} = f(t, x)$ where $f : \mathbf{R}^2 \to \mathbf{R}$ is C^1 and $\mid \frac{\partial f}{\partial x} \mid$ is bounded. Suppose $\phi(t)$ is a solution defined for $t \in [0, 1]$, and let $t_0 = 0$, $\xi_0 = \phi(0)$.
   Let (t_i, ξ_i), $i = 0, 1, \ldots n$ be the sequence of Euler's approximations

$$t_i = t_{i-1} + h, \quad t_n = 1$$

$$\xi_i = \xi_{i-1} + hf(t_{i-1}, \xi_{i-1})$$

   where $h = \frac{1}{n}$ is the step size. Prove that there exists a constant c independent of h such that for all $i = 0, 1 \ldots, n$

$$|\phi(t_i) - \xi_i| < ch$$

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF MARYLAND

WRITTEN GRADUATE QUALIFYING EXAM

ORDINARY DIFFERENTIAL EQUATIONS (PH.D. VERSION)

August 2006

Instructions. Answer all six questions. Your work on each question will be assigned a grade from 0 to 10. Carefully show all your steps, justify all your assertions, state precisely any definitions and theorems that you use, and explain your arguments in complete English sentences. Cross out any material that is not to be graded. Some problems may have multiple parts or ask you to do more than one thing. Be sure to go on to subsequent parts even if there is some part you cannot do.

1. Prove that for each real number c, the initial value problem

$$\frac{dx}{dt} = 1 + x^{1/3}, \qquad x(0) = c$$

has a unique solution defined for all real t.

2. Let $\mathbf{h} : \mathbf{R}^n \to \mathbf{R}$ be continuous and let A be an $n \times n$ real matrix. Assume that all solutions of $\dot{\mathbf{x}} = A\mathbf{x}$ are bounded for $t \geq 0$, and that

$$\int_0^\infty \|\mathbf{h}(t)\| dt < \infty,$$

where $\|\mathbf{v}\|$ denotes the Euclidean norm of a vector $\mathbf{v}$. Prove that all solutions of $\dot{\mathbf{y}} = A\mathbf{y} + \mathbf{h}(t)$ are bounded for $t \geq 0$.

3. Consider the initial value problem

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} c \\ d \end{pmatrix}.$$

For which values of the real constants a, b, c, d is $\mathbf{x}(t)$ bounded for $t \geq 0$?

4. Let $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^n$ be C^1, and assume that the divergence of $\mathbf{f}$ is identically zero: $\nabla \cdot \mathbf{f}(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbf{R}^n$. Recall that this condition implies that the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is volume preserving. Prove that this system cannot have an asymptotically stable fixed point.

5. Prove that the differential equation

$$\ddot{x} + (2x^2 + \dot{x}^2 - 1)\dot{x} + x = 0$$

has a nonconstant periodic solution.

6. The origin is a fixed point of the system

$$\begin{aligned} \dot{x} &= 2x + z \\ \dot{y} &= -y + z^2 \\ \dot{z} &= z. \end{aligned}$$

Find the global stable and unstable manifolds of the origin, and for each manifold, identify its dimension and its tangent line or plane at the origin.

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January 2006

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1. Consider the differential equation $\dot{x} = Ax$ where

$$A = \begin{pmatrix} -2 & 0 & 2 \\ 0 & -2 & 2 \\ -1 & -1 & 2 \end{pmatrix}$$

   (a) Find the fundamental matrix solution $\Phi(t)$ satisfying $\dot{\Phi} = A\Phi$, $\Phi(0) = I$.

   (b) Find all x_0 such that solutions $x(t)$ satisfying $x(0) = x_0$ are bounded for $t \geq 0$.

2. Consider the two-dimensional system

$$\begin{aligned} \dot{x} &= -y + xg(x,y)(-2 + \cosh\sqrt{x^2+y^2}) \\ \dot{y} &= x + yh(x,y)(-2 + \cosh\sqrt{x^2+y^2}) \end{aligned}$$

   where $g(x,y)$, $h(x,y)$ are C^1 functions. Let $\phi(t)$ be the solution satisfying $\phi(0) = (1,0)$. Prove that $\phi(t)$ is defined for all $t \in \mathbf{R}$.

3. Consider the two-dimensional system:

$$\begin{aligned} \dot{x} &= -\tfrac{1}{2}x - (p - \tfrac{1}{2})y + 2xy \\ \dot{y} &= -(p - \tfrac{1}{2})x - \tfrac{1}{2}y + x^2 + y^2 \end{aligned}$$

   Determine all values of the parameter p such that the qualitative nature of the solutions near the origin changes as the parameter passes through those values. In each interval between the bifurcation values you found, and at the bifurcation values, sketch and describe the qualitative behavior of the solutions near the origin.

4. Let $\dot{x} = f(x)$ and $\dot{y} = g(y)$ be two C^1 systems in $\mathbf{R}^n$ such that all solutions are defined for $t \in \mathbf{R}$. Let ϕ_t be the time t map of the first system; that is,
$$\phi_t(x(0)) = x(t)$$
for every t and every solution $x(t)$. Let ψ_t similarly be the time t map of the second system:
$$\psi_t(y(0)) = y(t).$$
The two systems are called *differentiably equivalent* if there is a diffeomorphism $h : \mathbf{R}^n \to \mathbf{R}^n$ satisfying
$$h \circ \phi_t = \psi_t \circ h.$$
They are called *linearly equivalent* if there is a an h satisfying the above equation such that $h(x) = Mx$ for some invertible matrix M. Prove that two linear systems with constant coefficients $\dot{x} = Ax$ and $\dot{y} = By$ are differentiably equivalent, if and only if they are linearly equivalent.

5. Prove that every solution $x(t)$ of
$$\ddot{x} + e^{\cos x}\dot{x} + \sin x = 0$$
satisfies
$$\lim_{t\to\infty} x(t) = n\pi$$
for some integer n.

6. Consider the second order ODE
$$\ddot{y} + (a^2 + bp(t))y = 0$$
where $p(t)$ is a continuous real function of period π, and a, b are real constants. Fix a value of a that is not an integer. Prove that if $|b|$ is sufficiently small, then all solutions of the ODE above are bounded.

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1. Let $f : \mathbf{R} \to \mathbf{R}$ be an odd continuous function such that $f(x) > 0$ for $x > 0$ and
$$\int_0^1 \frac{1}{f(x)} dx = 1$$
Consider a differential equation
$$\frac{dx}{dt} = f(x) \tag{1}$$
Prove that there exists a solution ϕ_a of (1) satisfying both $\phi_a(0) = 0$ and $\phi_a(1) = a$, if and only if $|a| \leq 1$.

2. Let $\alpha(t)$, $\beta(t)$, $\gamma(t)$ be three linearly independent solutions on $(0, \infty)$ of
$$y''' + p(t)y' + q(t)y = 0$$
where $p(t)$ and $q(t)$ are continuous on $0 < t < \infty$. Suppose that
$$\lim_{t\to\infty} (|\alpha(t)\beta'(t) - \beta(t)\alpha'(t)| + |\alpha'(t)\beta''(t) - \beta'(t)\alpha''(t)| + |\alpha(t)\beta''(t) - \beta(t)\alpha''(t)|) = 0$$
Prove that
$$\lim_{t\to\infty} (|\gamma(t)| + |\gamma'(t)| + |\gamma''(t)|) = \infty$$

3. Let Ω be an open domain in $\mathbf{R}^{2n}$ and let $H : \Omega \to \mathbf{R}$ be a C^2 function. Recall that an autonomous system of differential equations satisfying for $i = 1, 2, \ldots n$
$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned} \tag{2}$$
is called Hamiltonian.
   (a) Prove that a fixed point of a Hamiltonian system cannot be asymptotically stable.
   (b) Give an explicit example of a Hamiltonian system which has both stable and unstable fixed points.

4. Prove that the system

$$\begin{aligned} \dot{x} &= x - y - x^5 \\ \dot{y} &= x + y - y^5 \end{aligned} \tag{3}$$

has a periodic non-constant solution.

5. Recall that two systems of ODE defined in $\mathbf{R}^n$ are called *topologically equivalent* if there is a homeomorphism $h : \mathbf{R}^n \to \mathbf{R}^n$ which maps orbits of one system onto orbits of another system and preserves the direction of time. Notice that it is not required to preserve parametrization of orbits by time .
Consider a family of differential equations

$$\dot{x} = f(x, \mu) \qquad (F_\mu) \tag{4}$$

in $\mathbf{R}^1$ depending on the real parameter μ.
For every real ϵ consider the ϵ-perturbation:

$$\dot{x} = f(x, \mu) + \epsilon \qquad (F_\mu(\epsilon)) \tag{5}$$

Call the family (F_μ) *stable* if there is an $\epsilon_0 > 0$ such that for each ϵ satisfying $|\epsilon| < \epsilon_0$ there exists a continuous map $p_\epsilon : \mu \to p_\epsilon(\mu)$ such that for every μ the system $F_\mu(\epsilon)$ is topologically equivalent to $F_{p_\epsilon(\mu)}$.
Consider:

$$\dot{x} = -x^2 + \mu x + k \qquad (F_\mu^k) \tag{6}$$

(a) Prove: The family (F_μ^0) is not stable.

