

UMCP Department of Mathematics Qualifying Exam  
Partial Differential Equations, January 2011

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- (1) Solve all six problems. Each main problem will be assigned a grade from zero to ten.
  - (2) Begin your answer to each question on a separate sheet.
  - (3) Write your code number on each page of your answer sheets. Do *not* use your name.
  - (4) Keep any scratch work on separate sheets, which should not be submitted.
  - (5) Carefully explain all your steps. If you invoke a “well-known” theorem, you must make clear which theorem you are using and justify its use.
- 

1. Consider the conservation law

$$u_t + f(u)_x = 0 \quad \text{in } \mathbb{R} \times (0, \infty), \quad (1)$$

where  $f \in C^1(\mathbb{R})$ .

- (a) Define an integral solution of (1).
  - (b) Derive the jump (Rankine-Hugoniot) condition satisfied by a piecewise smooth integral solution  $u$  across a  $C^1$  curve where this  $u$  has a discontinuity
  - (c) Find an integral solution to (1) when  $f(u) = u^2 + u$  with  $u(x, 0) = 1$  if  $x < 0$ ,  $u(x, 0) = -3$  if  $x > 0$ .
2. Prove or disprove: Let  $U \subset \mathbb{R}^n$  ( $n \geq 2$ ) be an open, bounded set with  $C^1$  boundary. Suppose  $u \in W^{1,n}(U)$ . Then,  $u \in L^\infty(U)$  and there exists a constant  $C > 0$  such that

$$\|u\|_{L^\infty(U)} \leq C \|u\|_{W^{1,n}(U)}$$

3. Consider  $u : \bar{U} \rightarrow \mathbb{R}$  where  $u \in C^2(U) \cap C(\bar{U})$  and the set  $U \subset \mathbb{R}^n$  is open, bounded. This  $u$  satisfies  $Lu = 0$  in  $U$  where

$$Lu := - \sum_{i,j=1}^n a^{ij}(x) u_{x_i x_j} + \sum_{i=1}^n b^i(x) u_{x_i} + c(x) u$$

Assume that the operator  $L$  is uniformly elliptic; and  $a^{ij}(x)$ ,  $b^i(x)$  and  $c(x)$  are smooth with  $c(x) \geq 0$  in  $U$ .

Let  $v := \phi(u) \geq 0$  where  $\phi : \mathbb{R} \rightarrow \mathbb{R}_+$  is  $C^2$  and convex with  $\phi(0) = 0$ . Show that

$$\max_{\bar{U}} v = \max_{\partial U} v$$

4. Show that there is at most one smooth (up to the boundary) solution to the equation  $u_{tt} - u_{xx} = u_t - u^3$  in the domain  $(0, 1) \times (0, \infty)$  with initial conditions  $u(x, 0) = g(x)$ ,  $u_t(x, 0) = h(x)$  and boundary conditions  $u(0, t) = u(1, t) = 0$ .

CONTINUED ON REVERSE

5. Let  $\mathcal{D}$  be the unit disc in  $\mathbb{R}^2$ , i.e.,  $\mathcal{D} = B^0(0, 1) \subset \mathbb{R}^2$ . Given  $f, g \in L^2(\mathcal{D})$ , let  $\begin{pmatrix} u \\ v \end{pmatrix} \in H^1(\mathcal{D}) \times H^1(\mathcal{D})$  satisfy the boundary value problem (BVP)

$$\begin{cases} -\Delta u = f, & -\Delta v = g & \text{for } x = (x_1, x_2) \in \mathcal{D}, \\ u_{x_2} = v_{x_1}, & u_{x_1} = -v_{x_2} & \text{on } \partial\mathcal{D}. \end{cases} \quad (*)$$

The weak formulation for this problem is defined by the relation

$$B \left[ \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} U \\ V \end{pmatrix} \right] = \int_{\mathcal{D}} (fU + gV) dx \quad \text{for all } \begin{pmatrix} U \\ V \end{pmatrix} \in H^1(\mathcal{D}) \times H^1(\mathcal{D}), \quad (**)$$

where the bilinear form  $B[\cdot, \cdot]$  is

$$B \left[ \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} U \\ V \end{pmatrix} \right] := \int_{\mathcal{D}} ((u_{x_2} - v_{x_1})(U_{x_2} - V_{x_1}) + (u_{x_1} + v_{x_2})(U_{x_1} + V_{x_2})) dx.$$

- (a) Check that if  $u, v \in C^\infty(\overline{\mathcal{D}})$  and satisfy BVP (\*), then relation (\*\*) is true for all  $U, V \in C^\infty(\overline{\mathcal{D}})$ .  
 (b) Prove or disprove: There exist constants  $\gamma > 0, C > 0$  such that

$$\gamma (\|u\|_{L^2(\mathcal{D})}^2 + \|v\|_{L^2(\mathcal{D})}^2) + B \left[ \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right] \geq C (\|u\|_{H^1(\mathcal{D})}^2 + \|v\|_{H^1(\mathcal{D})}^2),$$

for all  $\begin{pmatrix} u \\ v \end{pmatrix} \in H^1(\mathcal{D}) \times H^1(\mathcal{D})$ .

6. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be smooth, and define the domains  $U_\pm \subset \mathbb{R}^2$  by  $U_+ = \{y > f(x)\}$  and  $U_- = \{y < f(x)\}$ . Let  $\Gamma = \{y = f(x)\}$  be the common boundary of these domains, and define the vector  $N = (-f'(x), 1)$ .

Consider functions  $u_\pm : U_\pm \rightarrow \mathbb{R}$ . Assume that  $u_\pm$  are (or have extensions) smooth up to the boundary and are harmonic. Define

$$u(x, y) = \begin{cases} u_-(x, y) & \text{if } y < f(x) \\ u_+(x, y) & \text{if } y \geq f(x) \end{cases}.$$

Prove or disprove: If  $u_+ = u_-$  on  $\Gamma$  and  $N \cdot \nabla u_+ = N \cdot \nabla u_-$  on  $\Gamma$ , then  $u$  is harmonic (i.e.,  $u$  is  $C^2(\mathbb{R}^2)$  and  $\Delta u = 0$  in  $\mathbb{R}^2$ ).

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1. A superharmonic  $u \in C^2(\bar{U})$  satisfies  $-\Delta u \geq 0$  in  $U$ , where here  $U \subset \mathbb{R}^n$  is open, bounded.
- (a) Show that if  $u$  is superharmonic, then

$$u(x) \geq \int_{B(x,r)} u \, dy \quad \text{for all } B(x,r) \subset U,$$

where  $B(x,r)$  is the closed ball with center  $x$  and radius  $r > 0$ , and  $\int_B f \, dy$  denotes the average of  $f$  over  $B$ .

- (b) Prove that if  $u$  is superharmonic, then  $\min_{\bar{U}} u = \min_{\partial U} u$ .
- (c) Suppose  $U$  is connected. Show that if there exists  $x_0 \in U$  such that  $u(x_0) = \min_{\bar{U}} u$  then  $u$  is constant in  $U$ .
2. (a) Let  $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$  be a smooth function of  $(p, z, x)$ .

Prove that, if the following statements are satisfied:

- $u$  is a smooth solution to  $F(Du, u, x) = 0$ ,
- $x(s)$  is a smooth curve,  $p(s) = Du(x(s))$ ,  $z(s) = u(x(s))$  and  $x'_i(s) = \frac{\partial F}{\partial p_i}$ ,

then  $p'_i(s) = -\frac{\partial F}{\partial x_i} - \frac{\partial F}{\partial z} p_i$

- (b) By using the method of characteristics, find an explicit local solution to  $u_t + \frac{(u_x)^2 + x^2}{2} = 0$  if  $t > 0$ ,  $x \in \mathbb{R}$ , with initial condition  $u(x, 0) = \frac{x^2}{2}$ .

3. Suppose  $u : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$  is a smooth solution of

$$u_{tt} + Lu = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty),$$

where  $Lu = -\sum_{i,j=1}^n a^{ij}(x) u_{x_i x_j} + c(x)u$  with  $a^{ij} = a^{ji}$ , smooth  $a^{ij}$  and  $c$ , and  $c(x) \geq 0$ . Assume that  $\sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2$  for some  $\theta > 0$ , and all  $x, \xi \in \mathbb{R}^n$ .

For any fixed  $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$ , consider the region  $C_t := \{x \mid q(x) < t_0 - t, 0 \leq t \leq t_0\}$  where  $q$  is smooth in  $\mathbb{R}^n - \{x_0\}$  and solves  $\sum_{i,j=1}^n a^{ij} q_{x_i} q_{x_j} = 1$ ,  $q > 0$  in  $\mathbb{R}^n - \{x_0\}$  and  $q(x_0) = 0$ . Prove that if  $u \equiv u_t \equiv 0$  on  $C_0$ , then  $u \equiv 0$  in  $C := \{(x, t) \mid q(x) < t_0 - t\}$ .

**Hint:** Define an appropriate energy  $E(t) = (1/2) \int_{C_t} u_t^2 + \dots \, dx$  and consider its derivative.  
**Note:** Partial credit will be given for treating only the case with  $c \equiv 0$ . For full credit, include a  $c(x) \geq 0$ .