(b) Prove: If $k \neq 0$ is fixed, then the family (F_μ^k) is stable.

6. Let $\dot{x} = f(x)$, $x \in \mathbf{R}^n$ where $f(x)$ is C^1 and every solution is defined for $t \in \mathbf{R}$. Let $(x, t) \mapsto \phi_t(x)$ be the respective flow: $\phi_0(x) = x$ for all x and, for each fixed x, the function $t \mapsto \phi_t(x)$ is a solution to the differential equation.

A non-empty set $A \subset \mathbf{R}^n$ is called *minimal* if for each $x \in A$ the orbit closure $\overline{\mathcal{O}}(x)$ coincides with A.

A point y is called *almost periodic* if for each $\epsilon > 0$ there exists $T > 0$ such that any time interval of length T contains at least one time τ satisfying $|\phi_\tau(y) - y| < \epsilon$.

Assume the orbit closure $\overline{\mathcal{O}}(x)$ is compact.

(a) Prove : If $\overline{\mathcal{O}}(x)$ is minimal, then x is almost periodic.

(b) Prove : If x is almost periodic, then $\overline{\mathcal{O}}(x)$ is minimal.

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January 2005

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1. Let $f : [0,\infty) \to \mathbf{R}$ be continuous. Prove that for every x_0 the solution of the initial value problem

$$x' = f(t) - x^5, \quad x(0) = x_0$$

is defined for $0 \le t < \infty$.

2. Consider the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + y(x^2+y^2-1) \\ \dot{z} &= z(x^2+y^2-1) \end{aligned} \tag{1}$$

Find the ω-limit set for all initial conditions in $\mathbf{R}^3$.

3. Solve the initial value problem $\dot{x} = Ax$, $x(0) = x_0$ where

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 2 & 0 & 1 & 0 \end{pmatrix}$$

and determine all x_0 such that solution is bounded.

4. Consider the system

$$\begin{aligned} \dot{x} &= -x \\ \dot{y} &= -y + x^2 \\ \dot{z} &= z - 30y^2 \end{aligned} \tag{2}$$

Find the global stable and the global unstable invariant manifolds for the fixed point at the origin.

5. (a) Show that any solution of

$$\ddot{x} + t^5 x = 0$$

has an infinite number of zeroes on the interval $[0, \infty)$.

(b) Consider the Sturm-Liouville problem

$$\ddot{x} + \lambda t^5 x = 0, \quad x(0) = x(1) = 0$$

Show that there are no negative λ for which this problem has a nontrivial solution, but there are infinitely many positive values of λ, for which this problem has a nontrivial solution.

6. Let $\dot{x} = f(t, x), \quad x \in \mathbf{R}^n$ be a system such that every solution exists for $t \geq 0$ and is unique.
Recall that a solution $\phi(t)$ is called Lyapunov stable on $[0, \infty)$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any solution $\psi(t)$ of our system, if $|\psi(0) - \phi(0)| < \delta$, then $|\phi(t) - \psi(t)| < \epsilon$ for all $t \geq 0$.
A solution $\phi(t)$ is called uniformly stable on $[0, \infty)$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any $t_0 \geq 0$ any solution $\psi(t)$ of our system satisfying $|\psi(t_0) - \phi(t_0)| < \delta$, satisfies $|\phi(t) - \psi(t)| < \epsilon$ for all $t \geq t_0$.
Give an example of a system that has a Lyapunov stable solution which is not uniformly stable.

DEPARTMENT OF MATHEMATICS
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WRITTEN QUALIFYING EXAMINATION
ORDINARY DIFFERENTIAL EQUATIONS (PH. D. VERSION)

AUGUST 2004

Instructions. Answer each the following six questions. Use a different answer sheet (or a different set of sheets) for each question. Write the problem number and your *code number* (not your name) on the top of each answer sheet. Keep scratch work on separate sheets.

Your work on each question will be assigned a grade from 0 to 10. Some problems have multiple parts or ask you to do more than one task. Be sure to go on to subsequent parts even if there is some part that you cannot do. Parts of a question need not have the same weight.

Carefully show all your steps, justify all your assertions, state precisely any definitions and theorems that you use, and explain your arguments in complete English sentences. Cross out any material that is not to be graded.

1. Let a fundamental matrix $\boldsymbol{\Phi}$ for the $n \times n$ system $\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x}$ satisfy

$$|\boldsymbol{\Phi}(t)\boldsymbol{\Phi}(s)^{-1}| \leq m \quad \text{for} \quad 0 \leq s \leq t.$$

Let the $n \times n$ matrix-valued function $\boldsymbol{B}$ be continuous on $[0,\infty)$ and satisfy

$$\int_0^\infty |\boldsymbol{B}(t)|\, dt < \infty.$$

Prove that (every solution of) the differential equation

$$\dot{\boldsymbol{y}} = [\boldsymbol{A}(t) + \boldsymbol{B}(t)]\boldsymbol{y}$$

is stable on $[0,\infty)$.

2. Let $[0,\infty) \times \mathbb{R}^n \ni (t,\boldsymbol{x}) \mapsto \boldsymbol{f}(t,\boldsymbol{x}) \in \mathbb{R}^n$ be continuously differentiable with

$$|\boldsymbol{f}(t,\boldsymbol{x})| \leq g(t)h(|\boldsymbol{x}|)$$

where g is continuous on $[0,\infty)$, $g(t) \geq 0$ for all $t \geq 0$, h is continuous on $[0,\infty)$, $h(u) \geq 1$ for all $u \geq 0$, and

$$\int_0^\infty \frac{du}{h(u)} = \infty.$$

Prove that for every $\boldsymbol{\xi}$ in $\mathbb{R}^n$ the initial-value problem $\dot{\boldsymbol{x}} = \boldsymbol{f}(t,\boldsymbol{x})$, $\boldsymbol{x}(0) = \boldsymbol{\xi}$ has a solution that exists for all $t \geq 0$.

3. Find the ω-limit set for each solution (corresponding to each initial condition) of the 2×2 system

$$\dot{x} = x[1 - \sqrt{x^2 + y^2}\,] - y[1 - \sqrt{x^2 + y^2}\,]^2 - \frac{y^3}{x^2 + y^2},$$
$$\dot{y} = y[1 - \sqrt{x^2 + y^2}\,] + x[1 - \sqrt{x^2 + y^2}\,]^2 + \frac{xy^2}{x^2 + y^2}$$

on $\mathbb{R}^2$. (At the origin the quotients in these equations are replaced by their limiting values 0.)

4. Consider $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{h}(t, \boldsymbol{x})$, where $\boldsymbol{A}$ is a constant symmetric matrix with all of its eigenvalues positive, $\boldsymbol{h}$ is continuously differentiable and has period 1 in t, and $\|\boldsymbol{h}(t, \boldsymbol{x})\| \leq \sqrt{\|\boldsymbol{x}\|} + 2004$. Prove that there is a periodic solution.

5. Consider the equation

$$\tfrac{d}{dt}[p(t)\tfrac{d}{dt}x] + [aq(t) + br(t)]x = 0$$

where $p, \frac{d}{dt}p, q, r$ are continuous functions of period 1 on $\mathbb{R}$ with p everywhere positive, $p(0) = 1$, and where a and b are real parameters. Let $t \mapsto \varphi(t; a, b)$ and $\mapsto \psi(t; a, b)$ be solutions of this equation satisfying the initial conditions

$$\varphi(0; a, b) = 1, \quad \tfrac{d}{dt}\varphi(0; a, b) = 0, \quad \psi(0; a, b) = 0, \quad \tfrac{d}{dt}\psi(0; a, b) = 1.$$

The functions φ and ψ are presumed known. For parameters a and b lying in a certain region, all solutions of the differential equation are bounded, and for parameters lying in another region, there is at least one unbounded solution. Find equations, in terms of φ and ψ, for those parameters on the boundary between these two regions.

6. Find the critical (= equilibrium) points for the scalar equation

$$\ddot{x} + 2b\dot{x} + \sin(x^3) = 0$$

where b is a positive constant. The linearization of this equation about a stable critical point $(\xi, 0)$ in the $(x, \dot{x})$-plane with $\xi > 0$ has three qualitatively different kinds of phase portraits, one for small positive b, one for large positive b, and one for a single positive value of b, with these ranges depending on ξ. Give a careful sketch of such a phase portrait for a large value of b. Be sure to identify important directions with their slopes and include arrows.

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ORDINARY DIFFERENTIAL EQUATIONS (PH. D. VERSION)
JANUARY 2004

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Carefully show all your steps, justify all your assertions, state precisely any definitions and theorems that you use, and explain your arguments in complete English sentences. Cross out any material that is not to be graded.

1. Let $[0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \ni (t, \mathbf{x}, \mathbf{y}) \mapsto \mathbf{f}(t, \mathbf{x}, \mathbf{y}) \in \mathbb{R}^n$ and $[-1, 0] \ni s \mapsto \mathbf{g}(s) \in \mathbb{R}^n$ be prescribed continuously differentiable functions. Prove that there is an interval $[0, a]$, $a > 0$, and a unique continuously differentiable function $[0, a] \ni t \mapsto \mathbf{x}(t) \in \mathbb{R}^n$ satisfying the differential-delay equation

$$\frac{d\mathbf{x}}{dt}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}(t-1)) \quad \text{for} \quad t \geq 0,$$

and the "initial condition"

$$\mathbf{x}(t) = \mathbf{g}(t) \quad \text{for} \quad -1 \leq t \leq 0.$$

2. (a) Prove that all solutions of initial-value problems for the nonlinear scalar equation

$$\ddot{x} + t^3 \tanh \dot{x} + \sinh x + x^5 = 0, \quad x \in \mathbb{R},$$

with initial data given for $t = 0$, are defined for all $t \geq 0$.

(b) Prove that all solutions of initial-value problems for the nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} \sin x_1 \\ \cos x_2 \\ x_3 \sin x_2 \end{bmatrix} + \begin{bmatrix} t \\ t^2 \\ \tanh \sqrt{1+t^2} \end{bmatrix}$$

with initial data given for $t = 0$, are defined for all $t \geq 0$.

3. Let $\lambda \in [0, \infty)$. Prove that the boundary-value problem

$$\frac{d}{dt}\left[(1+t^4)\frac{dx}{dt}\right]+[\lambda^2(1+t^2)+\lambda^3 \sin^2 t]x = 0 \quad \text{for} \quad 0 < t < 1, \qquad x(0) = 0 = x(1)$$

for a real-valued function x has nontrivial solutions exactly when λ belongs to a sequence $\{\lambda_k\}$ of real numbers, $k = 0, 1, 2, \ldots$ with $0 < \lambda_0 < \lambda_1 < \cdots$, and that the solution corresponding to λ_k has exactly k zeros on $(0, 1)$.

4. Find $e^{t\mathbf{A}}$ for

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 4 \\ 0 & 1 & 0 \\ -8 & -1 & -7 \end{bmatrix}.$$

5. Show that there is a real number $\mu \neq 0$ for which the system

$$\dot{x} = \mu x + y + xy - xy^2,$$
$$\dot{y} = -x + \mu y - x^2 - y^3$$

has a nontrivial periodic solution.

6. Give a careful sketch of the phase portrait of the second-order equation

$$\ddot{x} + x\dot{x} + x(x-1)(x+1) = 0$$

showing singular points, separatrices, and typical trajectories. Put arrows on nonconstant trajectories showing the direction of motion with increasing time. The portrait should have enough detail for one to determine all the qualitatively distinct kinds of solutions.

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1. Consider the scalar differential equation

$$\dot{x} = f(t, x)$$

where f is defined and continuous everywhere and $f(t, x+h) \leq f(t, x)$ for each real t, real x, and positive h. Assume that ϕ_1 and ϕ_2 are two solutions defined on an interval (a, b) and $\phi_1(t) = \phi_2(t)$ for *some* t in (a, b). Prove: $\phi_1(s) = \phi_2(s)$ for all s in (t, b).

2. Consider the linear system

$$\epsilon\dot{x} = Ax + b,$$

where $\epsilon > 0$, A and b are constant, and $\mathrm{Re}(\lambda) < a < 0$ for each eigenvalue λ of A. Let $x(t, x_0, \epsilon)$ denote the solution with initial value x_0.

Prove that, for every $s > 0$,

$$\lim_{\epsilon \to 0^+} \sup_{t \geq s} |x(t, x_0, \epsilon) + A^{-1}b| = 0,$$

3. Let $\phi(t)$ be the solution of the initial value problem

$$\ddot{y} + \dot{y}^3 + h(y) = 0, \; y(0) = y_0, \; \dot{y}(0) = \dot{y}_0,$$

where h is a smooth odd function on R and $yh(y) > 0$ for all $y \neq 0$.

Prove that either $\lim_{t\to\infty} \phi(t) = 0$ or $\lim_{t\to\infty} |\phi(t)| = \infty$.

4. Observe that the unit circle is an orbit of the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + \epsilon(1 - x^2 - y^2)y \end{aligned}$$

By continuity, there is a $\delta > 0$ such that if $|x-1| < \delta$ then there is a least positive time $t = t(x)$ such that the solution starting at $(x, 0)$ returns to the positive x-axis, at the point $(\phi(x), 0)$ at time $t(x)$. Calculate $\phi'(1)$.

5. Consider the system, for $b \neq 0$,

$$\begin{aligned} \dot{x}_1 &= ax_1 + bx_2 - x_1(x_1^4 + 2x_2^4) \\ \dot{x}_2 &= -bx_1 + ax_2 - x_2(x_1^4 + 2x_2^4) \end{aligned}$$

   (a) Prove that for any value of a there are no equilibrium points except $(0,0)$.

   (b) Prove that for any $a < 0$ there are no periodic orbits (except $(0,0)$).

   (c) Prove that for any $a > 0$ there is a non-trivial periodic orbit (different from $(0,0)$).

6. Consider the systems (1) $\dot{x} = A(t)x$ and (2) $\dot{y} = A(t)y + h(t)$, where A and h are continuous and periodic with period $T > 0$ and $x \in \mathrm{R}^n$. Assume that all solutions of (1) are periodic with period T and let Φ be a fundamental matrix solution of (1).

   Prove that all solutions of (2) are periodic with period T if and only if $\int_0^T \Phi^{-1}(s)h(s)ds = 0$.

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January 2003

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1. Consider the equation $\ddot{y} + \dfrac{t^2}{t^2+1} y = 0$. Prove that there is a constant M such that for every solution and for every $t \geq 0$,

$$\sqrt{\left(y(t)\right)^2 + \left(\dot{y}(t)\right)^2} \leq M \sqrt{\left(y(0)\right)^2 + \left(\dot{y}(0)\right)^2}.$$

2. Consider the equation $\ddot{y} + y^2 \dot{y}^5 + P(y) = 0$, where $P(y) = y^m + a_{m-1} y^{m-1} + \cdots + a_1 y + a_0$ and m is positive and odd. Prove that for every solution $y(t)$ there is a root y^* of P such that $\lim_{t\to\infty} y(t) = y^*$.

3. Consider the system

$$\begin{cases} \dot{x}_1 &= 2x_1 - x_1^2 - x_1 x_2 \\ \dot{x}_2 &= 4x_2 - x_1 x_2 - 2x_2^2 \end{cases}$$

a) Show that every solution, which starts in the first quadrant, stays in the first quadrant.
b) Find the ω-limit set for each solution that starts in the first quadrant.

4. Consider the equation $\dot{x} = Ax + h(t)$ in $\mathbf{R}^n$, where A has n distinct eigenvalues with negative real parts and h is continuous and bounded on $\mathbf{R}$.
a) Prove that there is a unique solution bounded on $(-\infty, \infty)$.
b) Prove that if in addition $\|h(t)\| \to 0$ as $t \to \infty$, then every solution tends to 0 as $t \to \infty$.

5. Prove that the system

$$\begin{cases} \dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2)\cos^2(x_1^2 + x_2^2) \\ \dot{x}_2 &= -x_1 + x_2 - x_2(x_1^2 + x_2^2)\cos^2(x_1^2 + x_2^2) \end{cases}$$

has infinitely many geometrically different periodic solutions.

6. Let $\mathcal{S}$ be a surface in $\mathbf{R}^3$ given by $x_3 = G(x_1, x_2)$ for all $(x_1, x_2) \in \mathbf{R}^2$, and suppose that G is C^1. Let $f(x)$ be a C^1 vector field in $\mathbf{R}^3$ that is tangent to $\mathcal{S}$ at every point of $\mathcal{S}$. Let ϕ be a solution to $\dot{x} = f(x)$. Prove: If $\phi(0) \in \mathcal{S}$ then $\phi(t) \in \mathcal{S}$ for all t.

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF MARYLAND

WRITTEN GRADUATE QUALIFYING EXAM

ORDINARY DIFFERENTIAL EQUATIONS (PH.D. VERSION)

August 2002

Instructions: Answer all six questions. Explain your answers fully. If you cite a theorem, explicitly explain why it applies.

1. Consider the differential equation $\dot{x} = Ax$, where

$$A = \begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & 2 & -5 \\ 1 & 0 & 1 & -2 \end{pmatrix}.$$

Find all real (a, b) such that every solution is bounded on $[0, \infty)$.

2. Consider, for $t > 0$, the equation

$$\ddot{x} + \frac{\lambda}{t^{2002}} x = 0.$$

(a) Prove that for any interval $(a, b) \subset (0, \infty)$ there exists λ such that if $\phi(t)$ is a solution of the above equation, then there exists $t_0 \in (a, b)$ satisfying $\phi(t_0) = 0$.

(b) Prove that if $\lambda < 0$, then any solution $\phi(t)$ which is not identically zero has no more than one root.

3. Consider the two-dimensional system:

$$X' = (A + B(t))X$$

where

$$A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix},$$

$$B(t) = \begin{pmatrix} e^{-t^2} & \frac{1}{1+t^2} \\ \frac{1}{1+t^{2002}} & e^{-t} \end{pmatrix}.$$

Prove that $\lim_{t \to \infty} X(t) = 0$ for every solution $X(t)$.

4. Consider the system:

$$
\begin{aligned}
\dot{x} &= x - x\sqrt{x^2+y^2} - y\sqrt{x^2+y^2} + xy, \\
\dot{y} &= y - y\sqrt{x^2+y^2} + x\sqrt{x^2+y^2} - x^2.
\end{aligned}
$$

Observe that the positive x-axis is invariant.
Find the ω-limit set $\omega(p)$ for every point p in the plane.