CONTINUED ON REVERSE

4. Consider an open, bounded  $U \subset \mathbb{R}^n$  with a smooth boundary  $\partial U$ .
- (a) Give the definitions of the Sobolev spaces  $W^{2,p}(U)$  and  $W_0^{1,p}(U)$  where  $1 \leq p < \infty$ . Define the corresponding norms
- (b) Prove or disprove: For any  $u \in W^{2,p}(U) \cap W_0^{1,p}(U)$  and  $2 \leq p < \infty$ ,  $\|Du\|_{L^p(U)}^2 \leq C \|u\|_{L^p(U)} \|D^2u\|_{L^p(U)}$ . **Hint:** Consider a (smooth) approximation for  $u$  and first compute  $\operatorname{div}(u Du |Du|^{p-2})$ .
- (c) Prove or disprove: For any  $u \in H_0^2(U)$ , there exists  $C > 0$  such that  $\|u\|_{H_0^2(U)} \leq C \|\Delta u\|_{L^2(U)}$
5. Consider the boundary value problem (BVP)

$$\begin{aligned} \Delta^2 u + \gamma \Delta u &= f && \text{in } U, \\ u = \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial U, \end{aligned}$$

where  $U \subset \mathbb{R}^n$  is open, bounded with smooth boundary,  $f \in L^2(U)$ , and  $\gamma < 0$  is a constant

- (a) Provide a weak formulation for this BVP if  $u \in H_0^2(U)$ .
- (b) Use the weak formulation of part 5(a) to prove the existence of a unique weak solution to the above BVP.
6. Let  $\Omega \subset \mathbb{R}^n$  be open, bounded and consider  $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$  with  $f, g \in C^1(\mathbb{R}^2)$ . Suppose  $u, v : \Omega \times (0, T) \rightarrow \mathbb{R}$  satisfy the reaction-diffusion equations

$$\begin{aligned} u_t &= \Delta u + f(u, v), & v_t &= \Delta v + g(u, v) && \text{in } \Omega \times (0, T), \\ u = v &= h > 0 && && \text{on } \partial\Omega. \end{aligned}$$

Assume that  $f(0, v) > 0$  for all  $v \in \mathbb{R}$  and  $g(u, 0) > 0$  for all  $u \in \mathbb{R}$ . Prove that if  $u(x, 0) > 0$  and  $v(x, 0) > 0$  for all  $x \in \Omega$ , then  $u > 0$  and  $v > 0$  for all  $(x, t) \in \Omega \times (0, T)$ . **Hint:** For partial credit, first consider the given problem for  $n = 0$ , i.e., without any  $x$  dependence of  $u, v$  (so that the PDEs reduce to ODEs).

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1. Let  $U = \{x \in \mathbb{R}^n : |x| > 1\}$ . Suppose that  $u \in C^2(U) \cap C(\bar{U})$  is a bounded solution of the following exterior Dirichlet problem:  $\Delta u = 0$  in  $U$ , and  $u = f$  on  $\Gamma = \{x \in \mathbb{R}^n : |x| = 1\}$  where  $f(x)$  is continuous on  $\Gamma$ .

(a) Consider  $n = 2$ . Show that there exists at most one solution of the above problem.

**Hint:** First, you might want to consider an appropriate maximum principle in  $U$  by using  $v = u \pm \epsilon \ln |x|$ .

(b) Now consider  $n = 3$ . Show that it is possible to have more than one bounded solutions of the above problem. What additional condition should you impose so that the solution  $u$  be unique in this case?

2. Consider the initial value problem

$$\begin{cases} (u_{x_1})^2 + \frac{1}{2}(u_{x_2})^2 = \alpha + x_1^2 & \text{in } \mathbb{R}^2, \\ u = x_1^2/2 & \text{on } \mathbb{R} \times \{x_2 = 0\}, \end{cases}$$

with  $u_{x_2}(x_1, 0) > 0$  for all  $x_1 \in \mathbb{R}$ ;  $\alpha$  is a positive constant ( $\alpha > 0$ ).

Find a smooth solution  $u(x_1, x_2)$  of this problem for any given  $\alpha > 0$ . For what value(s) of  $\alpha$  is  $u \equiv 0$  on the parabola  $\Gamma = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = -x_1^2\}$ ?

3. Consider the following Cauchy problem for  $u(x, t)$ :

$$\begin{cases} u_{tt} - g(x)\Delta u + u^{2m+1} = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ u = f, \quad u_t = h & \text{on } \mathbb{R}^3 \times \{t = 0\}, \end{cases}$$

where  $f(x)$  and  $h(x)$  are smooth with compact support,  $g(x) \in C(\mathbb{R}^3)$  with  $0 < d_1 \leq g(x) \leq d_2$  for all  $x \in \mathbb{R}^3$ , and  $m$  is a positive integer. Suppose that  $u$  is  $C^2$ .

(a) Find a conserved energy for this problem, assuming that  $u$  has compact support in  $x$  for each  $t$ . **Hint:** You are asked to find an expression of the form  $E(t) = \int_{\mathbb{R}^3} \rho(u, u_t, Du) dx$  that is conserved, where  $\rho \geq 0$  and  $\rho_t + \operatorname{div}_x q = 0$  for appropriate vector valued  $q$  (which you should find). You may want to first divide the given PDE by  $g$ .

(b) Assume that  $f$  and  $h$  are compactly supported in  $B(0, a)$ , i.e., in the ball of radius  $a$  ( $a > 0$ ). Show that  $u(x, t) \equiv 0$  for  $|x| > a + t\sqrt{d_2}$  for  $t \geq 0$ . **Hint:** You may use the result of part 3(a). You may want to appropriately integrate over the intersection of any slice  $\mathbb{R}^3 \times [0, T]$  with the exterior of the spacetime cone  $|x| - t\sqrt{d_2} = a$ ; and thereby derive a suitable bound for  $E(T)$ .

4. Let  $\Omega \subset \mathbb{R}^3$  be open, bounded and contain the origin. Show that  $u(x) = |x|^{-1}$  belongs to the Sobolev space  $W^{1,1}(\Omega)$ . **Note:** For full credit, you should study the weak derivative of  $u(x)$  by including the origin, not just the usual calculus derivative (away from 0).
5. Let  $\Omega$  be a smoothly bounded subset of the unit ball centered at  $x_0$  in  $\mathbb{R}^n$ .
- a) If  $x_0 = 0$ , prove that for every  $f \in L^2(\Omega)$  there exists a unique solution  $u \in H_0^1(\Omega)$  to the equation

$$-\frac{1}{9}\Delta u + 9u + x \cdot Du = f \quad (1)$$

- b) What happens if  $x_0$  is different from 0 and possibly large? Is it still true that for every  $f \in L^2(\Omega)$  there exists a unique solution  $u \in H_0^1(\Omega)$  to problem (1)? Prove your claim, or give a counterexample.
6. Consider the following initial boundary value problem for the vertical displacement  $u(x, t)$  of a thin plate:

$$\begin{cases} u_{tt} + \Delta^2 u = 0 & \text{in } \Omega \times (0, T), \\ u = \partial u / \partial \nu = 0 & \text{on } \partial \Omega \times [0, T], \\ u = g, u_t = h & \text{on } \bar{\Omega} \times \{t = 0\}, \end{cases}$$

where  $\Omega \subset \mathbb{R}^2$  is open, smoothly bounded with outward normal  $\nu$ ; and  $g \in H_0^2(\Omega)$ ,  $h \in L^2(\Omega)$

Give a weak formulation for this problem where  $u$  and the test function  $v$  satisfy  $\|D_t^\alpha u\|_{L^2} + \|D_x^\beta u\|_{L^2} \leq C$  and  $\|D_t^\alpha v\|_{L^2} + \|D_x^\beta v\|_{L^2} \leq C$  if  $|\alpha| \leq 1$  and  $|\beta| \leq 2$  together with additional boundary conditions which you must find and which must make sense in this low regularity setting.

Department of Mathematics Qualifying Exam  
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- (1) Let  $f \in C_c(\mathbb{R})$ . For every  $(x, y) \in \mathbb{R} \times \mathbb{R}_+$  let

$$u(x, y) = \int_{-\infty}^{\infty} \frac{y f(z)}{(x-z)^2 + y^2} dz.$$

Show that  $u$  satisfies

$$\begin{aligned} \Delta_x u &= 0, & \text{over } (x, y) \in \mathbb{R} \times \mathbb{R}_+, \\ \lim_{y \rightarrow 0^+} u(x, y) &= f(x), & \text{for every } x \in \mathbb{R}. \end{aligned}$$

- (2) Let  $\Omega \subset \mathbb{R}^D$  be a smooth bounded domain and  $T > 0$ . Let  $f \in C(\bar{\Omega} \times [0, T])$ ,  $g \in C(\partial\Omega \times [0, T])$ , and  $h_1, h_2 \in C(\bar{\Omega})$  with  $h_2(x) = h_1(x) = g(x, 0)$  for every  $x \in \partial\Omega$ . Suppose that  $u_1, u_2 \in C^2(\Omega \times (0, T)) \cap C^1(\bar{\Omega} \times [0, T])$  solve

$$\begin{aligned} \partial_t u_j - \Delta_x u_j &= f, & \text{over } (x, t) \in \Omega \times (0, T), \\ u_j &= g, & \text{over } (x, t) \in \partial\Omega \times (0, T), \\ u_j(x, 0) &= h_j(x), & \text{for every } x \in \Omega, \end{aligned}$$

Let  $v = (u_2 - u_1)/M$  where  $M = \max\{|h_2(x) - h_1(x)| : x \in \bar{\Omega}\}$ . Given the fact that  $w = \log(1 + v^2)$  satisfies

$$\partial_t w = \Delta_x w + 2 \frac{1 - v^2}{(1 + v^2)^2} |\nabla_x v|^2,$$

show that if  $u_1(x, T) = u_2(x, T)$  for every  $x \in \Omega$  then

$$u_1(x, t) = u_2(x, t), \quad \text{for every } (x, t) \in \Omega \times [0, T].$$

- (3) Let the function  $G$  be defined over  $\mathbb{R}^2 \times \mathbb{R}$  by

$$G(x, t) = \begin{cases} \frac{1}{2\pi} \frac{1}{\sqrt{t^2 - |x|^2}} & \text{for } t > |x|, \\ 0 & \text{otherwise.} \end{cases}$$

Show that  $\partial_t^2 G - \Delta_x G = \delta(x) \delta(t)$  — i.e. show that for every  $\phi \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R})$

$$\iint_{\mathbb{R}^2 \times \mathbb{R}} G (\partial_t^2 \phi - \Delta_x \phi) dx dt = \phi(0, 0)$$

You may take as given that  $G(x, t)$  satisfies the wave equation in the region  $t > |x|$ .  
Hint:  $G(x, t)$  is singular where  $t = |x|$ , so be careful when integrating by parts.