5. (a) Check that $(x = 3\cos 3t,\ y = \sin 3t)$ is a periodic solution of the system

$$
\begin{aligned}
\dot{x} &= -9y + x(1 - \tfrac{x^2}{9} - y^2), \\
\dot{y} &= x + y(1 - \tfrac{x^2}{9} - y^2).
\end{aligned}
$$

(b) Find the derivative (eigenvalue) of the Poincaré first return map to the real axis at $(3, 0)$.

6. Consider the two-dimensional system:

$$
\begin{aligned}
\dot{x} &= px + y - 2x + x^2 + y^2, \\
\dot{y} &= py + x - 2y + 2xy.
\end{aligned}
$$

(a) Find those real values of p such that the origin is a hyperbolic fixed point. For each such p, determine the type and stability of the origin.

(b) For those values of p such that the origin is *not* a hyperbolic fixed point, sketch the global phase portrait of the system.

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF MARYLAND

WRITTEN GRADUATE QUALIFYING EXAM

ORDINARY DIFFERENTIAL EQUATIONS (PH.D. VERSION)

August 2000

Instructions Answer all six questions. Your work on each question will be assigned a grade from 0 to 10. Your grade will be based on the work that is shown as well as your answer. Some problems have multiple parts or ask you to do more than one thing. Be sure to go on to subsequent parts even if there is some part you cannot do. If you are asked to prove a result and then apply it to a given situation you may receive partial credit for the correct application.

1. Consider the Differential Equation

$n = 1$

$$\dot{x} = f(x, t)$$

Suppose that $f : \mathrm{R}^n \times \mathrm{R} \to \mathrm{R}$ is a C^1 function. Suppose that $\phi : (a, b) \to \mathrm{R}$ is a solution to the DE that is bounded on the finite interval (a, b).

   (a) Prove that ϕ is uniformly continuous on (a, b) and that therefore $L = \lim_{t \to b^-} \phi(t)$ exists.

   (b) Prove that there is a number $c > b$ and a solution ψ to the DE defined on (a, c) such that $\psi(t) = \phi(t)$ for all t in the interval (a, b).

   (c) Suppose that a solution θ of the DE is defined on a finite interval (u, v) and that θ cannot be extended to a solution on any interval (u, w) with $w > v$. Prove that $\|\theta\| \to \infty$ as $t \to v^-$.

2. Consider the two dimensional system

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y). \end{aligned}$$

Suppose that f and g are C^1 throughout the xy-plane, that $f(0,0) = g(0,0) = 0$ and that $\begin{pmatrix} f_x(0,0) & f_y(0,0) \\ g_x(0,0) & g_y(0,0) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ Provide explicit examples of functions f and g satisfying the above conditions and such that the stated additional property holds. You do not need to prove that the system has the stated properties.

   (a) Every solution converges to 0 as $t \to \infty$

   (b) Every non-constant solution diverges to ∞ as $t \to \infty$.

   (c) There is a unique periodic solution (lying on a simple closed curve γ) and each non-constant solution has γ as its ω-limit set.

3. Consider the Initial Value Problem:

$$\dot{x} = f(x,t), \quad x(0) = x_0$$

Suppose that $f : \mathrm{R}^n \times \mathrm{R} \to \mathrm{R}^n$ is a C^1 function and suppose there is a continuous function $M : \mathrm{R} \to \mathrm{R}$ such that $\|f(x,t)\| \leq M(t)\,\|x\|$ for all $(x,t) \in \mathrm{R}^n \times \mathrm{R}$.
Prove: For every $x_0 \in \mathrm{R}^n$ there is a solution to the IVP that is defined on $(-\infty, \infty)$.

4. Consider the two-dimensional system:

$$\begin{aligned} \dot{x} &= 3x \\ \dot{y} &= 5x - 2y + 8x^2 \end{aligned}$$

(a) Solve this system assuming that $x(0) = A$ and $y(0) = B$.

(b) For which choices of A and B does the solution lie on the stable manifold for the critical point $(0,0)$?

(c) For which choices of A and B does the solution lie on the unstable manifold for the critical point $(0,0)$?

(d) Find an equation in x and y whose graph is this unstable manifold.

5. Consider the Differential Equation $\dot{x} = f(x)$ where $f : \mathrm{R}^n \to \mathrm{R}^n$ is C^1 Suppose that $V : \mathrm{R}^n \to \mathrm{R}^1$ is continuously differentiable and suppose that $\nabla V(x) \cdot f(x) \leq 0$ for every $x \in \mathrm{R}^n$. Prove that V is constant on the ω-limit set of any solution to the DE.

6. Consider the Initial Value Problem:

$$\ddot{x} + \frac{1}{x^2} - \frac{1}{x^3} = 0, \quad x(0) = x_0 > 0, \quad \dot{x}(0) = v_0.$$

(a) Find an algebraic expression $E(x,v)$, $(v = \dot{x})$, and a constant C_0 such that the solution to the IVP is bounded if and only if $E(x_0, v_0) < C_0$.

(b) Sketch the phase portrait in the right-half $(x > 0)$ of the xv phase plane and indicate the regions corresponding to $E(x,v) < C_0$, $E(x,v) = C_0$, and $E(x,v) > C_0$.

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF MARYLAND

WRITTEN GRADUATE QUALIFYING EXAM

ORDINARY DIFFERENTIAL EQUATIONS (PH.D. VERSION)

January 2000

Instructions

Answer all six questions. Your work on each question will be assigned a grade from 0 to 10. Some problems have multiple parts or ask you to do more than one thing. Be sure to go on to subsequent parts even if there is some part you cannot do. If you are asked to prove a result and then apply it to a given situation you may receive partial credit for the correct application.

Notation: $\|\cdot\|$ is the Euclidean norm.

1. Consider the initial value problem
$$\left\{\begin{array}{rcl} \dot{x} & = & x^2 + y\sin x \\ \dot{y} & = & -1 + xy + \cos y \end{array}\right\}, \quad x(0) = x_0 > 0,\ y(0) = y_0 > 0.$$
Prove that the solution to this IVP lies entirely in the first quadrant.

2. Suppose that $f : R^n \to R^n$ is C^1 with $\|f(x)\| \le M\|x\|$ for some constant M. Note that this implies $f(0) = 0$ and that every solution to $\dot{x} = f(x)$ is defined on $(-\infty, \infty)$. Suppose $V : R^n \to R$ is C^1 and that its gradient ∇V satisfies $\nabla V(x) \cdot f(x) \ne 0$ for all $x \ne 0$.
Prove that a solution ϕ to $\dot{x} = f(x)$ that is bounded on $[0, \infty)$ satisfies $\lim\limits_{t\to\infty} \phi(t) = 0$

3. Suppose that $f : R^n \to R^n$ is C^1 and that $T : R^n \to R^n$ is a linear map satisfying $T^2 = I$ (the identity map) and $f(Tx) \equiv -Tf(x)$. Assume for convenience that all solutions are defined for $-\infty < t < \infty$.

   (a) Prove: If $\phi(t)$ is a solution to $\dot{x} = f(x)$ so is $\psi(t)$ defined by $\psi(t) = T(\phi(-t))$.

   (b) Prove: If $\phi(t)$ is a solution satisfying $T(\phi(t_1)) = \phi(t_1)$ and $T(\phi(t_2)) = \phi(t_2)$ for some $t_1 < t_2$ then $\phi(t)$ is periodic (or constant).

4. Suppose that $f : R^n \to R^n$ is C^1 and there is a real constant M such that $f(x) \cdot x \le M\|x\|^2$ for all x.

   (a) Prove that every solution ϕ to $\dot{x} = f(x)$ satisfies $\|\phi(t)\| \le \|\phi(0)\| e^{Mt}$ for all $t \ge 0$.

   (b) Show that every solution ϕ to $\left\{\begin{array}{rcl} \dot{x} & = & 8x + 2y + y^2 \\ \dot{y} & = & 2x + 5y - xy \end{array}\right\}$ satisfies $\|\phi(t)\| \le \|\phi(0)\| e^{9t}$ for all $t \ge 0$

5. Consider the system S: $\left\{ \begin{array}{rcl} \dot{x} & = & x - y - x^3 \\ \dot{y} & = & x + y - y^3 \end{array} \right\}$.

(a) Determine the nature of the critical point $(0,0)$.

(b) Either by considering the graphs of the equations $\left\{ \begin{array}{rcl} y & = & x - x^3 \\ x & = & y^3 - y \end{array} \right\}$ or by a method of your choice, show that $(0,0)$ is the only critical point of the system.

(c) Show that for every x_0, the solution ϕ to S with $\phi(0) = x_0$ is bounded on the interval $[0, \infty)$.

(d) Show that there is a (non-constant) periodic solution to S.

6. Consider the parameterized system S : $\left\{ \begin{array}{rcl} \dot{x} & = & -y + x(\mu - x^2 - y^2) \\ \dot{y} & = & x + y(\mu - x^2 - y^2) \end{array} \right\}$.

Determine the changes in the qualitative features of the phase portrait of all solutions to the system as μ passes from positive to negative values. Consider the 3 cases: $\mu < 0$, $\mu > 0$, $\mu = 0$.