- (4) Let  $\Omega \subset \mathbb{R}^D$  be a smooth bounded domain. Let  $\partial\Omega = \Gamma_0 \cup \Gamma_1$  where  $\Gamma_0$  and  $\Gamma_1$  each have positive surface area and a smooth boundary. Consider the boundary-value problem

$$\begin{aligned} -\nabla_x \cdot (A(x)\nabla_x u) + c(x)u &= f(x) && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ n(x) (A(x)\nabla_x u) &= 0 && \text{on } \Gamma_1, \end{aligned}$$

where  $A$ ,  $c$ , and  $f$  are smooth over  $\bar{\Omega}$ ,  $n$  is the outward unit normal on  $\partial\Omega$ ,  $c$  is positive, and the  $D \times D$  matrix-valued function  $A$  is symmetric and satisfies the uniformly ellipticity condition — namely, that there exists an  $\alpha > 0$  such that

$$\xi^T A(x)\xi > \alpha|\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^D \text{ and } x \in \bar{\Omega}.$$

Give a weak formulation of this problem and use the Lax-Milgram theorem to show the existence of a weak solution in  $H^1(\Omega)$ .

- (5) Consider the initial-value problem

$$\begin{aligned} \partial_t u - \Delta_x u &= \sin(u), && \text{over } \mathbb{R}^D \times \mathbb{R}_+, \\ u|_{t=0} &= h, && \text{over } \mathbb{R}^D, \end{aligned}$$

where  $h \in L^2(\mathbb{R}^D)$ . Show that for every  $T > 0$  there exists a unique mild solution to this problem in  $C([0, T]; L^2(\mathbb{R}^D))$ . Recall that a mild solution is one that satisfies the integral equation

$$u(x, t) = g_t * h(x) + \int_0^t g_s * \sin(u(\cdot, t-s))(x) ds,$$

where  $*$  denotes convolution over  $\mathbb{R}^D$  and  $g_t$  is the fundamental solution for the linear heat equation, which is given by

$$g_t(x) = \frac{1}{(4\pi t)^{\frac{D}{2}}} \exp\left(-\frac{|x|^2}{4t}\right), \quad \text{for every } (x, t) \in \mathbb{R}^D \times \mathbb{R}_+$$

- (6) Let  $u$  be an absolutely continuous function over  $\mathbb{R}_+$  with a derivative  $u'$  such that

$$\int_0^\infty xu^2 dx < \infty, \quad \int_0^\infty x^3(u')^2 dx < \infty.$$

Show that

- $u$  is uniformly Hölder continuous away from the origin;
- $\lim_{x \rightarrow \infty} xu(x) = 0$ ;
- $x|u(x)|$  is uniformly bounded over  $\mathbb{R}_+$

Hint: Because  $u$  is absolutely continuous, the fundamental theorem of calculus implies

$$u(x_2) - u(x_1) = \int_{x_1}^{x_2} u'(x) dx.$$



DEPARTMENT OF MATHEMATICS  
PDE EXAM JANUARY 12 2009

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Instructions

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1 : a) Find the function  $G(x)$  that satisfies

$$-G'' + G = \delta(x) , \quad -\infty < x < +\infty \quad (1)$$

$$G(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty \quad (2)$$

where the first equation is understood in the sense of distributions. Explain, which problem you solve in order to find the proposed solution, and next show that the function  $G(x)$  that you obtained satisfies (1) in the sense of distributions.

b) Use part (a) to write down a formula for the solution of

$$-u'' + u = f(x) , \quad -\infty < x < +\infty \quad (3)$$

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2 : Consider the equation

$$u_{tt} - \Delta u + V(x)u = h(x) \quad x \in \Omega , \quad t > 0 \quad (1)$$

where  $\Omega$  is a bounded smooth domain and  $V(x)$  and  $h(x)$  are smooth functions with  $V(x) \geq 0$  on  $\Omega$ . The initial conditions are

$$u(x, 0) = f(x) , \quad u_t(x, 0) = g(x) \quad (2)$$

and the boundary conditions are

$$u + \frac{\partial u}{\partial n} = 0 , \quad x \in \partial\Omega . \quad (3)$$

- a) Assume that  $h = 0$  and derive an energy equation for an appropriately defined energy.  
 b) State and prove a uniqueness theorem using the energy equation in part (a).  
 c) Repeat parts (a) and (b) with  $h \neq 0$ .
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**3 :** Let  $\Omega \subset \mathbb{R}^n$  be a bounded open set with smooth boundary  $\partial\Omega$ . Show that  $C^2$  solutions of the initial-boundary value problem

$$u_t - \Delta u + \cos(u) = 0 \quad \text{in } \Omega \times (0, +\infty) \quad (1)$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, +\infty) \quad (2)$$

$$u(x, 0) = u_0(x) \quad \text{in } \Omega \quad (3)$$

are unique.

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**4 :** Prove that the function

$$u(x, y) = \frac{1}{2\pi} \log(\sqrt{x^2 + y^2}) \quad (1)$$

satisfies in the sense of distributions the equation,

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u = \delta_{2d}((x, y)) \quad (2)$$

where  $\delta_{2d}((x, y))$  is the Dirac measure in 2-space dimensions defined by its action on test functions

$$\left( \delta_{2d}((x, y)), \phi \right) = \phi(0, 0) \quad \forall \phi \in C_c^\infty(\mathbb{R}^2) \quad (3)$$


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**5 :** Let  $L$  be a strongly elliptic operator in  $\mathbb{R}^n$  with  $C^\infty$  coefficients. To keep notation simple, assume

$$Lu = - \sum_{i,j} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + c(x)u \quad (1)$$

and assume  $\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq |\xi|^2$  for all  $x$  and  $\xi$ .

- a) Prove there exists  $U$  a ball of radius  $\epsilon > 0$  centered at 0 such that for every  $f \in L^2(U)$ , the equation  $Lu = f$  has a unique solution in  $u \in H_0^1(U)$ .  
 b) Explain briefly why this result might not be true if  $\epsilon = 1$ .
- 

**6 :** Let  $B$  be the unit disc in  $\mathbb{R}^2$ ,  $\partial B$  its boundary.

a) Prove directly that there exist constant  $C$  such that for any  $u$  smooth up to the boundary of  $B$

$$\|u\|_{L^2(\partial B)}^2 \leq C \left( \|u\|_{L^2(B)}^2 + \|u\|_{L^2(B)} \left\| \frac{\partial u}{\partial r} \right\|_{L^2(B)} \right) \quad (1)$$

b) Prove that the trace operator  $T : H^1(B) \mapsto L^2(\partial B)$  is compact.

HINT : For part (b) you can assume the estimate

$$\|T(u)\|_{L^2(\partial B)}^2 \leq C \left( \|u\|_{L^2(B)}^2 + \|u\|_{L^2(B)} \|\nabla u\|_{L^2(B)} \right) \quad (2)$$

for all  $u \in H^1(B)$ , which follows easily from part (a) of this problem.

Qualifying PDE Exam, August 2008

1. Answer all six questions. Each will be assigned a grade from zero to ten.
2. Begin your answer to each question on a separate sheet. Write your code number on each page of your answer sheets. Do not use your name.
3. Keep scratch work on separate sheets, which should not be submitted.

1. Let  $B$  be the unit ball in  $\mathbf{R}^n$  and let  $u : [0, \infty) \times \bar{B} \rightarrow [0, M]$  be sufficiently smooth up to the boundary.  $M$  is a fixed number. Assume  $u$  satisfies  $u_t - \Delta u + u^{1/2} = 0$  in  $(0, \infty) \times B$  with Dirichlet conditions on the side boundary,  $u = 0$  on  $\partial B \times (0, \infty)$ . Prove there exists  $T$  such that  $u = 0$  on  $B \times [T, \infty)$ .

HINT. Let  $v$  solve  $v_t + v^{1/2} = 0$ ,  $v(0) = M$ , and look at  $w = v - u$ .

2. Find a smooth solution to  $u_t + (u_x)^4 = 0$ ,  $x \in \mathbf{R}$ ,  $t > 0$ , with initial conditions  $u(x, 0) = \frac{3}{4}x^{4/3}$ .

3. Show that there is at most one smooth (up to the boundary) solution to the equation  $u_{tt} + cu_t - u_{xx} = f(x, t)$  in the domain  $(0, 1) \times (0, \infty)$  with initial conditions  $u(x, 0) = g(x)$ ,  $u_t(x, 0) = h(x)$  and boundary conditions  $u(0, t) = u(1, t) = 0$ . Three points credit will be given for the easy case  $c \geq 0$ , but the case  $c < 0$  is also required.

4. Let  $U$  be a smoothly bounded domain in  $\mathbf{R}^n$ .

(a) Find the weak formulation of  $u - \Delta u = 0$  in  $U$  with boundary conditions  $u_n = g$  on  $\partial U$ . ( $u_n =$  normal derivative). Your weak formulation for  $u \in H^1(U)$  should be equivalent to the classical formulation under the additional assumption that all functions considered are smooth up to the boundary.

(b) Prove that for every  $g \in L^2(\partial U)$  the above weak formulation has a unique solution  $u \in H^1(U)$ .