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF MARYLAND

WRITTEN GRADUATE QUALIFYING EXAM

ORDINARY DIFFERENTIAL EQUATIONS (PH.D. VERSION-3)

August 1999

Instructions

- Your work on each question will be assigned a grade from 0 to 10. Some problems have multiple parts or ask you to do more than one thing. Be sure to go on to subsequent parts even if there is some part you cannot do.
- Use a different sheet (or different set of sheets) for each question. Write the problem number and your *code number* (not your name) on the top of every sheet.
- Keep scratch work on separate sheets.
- Unless otherwise stated, you may appeal to a "well-known theorem" in your solution to a problem. However, it is your responsibility to make it clear exactly which theorem you are using and to justify its use.

1. Prove the following *Gronwall inequality* .

   Let K be a nonnegative constant and let f and g be continuous nonnegative functions on some interval $a \leq t \leq b$ satisfying the inequality

$$f(t) \leq K + \int_a^t f(s)g(s)ds \qquad \text{for } a \leq t \leq b. \tag{I}$$

   Then

$$f(t) \leq K \exp\left(\int_a^t g(s)ds\right) \qquad \text{for } a \leq t \leq b. \tag{II}$$

2. Give two explicit examples of parametrized C^1 systems

$$\begin{aligned} \dot{x} &= f(x, y, \mu) \\ \dot{y} &= g(x, y, \mu) \end{aligned} \tag{III}$$

   with the following properties .

   (a) In both examples the origin is a stable critical point for $\mu < 0$ and it is an unstable critical point for $\mu > 0$.

   (b) In example 1 the system has no periodic orbits for $\mu \leq 0$ and for every $\mu > 0$ there is one stable periodic orbit.
   Draw the bifurcation diagram of Example 1.

   (c) In example 2 the system has no periodic orbits for $\mu > 0$, no periodic orbits for $\mu < 0$, but there are periodic orbits for $\mu = 0$.

   Note. You need to prove that your examples satisfy the required properties.

3. Consider the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + \left(\frac{4 - x^2 - y^2}{4 + x^2 + y^2}\right) y \end{aligned} \tag{IV}$$

   Find the ω-limit set for each trajectory.

4. Consider an ODE

$$\ddot{u} + \alpha \dot{u} + g(u) = 0 \tag{V}$$

where $\alpha > 0$ and g is a C^1 function with $ug(u) > 0$ for $u \neq 0$, $\int_0^{-\infty} g(u)du = \infty$ and $\int_0^{\infty} g(u)du = c < \infty$.

(a) Prove that every bounded solution $u(t)$ to (V) satisfies

$$\lim_{t\to\infty} u(t) = \lim_{t\to\infty} \dot{u}(t) = 0.$$

(b) Prove that every solution $u(t)$ to (V) is bounded on $(0, \infty)$.

5. Consider the ODE

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned} \tag{VI}$$

(a) Prove: Suppose that f and g are C^1 throughout the x-y plane, that $f(-x, y) \equiv f(x, y)$ and $g(-x, y) \equiv -g(x, y)$. Then the IVP (VI) with $x(0) = 0$, $y(0) = y_0$ has a solution whose maximal domain is symmetric about the time $t = 0$ and satisfies $(x(-t), y(-t)) \equiv (-x(t), y(t))$.

(b) Suppose in particular that

$$\begin{aligned} f(x, y) &= -y + x^2 y \\ g(x, y) &= x + x^3 y \end{aligned}$$

Prove: There is a $\delta > 0$ such that if $0 < x_0^2 + y_0^2 < \delta^2$ then the solution to (VI) through (x_0, y_0) is periodic.

6. Suppose that $y(t)$ is a solution of the integral equation

$$y(t) = (\cos t) + \alpha \int_t^{\infty} \sin(t - s) \frac{y(s)}{s^2}\, ds \qquad (t > 0) \tag{VII}$$

(a) Assuming that $\int_t^{\infty} \frac{|y(s)|}{s^2}\, ds < \infty$ find a second order homogeneous linear ordinary differential equation with variable coefficients satisfied by y.

(b) Consider the successive approximations on the interval $0 < t < \infty$:

$$\phi_0(t) \equiv 0$$

$$\phi_n(t) = (\cos t) + \alpha \int_t^{\infty} \sin(t - s) \frac{\phi_{n-1}(s)}{s^2}\, ds \qquad (\alpha > 0)$$

Show that $\{\phi_n\}$ converge uniformly on any interval (a, ∞), with $a > 0$, to a solution $\phi(t)$ of (VII) and that $| \phi(t) | \leq e^{\frac{\alpha}{t}}$ for all $t \in (0, \infty)$.

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF MARYLAND

WRITTEN GRADUATE QUALIFYING EXAM

ODE Differential Equations (PH.D. VERSION)

January 1999

Instructions

i Answer all six questions. Your work on each question will be assigned a grade from 0 to 10. Some problems have multiple parts or ask you to do more than one thing. Be sure to go on to subsequent parts even if there is some part you cannot do. If you are asked to prove a result and then apply it to a given situation you may receive partial credit for a correct application even though you do not give a correct proof.

ii Use a different sheet (or different set of sheets) for each question. Write the problem number and your **code number** (not your name) on the top of every sheet.

iii Keep scratch work on separate sheets.

iv Unless otherwise stated, you may appeal to a "well-known theorem" in your solution to a problem. However, it is your responsibility to make it clear exactly which theorem you are using and to justify its use.

1. Consider the initial value problem

$$\dot{x} = Ax \qquad x(0) = (1, 2, 3, 4)^T \tag{1}$$

where $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & k & 0 & 0 \\ 2 & 3 & 0 & 1 \\ 4 & 5 & -1 & 0 \end{pmatrix}$

Determine the values of k for which the solution of (1) is unbounded as $x \to \infty$

2. Prove the fundamental existence theorem: If $f(t, y)$ is a continuous function on an open set $D \subset \mathbf{R}^2$ then, for any $(t_0, y_0) \in D$, there is an open interval I containing t_0 and a continuously differentiable function $\phi(t)$ defined on I such that $\phi(t_0) = y_0$ and $\frac{d}{dt}\phi(t) = f(t, \phi(t))$ for all t in I.

3. Suppose that $f(t, y) = -\sqrt{ty}$ for $t > 0,\ y > 0$ and $f(t, y) = 0$ otherwise. Determine for which values of t_0 and y_0 the initial value problem $\dot{y} = f(t, y)$, $y(t_0) = y_0$ has a locally unique solution and for which values it does not have a locally unique solution. Note: We say there is a locally unique solution at (t_0, y_0) if for any sufficiently small open interval I there is exactly one function ϕ defined on I with $\phi(t_0) = y_0$ and $\dot{\varphi}(t) \equiv f(t, \phi(t))$ on I.

4. Let A be a constant $n \times n$ matrix such that all eigenvalues have negative real parts. Let $B(t)$ be a continuous matrix valued function such that $\int_0^\infty \|B(t)\| \, dt < \infty$.

   Prove that the zero solution to $\dot{y} = (A+B(t))y$ is asymptotically stable.

5. Suppose that $f(x, y)$ and $g(x, y)$ are C^1 functions defined throughout R^2. Assume that the autonomous system $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$ has exactly 1 non-constant periodic solution and let $\mathcal{D}$ be the domain interior to the closed curve given by this periodic solution. Suppose that there is exactly one point (x_0, y_0) in $\mathcal{D}$ where $f(x_0, y_0) = g(x_0, y_0) = 0$ and suppose $\Delta = f_x(x_0, y_0)g_y(x_0, y_0) - f_y(x_0, y_0)g_x(x_0, y_0) \neq 0$.

   Prove that either

   (a) For every $(x_1, y_1) \neq (x_0, y_0)$ in $\mathcal{D}$, the ω limit set of the solution through (x_1, y_1) is the fixed point and the α limit set is the periodic orbit

   or

   (b) For every $(x_1, y_1) \neq (x_0, y_0)$ in $\mathcal{D}$, the α limit set of the solution through (x_1, y_1) is the fixed point and the ω limit set is the periodic orbit

6. Consider the system of differential equations $\dot{x} = x(1 - ax - by)$ and $\dot{y} = y(1 - cx - dy)$ with constants $0 < c < a$ and $0 < b < d$ and initial data $x(0) = x_0 > 0$ and $y(0) = y_0 > 0$. Draw the phase portrait in the first quadrant and determine the long term behavior of solutions to this system.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM, PH.D.
ORDINARY DIFFERENTIAL EQUATIONS
AUGUST 1998

a. Your answer to each of the questions will be assigned a score from zero to ten. Begin each problem on a separate answer sheet. Write the problem number and your code number (NOT your name) on the answer sheets. Clearly indicate the pages that are to be regarded as scratch work and the pages that are part of the solution.

b. Justify your answers. Provide careful statements of any theorems you invoke.

c. The student is completely responsible for solving the problem and presenting the solution in concise, comprehensible and mathematically meaningful manner. No credit will be given for arguments that do not directly lead to a solution.

---

1. Let $\dot{x} = u(x, y)$, $\dot{y} = v(x, y)$ be a C^1 system in the plane. Suppose $u(x, y)$ is even in x and $v(x, y)$ is odd in x. Let $\phi(t) = (x(t), y(t))$ be a solution that is not a fixed point.
(a) Show that $\psi(t) = (-x(-t), y(-t))$ is also a solution.
(b) Prove that $\phi(t)$ is a periodic solution if $x(0) = x(T) = 0$ for some $T > 0$.

2. Let $q : \mathbb{R} \to \mathbb{R}$ be a continuous nonnegative function. Let ϕ be a solution of

$$\ddot{x} - q(t)x = 0.$$

Show that if $\phi(t_1) = \phi(t_2) = 0$ for some $t_1 \neq t_2$, then ϕ is identically zero.

3. Prove that any integral curve of $\frac{dy}{dx} = x^2 - y^2$ intersects the line $y = x$ at least once.

4. Discuss the stability and bifurcation phenomena for the following equation. Sketch the bifurcation diagram.
$$\dot{x} = x(\mu - 2x + x^2).$$
5. Let $f(t, x)$ be a C^1 real-valued function on $\mathbb{R}^2$ and suppose
$$\left|\frac{\partial f}{\partial x}\right| \leq K$$
on $\mathbb{R}^2$. Let $x(t, x_0)$ be the solution of the initial value problem
$$\dot{x} = f(t, x)$$
$$x(0) = x_0.$$

(a) Show that if $x(t, x_0)$ and $x(t, x_1)$ are defined, then
$$|x(t, x_0) - x(t, x_1)| \leq |x_1 - x_0| + K \int_0^t |x(s, x_0) - x(s, x_1)|\, ds$$
(b) Show that $\left|\frac{\partial x}{\partial x_0}(1, x_0)\right| \leq e^K$ whenever $x(1, x_0)$ is defined.

6. NOTE: The two parts of this problem can be done independently of each other.
Let I_n be the $n \times n$ identity matrix, and let $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.
Note that $J^2 = -I$. A real $2n \times 2n$ matrix A is *symplectic* if $AJ + JA^T = 0$. The associated linear system $\dot{x} = Ax$ is said to be a *linear Hamiltonian system.*
(a) Show that if λ is an eigenvalue of a real symplectic matrix A, then $-\lambda$, $\bar{\lambda}$, and $-\bar{\lambda}$ are also eigenvalues of A.
(b) Assume part (a) even if you can't prove it. Let A be a real symplectic matrix without repeated eigenvalues. Show that if the origin is a stable equilibrium point of the associated linear Hamiltonian system $\dot{x} = Ax$, then every solution of $\dot{x} = Ax$ is a finite linear combination of periodic functions.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM, PH.D.
ORDINARY DIFFERENTIAL EQUATIONS
JANUARY 1998

a. Your answer to each of the questions will be assigned a score from zero to ten. Begin each problem on a separate answer sheet. Write the problem number and your code number (NOT your name) on the answer sheets. Clearly indicate the pages that are to be regarded as scratch work and the pages that are part of the solution.

b. Justify your answers. Provide careful statements of any theorems you invoke.

c. The student is completely responsible for solving the problem and presenting the solution in concise, comprehensible and mathematically meaningful manner. No credit will be given for arguments that do not directly lead to a solution.

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1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f(y_0) = 0$ and $f(y) > 0$ for $y \neq y_0$.
State and prove a necessary and sufficient condition on f so that the solution of the initial value problem

$$y' = f(y), \; y(0) = y_0,$$

is unique.

2. Determine the set of values of k for which the system $x' = Ax$, $x \in \mathbb{R}^{2001}$, is stable, where

$$A = \begin{pmatrix} 1 & 2 & \dots & 2000 & k \\ 1 & 2 & \dots & 2000 & k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2 & \dots & 2000 & k \end{pmatrix} .$$

3. Let $f : [0, \infty) \to \mathbb{R}$ be continuous. Prove that for every x_0, the solution of the initial value problem

$$x' = f(t) - x^3, \quad x(0) = x_0,$$

is defined on the interval $[0, \infty)$.

4. Prove that every solution $x(t)$ of

$$\ddot{x} + e^{-\cos x}\dot{x} + \sin x = 0$$

satisfies $\lim_{t\to\infty} x(t) = n\pi$ for some integer n.

5. Consider the system

$$\begin{aligned}
\dot{x}_1 &= -x_2(x_3^2 + 1) \\
\dot{x}_2 &= x_1(x_3^2 + 1) \\
\dot{x}_3 &= x_3(x_1^2 + x_2^2 - 1)\,.
\end{aligned}$$

Determine the α- and ω-limit sets for all initial conditions $x(0) \in \mathbb{R}^3$.

6. Consider the equation:

$$\ddot{y} + \dot{y} + (1 + \varepsilon \cos 2t)y = 0\,.$$

a) Show that for all $\epsilon \neq 0$, the characteristic (Floquet) multipliers ρ_1, ρ_2 for the corresponding linear system satisfy $\rho_1\rho_2 = e^{-\pi}$.
b) Show that there is $\varepsilon_0 > 0$ such that for every ε with $|\varepsilon| < \varepsilon_0$, all solutions are bounded for $t \geq 0$.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM, PH.D.
ORDINARY DIFFERENTIAL EQUATIONS
AUGUST 1997

a. Begin each problem on a separate answer sheet. Write the problem number and your code number (NOT your name) on the answer sheets. Clearly indicate the pages that are to be regarded as scratch work and the pages that are part of the solution.

b. Provide careful statements of any theorems you invoke and give careful explanations of your procedures. Your answer to each of the questions will be assigned a score from zero to ten.

c. The student is completely responsible for solving the problem and presenting the solution in concise, comprehensible and mathematically meaningful manner. No credit will be given for an answer without a complete justification. No credit will be given for arguments that do not directly lead to a solution.

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1. Let $f : \mathbb{R} \to \mathbb{R}$ be an odd continuous function such that $f(0) = 0$, $f(x) > 0$ for $x > 0$ and $\int_0^{1997} \frac{1}{f(x)} dx = 1$.
Determine all values of a for which there is a solution $x(t)$ to $\dot{x} = f(x)$ with $x(0) = 0$, $x(1) = a$.

2. Determine, as a function of k, the dimension of the space of initial conditions x_0 for which the solution of

$$\dot{\mathrm{x}} = \begin{pmatrix} 0 & k & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{pmatrix} \mathrm{x}, \ \mathrm{x}(0) = \mathrm{x}_0,$$

is bounded for $t \geq 0$.

3. The unit circle $x(t) = \cos t$, $y(t) = \sin t$ is a periodic solution of the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -(x^2 + y^2)^2 y - x \end{aligned}$$

and the segment of the x-axis $S = \{(x, 0) : .9 < x < 1.1\}$ is a Poincaré section with the Poincaré return map $P : S \to S$.
Find the derivative of P at $(1, 0)$.

4. Prove that every solution of

$$\begin{aligned} \dot{x} &= (x + 1)y \\ \dot{y} &= \cos^2 y - xy^2 \end{aligned}$$

is unbounded as $t \to \infty$.

5. Consider the n-dimensional autonomous system $\dot{x} = v(x)$, where v is $C^1(\mathbb{R}^n)$ and all first derivatives of v are bounded.
Prove that there is a constant $M > 0$ such that

$$\|x(t) - y(t)\| \leq \|x(0) - y(0)\| \cdot e^{Mt}$$

for every two solutions $x(t)$ and $y(t)$ and all $t \geq 0$.

6. Consider the n-dimensional autonomous system $\dot{x} = f(x)$, where f is $C^1(\mathbb{R}^n)$. Suppose every solution is a bounded function of $t \in \mathbb{R}$. For $x \in \mathbb{R}^n$, let $\alpha(x)$ and $\omega(x)$ denote, respectively, the α- and ω-limit sets of x. Suppose for all $x, y \in \mathbb{R}^n$ that $x \in \omega(y)$ whenever $y \in \omega(x)$.
Prove that $\alpha(x) = \omega(x)$ for every $x \in \mathbb{R}^n$.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM, PH.D.
ORDINARY DIFFERENTIAL EQUATIONS
JANUARY 1997

a. Begin each problem on a separate answer sheet. Write the problem number and your code number (NOT your name) on the answer sheets. Clearly indicate the pages that are to be regarded as scratch work and the pages that are part of the solution.

b. Provide careful statements of any theorems you invoke and give careful explanations of your procedures. Your answer to each of the questions will be assigned a score from zero to ten. Some problems are broken into parts for convenience, but the score will be assigned to the problem as a whole.

c. No credit will be given for an answer without a complete justification.

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1. Determine whether the origin is stable, asymptotically stable, or unstable for the system

$$\dot{\mathrm{x}} = \begin{pmatrix} -3 & -3 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix} \mathrm{x}\,.$$

(Note: there are exactly two distinct eigenvalues.)

2. Consider the planar system

$$\begin{aligned} \dot{x} &= x(1-2x^2-y^2) - y \\ \dot{y} &= x + y(1-2x^2-y^2) \end{aligned}$$

a) Prove that every solution is bounded.
b) Determine all solutions whose ω-limit set contains the origin.
c) Prove that there is a nontrivial (i.e., not a fixed point) periodic orbit.