5. Let  $\theta \in C^\infty(\mathbf{R})$ ,  $\theta(x) = 1$  if  $|x| > 2$ ,  $\theta(x) = 0$  if  $|x| < 1$ . Let  $\theta_\epsilon(x) = \theta(\frac{|x|}{\epsilon})$ . Let  $h \in L^2(\mathbf{R}^n)$ , smooth away from 0, and assume  $\|\theta_\epsilon \nabla h\|_{L^2(\mathbf{R}^n)} \leq 1$  for all  $\epsilon > 0$ .

Prove or disprove:  $h \in H^1(\mathbf{R}^n)$

(a) in the case  $n = 1$

(b) in the case  $n = 3$ .

HINT: You can use without proof Sobolev type inequalities

$\|u\|_{L^{p^*}(U_\epsilon)} \leq C \|u\|_{W^{1,p}(U_\epsilon)}$  where  $U_\epsilon = \{|x| > \epsilon\}$ ,  $0 < \epsilon < 1$ , and  $C$  independent of  $\epsilon$ .

6. Let  $U$  be an open set in  $\mathbf{R}^n$ , assume  $0 \in U$ .

(a) Prove that  $H = \{u \in L^2(U) \mid \Delta u = 0 \text{ in the sense of distribution theory}\}$  is a closed subspace of  $L^2(U)$ .

(b) Prove there exists a unique  $v \in H$  such that  $u(0) = \int_U u(x)v(x)dx$  for all  $u \in H$ . For part (b) you can assume the elements of  $H$  are automatically smooth.

(c) Find  $v$  explicitly if  $U$  is the unit ball centered at 0.

DEPARTMENT OF MATHEMATICS  
PDE EXAM JANUARY 12 2007

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Instructions

- 1 . Answer all six questions. Each will be assigned a grade from zero to ten.
- 2 . Begin your answer to each question on a separate answer sheet. Write your code number on each page of your answers sheets. Do not use your name.
- 3 . Keep scratch work on separate sheets, which should not be submitted.
- 4 . Carefully explain your steps. If you invoke a "well known" theorem, it is your responsibility to make clear which theorem you are using, and to justify its use.

1 : a) Find the function  $G(x)$  that satisfies

$$-G'' + G = \delta(x) , \quad -\infty < x < +\infty \quad (1)$$

$$G(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty \quad (2)$$

where the first equation is understood in the sense of distributions. Explain, which problem you solve in order to find the proposed solution, and next show that the function  $G(x)$  that you obtained satisfies (1) in the sense of distributions.

b) Use part (a) to write down a formula for the solution of

$$-u'' + u = f(x) , \quad -\infty < x < +\infty \quad (3)$$

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2 : Consider the equation

$$u_{tt} - \Delta u + V(x)u = h(x) \quad x \in \Omega , \quad t > 0 \quad (1)$$

where  $\Omega$  is a bounded smooth domain and  $V(x)$  and  $h(x)$  are smooth functions with  $V(x) \geq 0$  on  $\Omega$ . The initial conditions are

$$u(x, 0) = f(x) , \quad u_t(x, 0) = g(x) \quad (2)$$

and the boundary conditions are

$$u + \frac{\partial u}{\partial n} = 0 , \quad x \in \partial\Omega . \quad (3)$$

- a) Assume that  $h = 0$  and derive an energy equation for an appropriately defined energy.  
 b) State and prove a uniqueness theorem using the energy equation in part (a).  
 c) Repeat parts (a) and (b) with  $h \neq 0$ .
- 

**3 :** Let  $\Omega \subset \mathbb{R}^n$  be a bounded open set with smooth boundary  $\partial\Omega$ . Show that  $C^2$  solutions of the initial-boundary value problem

$$u_t - \Delta u + \cos(u) = 0 \quad \text{in } \Omega \times (0, +\infty) \quad (1)$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, +\infty) \quad (2)$$

$$u(x, 0) = u_0(x) \quad \text{in } \Omega \quad (3)$$

are unique.

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**4 :** Prove that the function

$$u(x, y) = \frac{1}{2\pi} \log(\sqrt{x^2 + y^2}) \quad (1)$$

satisfies in the sense of distributions the equation,

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u = \delta_{2d}((x, y)) \quad (2)$$

where  $\delta_{2d}((x, y))$  is the Dirac measure in 2-space dimensions defined by its action on test functions

$$\left( \delta_{2d}((x, y)), \phi \right) = \phi(0, 0) \quad \forall \phi \in C_c^\infty(\mathbb{R}^2) \quad (3)$$


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**5 :** Let  $L$  be a strongly elliptic operator in  $\mathbb{R}^n$  with  $C^\infty$  coefficients. To keep notation simple, assume

$$Lu = - \sum_{i,j} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + c(x)u \quad (1)$$

and assume  $\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq |\xi|^2$  for all  $x$  and  $\xi$ .

a) Prove there exists  $U$  a ball of radius  $\epsilon > 0$  centered at 0 such that for every  $f \in L^2(U)$ , the equation  $Lu = f$  has a unique solution in  $u \in H_0^1(U)$ .

b) Explain briefly why this result might not be true if  $\epsilon = 1$ .