3. Consider the following system in $\mathbb{R}^n$:

$$\dot{\mathrm{x}} = \|\mathrm{x}\|\mathrm{x} + \mathrm{h}(\mathrm{x}, t),$$

where h is C^1, bounded and $\mathrm{h}(\mathrm{x}, t+1) = \mathrm{h}(\mathrm{x}, t)$ for all $\mathrm{x} \in \mathbb{R}^n$ and $t \in \mathbb{R}$.
a) Show that if $\|\mathrm{x}(0)\|$ is large enough, then $\|\mathrm{x}(t)\| \to \infty$ as $t \to \infty$.
b) Prove that there is either a fixed point or a periodic solution of (not necessarily least) period 1.

4. For the second order initial value problem

$$(*) \qquad \ddot{u} + u(u-2) = 0\,; \ u(0) = u_0, \ \dot{u}(0) = 0,$$

find $u_0 \neq 0$ such that the solution of $(*)$ tends to 0 as $t \to \infty$.

5. Let $f(y) = \ln y \cdot \ln(y-1)$ if $y > 1$ and $f(y) = 0$ if $y \leq 1$. Determine whether the initial value problem

$$y' = f(y), \quad y(0) = 1\,,$$

has a unique solution.

6. Let $\mathrm{v} : \mathbb{R}^n \to \mathbb{R}^n$ be C^1 and bounded and let $\mathrm{x}(t)$ and $\mathrm{y}(t)$ be solutions of the system $\dot{\mathrm{x}} = \mathrm{v}(\mathrm{x})$ with initial conditions $\mathrm{x}(0) = \mathrm{x}_0$ and $\mathrm{y}(0) = \mathrm{y}_0$. Assume that $\mathrm{x}(t_k) \to \mathrm{y}_0 \in \mathbb{R}^n$ as $k \to \infty$ for a sequence $t_k \nearrow \infty$ with $t_{k+1} - t_k < 1$ for all k.
Prove that y_0 is a fixed or a periodic point.

WRITTEN QUALIFYING EXAMINATION
ORDINARY DIFFERENTIAL EQUATIONS
AUGUST 1996

Instructions: Answer each of the following six problems, giving careful statements of any theorems you invoke and giving careful explanations of your procedures. Your answer to each question will be assigned a grade from 0 to 10. Although some preblems are broken into parts, the parts need not have equal weight.
Begin each problem on a separate answer sheet. Write the problem number and your code number (NOT your name) on the answer sheets.
Clearly indicate the pages that are to be regarded as scratch work and the pages that are not part of the solution.

1. Let $f(x) = x\ln(1/x)$ when $0 < x < 1$ and $f(x) = 0$ when $x \leq 0$. Show that the scalar IVP $\{\dot{x} = f(x);\ x(0) = 0\}$ has a unique solution.

2. Consider the 2-dimensional system (1) $\dot{x} = f(x)$ where f is smooth, $f(0) = 0$, $x^t f(x) > 0$ in a punctured neighborhood of 0, and $f(x) \neq 0$ for all $x \neq 0$. If (1) does not possess a periodic orbit (except the fixed point at the origin), prove that all solutions of (1) except $x \equiv 0$ are unbounded on $[0, \infty)$.

3. Consider the n-dimensional system $\dot{x} = f(x)$, where f is smooth, and assume that there is a point x_0 whose positive semiorbit is compact. Prove that x_0 is either a fixed or a periodic point.

4. Consider the second order equation (1) $\ddot{y} + p(t)y = 0$, where p is continuous and T-periodic. Let y_1, y_2 be solutions of (1) which satisfy the initial conditions $y_1(0) = 1$, $\dot{y}_1(0) = 0$, $y_2(0) = 0$ and $\dot{y}_2(0) = 1$. Let ρ be a complex number.
Show that there is a nontrivial (possibly complex valued) solution y that satisfies $y(t+T) \equiv \rho y(t)$ if and only if ρ satisfies the equation $\rho^2 - (y_1(T) + \dot{y}_2(T))\rho + 1 = 0$.

5. Consider the second order equation $\ddot{y} + g(y, \dot{y}) = 0$, where g is smooth on $\mathbb{R} \times \mathbb{R}$, $g(0,0) = 0$ and $\dfrac{\partial g}{\partial y}(0,0)$, $\dfrac{\partial g}{\partial \dot{y}}(0,0) > 0$. Prove that $y(t) \equiv 0$ is asymptotically stable.

6. Consider the scalar second order IVP

$$\ddot{y} + \mu\dot{y} + h(y) = 0,\ y(0) = y_0 > 0,\ \dot{y}(0) = 0, \tag{1}$$

where $\mu > 0$, h is smooth and $yh(y) > 0$ for all $y \neq 0$.
a) Show that the solution of (1) is bounded above for $t \geq 0$.
b) Show that the solution of (1) is bounded below for $t \geq 0$.

WRITTEN QUALIFYING EXAMINATION

ORDINARY DIFFERENTIAL EQUATIONS

January 1996

Instructions. Answer each of the following six problems, giving careful statements of any theorems you invoke and giving careful explanations of your procedures. Your answer to each question will be assigned a grade from 0 to 10. Although some problems are broken into parts for convenience, the parts need not have equal weight.

Begin each problem on a separate answer sheet. Write the problem number and your code number (NOT your name) on the answer sheets.

Clearly indicate the pages that are to be regarded as scratch work and the pages that are not part of the solution.

1. Consider the planar system of differential equations

$$\dot{x} = f(x)$$

$$\dot{y} = h(x)g(y)$$

where f, g, and h are continuous real valued functions defined on $\mathbf{R}$ and f and g are Lipschitz on $\mathbf{R}$. Let $\varphi(t) = (\varphi_1(t), \varphi_2(t))$ and $\psi(t) = (\psi_1(t), \psi_2(t))$ be two solutions of this system defined on the same open interval I containing the closed interval $a \leq t \leq b$.

(a) Prove that if $\phi(a) = \psi(a)$, then $\phi(t) = \psi(t)$ on I.

(b) Prove that given $\epsilon > 0$, there exists $\delta > 0$ such that for $a \leq t \leq b$, $|\phi(t) - \psi(t)| < \epsilon$ when $\phi_1(a) = \psi_1(a)$ and $|\phi_2(a) - \psi_2(a)| < \delta$.

2. Show that if $x(t)$ is a solution of the second order differential equation

$$\ddot{x} + \dot{x}^3 + xe^{-x^2} = 0$$

satisfying $\dot{x}(0) = 0$, then

$$\lim_{t\to\infty} x(t) = 0.$$

3. Let $\dot{x} = f(x)$ be a C^1 system of differential equations in the plane with the following properties:

   (i) There are exactly two fixed points denoted by p and q and they are hyperbolic fixed points with stable manifolds of dimension one,

   (ii) there exists a point ξ such that $\omega(\xi) = \{p\}$ and $\alpha(\xi) = \{q\}$,

   (iii) every solution not on a stable or unstable manifold goes to infinity both as time increases and decreases.

   Sketch a phase portrait of a system satisfying these conditions and argue with the aid of pictures that there exist arbitrarily small C^1 perturbations of f in a neighborhood of ξ not containing p and q such that for the perturbed system there does not exist any point η satisfying $\omega(\eta) = \{p\}$ and $\alpha(\eta) = \{q\}$.

4. Let A be an $n \times n$ real matrix such that

$$-A^t = QAQ^{-1}$$

   for some invertible matrix Q. Consider the linear system $\dot{x} = Ax$.

   (a) Show that the origin is neither positively nor negatively asymptotically stable.

   Let S denote the symmetric matrix $Q + Q^t$, and thinking of x as an $n \times 1$ matrix, set

$$V(x) = x^t S x.$$

   (b) Show that the derivative of V along solutions is 0.

   (c) Show that if the eigenvalues of S are all positive, then the origin is stable.

5. Let φ be a continuously differentiable function on the closed interval $[-1, 0]$ with values in $\mathbb{R}^n$. Using the contraction mapping principle prove that for small positive a there exists a unique continuous $\mathbb{R}^n$ valued function $u(t)$ on the closed interval $[-1, a]$ such that for $-1 \leq t \leq 0$

$$u(t) = \varphi(t)$$

WRITTEN QUALIFYING EXAMINATION

ORDINARY DIFFERENTIAL EQUATIONS

AUGUST 1995

Instructions. Answer each of the following six problems, giving careful statements of any theorems you invoke and giving careful explanations of your procedures. Your answer to each question will be assigned a grade from 0 to 10. Although some problems are broken into parts for convenience, the parts need not have equal weight.

Begin each problem on a separate answer sheet. Write the problem number and your code number (NOT your name) on the answer sheets.

Clearly indicate the pages that are to be regarded as scratch work and the pages that are not part of the solution.

1. Suppose throughout that the eigenvalues of the matrix A for a linear system of differential equations with constant coefficients all have non-positive real parts.
   (a) Show that all the solutions are bounded in positive time, when all the eigenvalues are simple.
   (b) Give two examples demonstrating that when the eigenvalues are not all simple, the solutions may or may not all be bounded in positive time.
   (c) Prove that all the solutions are bounded in positive time when the matrix A is symmetric.
2. Given a smooth autonomous ordinary differential equation on the plane, prove that for a non-periodic point when the intersection of its alpha-limit set and omega-limit set is not empty, it contains only fixed points.

and for $0 < t \le a$ the function u has a derivative satisfying

$$\dot{u}(t) + u(t) + \int_0^1 u(t-\tau)d\tau = 0.$$

6. Let A be an $n \times n$ real matrix and suppose that the eigenvalues of A all have negative real parts. Let $B(t)$ be a continuous function defined on $\mathbf{R}$ with values in the $n \times n$ real matrices and suppose that $\|B(t)\| \le \mu$. Consider the linear differential equation $\dot{x} = (A + B(t))x$, and let $x(t, \xi)$ denote the solution satisfying $x(0, \xi) = \xi$.

   (a) Show that there exist positive constants C and α such that for $t > 0$
   $$\|x(t,\xi)\| \le C\|\xi\|e^{-\alpha t} + \int_0^t \mu C\|x(s,\xi)\|ds.$$

   (b) Show that for small enough positive μ, there exist positive constants M and δ such that for $t > 0$
   $$\|x(t,\xi)\| \le Me^{-\delta t}.$$

3. Consider the second order equation

$$\ddot{y} + p(t)\dot{y} + q(t)y = 0$$

where $p(t)$ and $q(t)$ are smooth positive functions on the real line and $q(t)$ is decreasing.

(a) Rewrite this equation as a linear system and show that the function

$$E(t, y, \dot{y}) = q(t)y^2 + (\dot{y})^2$$

is decreasing along the trajectories of the system.

(b) Show that the system is uniformly stable, if

$$\lim_{t\to\infty} q(t) > 0.$$

4. Show that there exist $\mu \neq 0$ for which the system

$$\dot{x} = \mu x + y + xy - xy^2$$
$$\dot{y} = -x + \mu y - x^2 - y^3$$

has periodic solutions.

5. Consider the spiral system

$$\dot{x}_1 = -x_2 + x_1(1 - r^2)$$
$$\dot{x}_2 = x_1 + x_2(1 - r^2)$$

where $r^2 = x_1^2 + x_2^2$. Let $\mathrm{x}(t, \tau, \xi)$ denote the solution which passes through ξ at time τ and note that $\mathrm{x}(t, \tau, (1, 0)) = (\cos(t-\tau), \sin(t-\tau))$.

(a) Explicitly calculate the linear system of differential equations for which the Jacobian matrix

$$\left(\frac{\partial \mathrm{x}}{\partial \xi}(t, 0, (1, 0))\right)$$

is a fundamental matrix solution.

(b) Use the given information to exhibit a periodic solution of period 2π for the linear system in part (a).

(c) Find the characteristic multipliers of the linear system in part (a).

6. Let K be a closed and bounded subset of $\mathbf{R}^n$ and let $F : K \to K$ be a continuous map, not necessarily either one-to-one or onto. A point p in K is said to be a wandering point for F if there exists a neighborhood U of p for which $\{F^{-m}(U) : m \geq 0\}$ is a pairwise disjoint family of sets. Let p be a point of K.

   (a) Given $M > 0$, prove that

   $$\{F^{-n}(\{p\}) : 0 \leq n < M\}$$

   is a pairwise disjoint family of sets, if $p \notin F^M(K)$. (It is possible that some of the sets in this family are in fact empty.)

   (b) Prove that p is a wandering point for F, when there exists $M > 0$ such that $p \notin F^M(K)$.

WRITTEN QUALIFYING EXAMINATION
ON ORDINARY DIFFERENTIAL EQUATIONS (670–671)
JANUARY 1995

Instructions. Answer each of the following six problems, giving careful statements of any theorems you invoke and giving careful explanations of your procedures. Your answer to each question will be assigned a grade from 0 to 10. Although some problems are broken into parts for convenience, the parts need not have equal weight.

Begin each problem on a separate answer sheet. Write the problem number and your code number (NOT your name) on the answer sheets.

Clearly indicate the pages that are to be regarded as scratch work and the pages that are part of the solution.

1. Consider the 2-dimensional autonomous system

$$\dot{x} = \cos\left(\frac{1}{1+x^4+y^4}\right),$$
$$\dot{y} = \sin\left(\frac{1}{1+x^4+y^4}\right)$$

defined on $\mathbb{R}^2$. Show that every orbit goes to infinity.

2. Consider the 2-dimensional system

$$\dot{x} = \sin x \cos y,$$
$$\dot{y} = \cos x \sin y.$$

Clearly the stationary (equilibrium, critical) points of this system are precisely

$$a_{(p,q)} = \left((p+\tfrac{1}{2})\pi, (q+\tfrac{1}{2})\pi\right), \quad d_{(p,q)} = (p\pi, q\pi)$$

for p and q ranging over all integers. Say that one stationary point s_1 *precedes* another one s_2 if there is an orbit $\gamma(t)$ with $\lim_{t\to-\infty}\gamma(t) = s_1$ and $\lim_{t\to+\infty}\gamma(t) = s_2$. Denote this by $s_1 \mapsto s_2$. Determine which of the stationary points precede which others.

3. Let the real-valued function f be continuously differentiable on the (t,x)-plane. Suppose that there exists a continuous, increasing function h defined on $\mathbb{R}$ such that

$$[h(t) - x]f(t,x) \geq 0$$

for all t, x. Let ϕ be a solution of $\dot{x} = f(t,x)$, and let its maximal interval of existence be (a,b). Prove:

(a) $f(t, h(t)) = 0$.
(b) If there is a τ such that $\phi(\tau) = h(\tau)$, then $\phi(t) \leq h(t)$ for $t \geq \tau$.
(c) $b = \infty$.

4. Parts a and b of this problem are independent.
(a) Let $\boldsymbol{w}$ be a continuously differentiable function with values in the Euclidean 3-space $\mathbb{E}^3$. Let $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$, each with values in $\mathbb{E}^3$, satisfy

$$\dot{\boldsymbol{u}}_k = \boldsymbol{w}(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, t) \times \boldsymbol{u}_k, \quad k = 1, 2, 3$$

where $\times$ is the cross product. Suppose that

$$\boldsymbol{u}_k(0) \cdot \boldsymbol{u}_l(0) = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{if } k \neq l. \end{cases}$$

Prove that

$$\boldsymbol{u}_k(t) \cdot \boldsymbol{u}_l(t) = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{if } k \neq l \end{cases}$$

for all t.
(b) Let $(x, y) \mapsto f(x, y)$ be a twice continuously differentiable mapping from $\mathbb{R}^2$ to $\mathbb{R}$. Consider the system

$$\text{(A)} \qquad \begin{aligned} \dot{x} &= af_x(x, y) + bf_y(x, y), \\ \dot{y} &= cf_x(x, y) + df_y(x, y) \end{aligned}$$

where a, b, c, d are constants. Let the solution of (A) satisfying the initial conditions $x(0) = \xi$, $y(0) = \eta$ be denoted by $t \mapsto (\hat{x}(t, \xi, \eta), \hat{x}(t, \xi, \eta))$. The flow of (A) is said to be *measure-preserving* if

$$\text{area}\,\{(x, y) = (\hat{x}(t, \xi, \eta), \hat{x}(t, \xi, \eta)) : (\xi, \eta) \in \mathcal{E}\} = \text{area}\,\mathcal{E}$$

for each (measurable) set $\mathcal{E}$ in $\mathbb{R}^2$ and for each t. Find the most general set of constants a, b, c, d for which the flow of (A) is measure-preserving for all twice continuously differentiable functions f.

5. Let A and B be real, constant $n \times n$ matrices. Let $t \mapsto \boldsymbol{\phi}(t, \boldsymbol{\xi})$ be the solution of the initial-value problem $\dot{\boldsymbol{x}} = A\boldsymbol{x}$, $\boldsymbol{x}(0) = \boldsymbol{\xi}$, and let $t \mapsto \boldsymbol{\psi}(t, \boldsymbol{\xi})$ be the solution of the initial-value problem $\dot{\boldsymbol{x}} = B\boldsymbol{x}$, $\boldsymbol{x}(0) = \boldsymbol{\xi}$. Suppose that there is a smooth diffeomorphism $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\boldsymbol{f}(\boldsymbol{\phi}(t, \boldsymbol{\xi})) = \boldsymbol{\psi}(t, \boldsymbol{f}(\boldsymbol{\xi}))$ for all $t \in \mathbb{R}$, $\boldsymbol{\xi} \in \mathbb{R}^n$. Show that e^{tA} and e^{tB} are similar for all t. Prove that there is a linear map $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\boldsymbol{g}(\boldsymbol{\phi}(t, \boldsymbol{\xi})) = \boldsymbol{\psi}(t, \boldsymbol{g}(\boldsymbol{\xi}))$ for all $t \in \mathbb{R}$, $\boldsymbol{\xi} \in \mathbb{R}^n$.

6. Let Ω be an open set of $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}^n$ be Lipschitz continuous. Let $V : \Omega \to \mathbb{R}$ be continuously differentiable and satisfy $\nabla V(x) \cdot f(x) > 0$ for each $x \in \Omega$. Let $x(t, \xi)$ be the solution of the initial-value problem $\dot{x} = f(x)$, $x(0) = \xi \in \Omega$, and let $\omega(\xi)$ be the omega limit set of of this solution.
(a) Show that $\omega(\xi) \cap \Omega = \emptyset$ for all $\xi \in \Omega$.
(b) Show that $\{\xi : V(x(t, \xi)) = 0 \text{ for some } t\}$ is an open subset of Ω.