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**6 :** Let  $B$  be the unit disc in  $\mathbb{R}^2$ ,  $\partial B$  its boundary.

a) Prove directly that there exist constant  $C$  such that for any  $u$  smooth up to the boundary of  $B$

$$\|u\|_{L^2(\partial B)}^2 \leq C \left( \|u\|_{L^2(B)}^2 + \|u\|_{L^2(B)} \left\| \frac{\partial u}{\partial r} \right\|_{L^2(B)} \right) \quad (1)$$

b) Prove that the trace operator  $T : H^1(B) \mapsto L^2(\partial B)$  is compact.

HINT : For part (b) you can assume the estimate

$$\|T(u)\|_{L^2(\partial B)}^2 \leq C \left( \|u\|_{L^2(B)}^2 + \|u\|_{L^2(B)} \|\nabla u\|_{L^2(B)} \right) \quad (2)$$

for all  $u \in H^1(B)$ , which follows easily from part (a) of this problem.

Department of Mathematics  
Qualifying Exam in Partial Differential Equations, January, 2008

Carefully explain your steps. If you invoke a "well-known" theorem, you must make clear which theorem you are using and justify its use. •

1. Let  $U$  be a smooth bounded domain in  $\mathbb{R}^n$  and let

$$Lu = - \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i}$$

be a uniformly elliptic operator with smooth coefficients. In particular,  $(a_{ij}(x))$  is positive definite.

Suppose  $L$  has a Green's function  $G(x, y)$ . In other words, if  $f \in C_0^\infty(U)$  and  $u(x) = \int_U G(x, y) f(y) dy$ , then  $u \in C^\infty(U)$ ,  $u$  is continuous up to the boundary and solves  $L(u) = f$ , with  $u = 0$  on the boundary of  $U$ . Assume  $G(x, y)$  is integrable in  $y$  for fixed  $x$ , and smooth for  $x \neq y$ .

Prove  $G(x, y) \geq 0$  for all  $x, y \in U$ ,  $x \neq y$ .

2. Let  $u$  be a  $C^2$  solution of  $u_{tt} - \Delta u = 0$  for  $t > 0$ ,  $x \in \mathbb{R}^2$ . Let  $H_r$  be the surface  $\{t^2 - |x|^2 = r^2, t > 0\}$ , with the usual surface measure of surfaces in  $\mathbb{R}^3$  (the kind you learned in Calculus). Assume the support of  $u$  intersects each  $H_r$ ,  $r \geq 1$ , in a compact set.

a/. Prove that the vector field

$$(V_0, V_1, V_2) = \left( \frac{1}{2}(u_t)^2 + \frac{1}{2}|\nabla u|^2, -u_t u_{x_1}, -u_t u_{x_2} \right)$$

satisfies

$$\partial_t V_0 + \partial_{x_1} V_1 + \partial_{x_2} V_2 = 0$$

b/. Find an "energy"

$$E(r) = \int_{H_r} \left( \frac{1}{2} ((u_t)^2 + |\nabla u|^2) \frac{t}{\sqrt{t^2 + |x|^2}} + \dots \right) dS$$

which is constant as  $1 \leq r < \infty$ . Prove that  $E(r) \geq 0$  for any such  $u$ .

3. Consider the equation

$$-(u_{x_1})^2 + (u_{x_2})^2 + x_2^2 = 0, \quad x_1 \in \mathbb{R}, x_2 > 0$$

with initial condition  $u(x_1, 0) = g(x_1)$ , with  $g(x_1) \in C^1$  strictly increasing. Also assume  $u_{x_2}(x_1, 0) \geq 0$ .

a) Find explicitly the characteristics  $x_1(y, s)$ ,  $x_2(y, s)$  starting at the point  $(y, 0)$ .

b) In the (admittedly easy) case  $g(x_1) = x_1$ , sketch the characteristics and write down an explicit solution. Your answer can involve  $I(s) = \int_0^s \sin^2(r) dr$  which you need not evaluate.



4. Let  $B = \{x \in \mathbb{R}^3 : |x| < \pi\}$ , and let  $u$  be smooth up to the boundary in  $B$ ,  $u = 0$  on the boundary of  $B$ . Let  $\Delta u + u = f$ .

Prove that

$$\int_B \frac{\sin |x|}{|x|} f(x) dx = 0$$

HINT. Show first that  $\frac{\sin |x|}{|x|}$  is an eigenfunction of  $\Delta$ . You may use without proof the expression of  $\Delta$  in spherical coordinates.

5. Let  $u \in W^{1,p}(\Omega)$ ,  $v \in W^{1,q}(\Omega)$  with  $1 < p, q < \infty$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ . Show that  $uv \in W^{1,1}(\Omega)$  and that its weak derivative satisfies the product rule

$$D(uv) = u Dv + v Du$$

6. Consider the boundary value problem

$$\begin{aligned} -\Delta u &= f & \text{in } \Omega \\ \frac{\partial u}{\partial n} + \alpha(x)u &= g(x) & \text{on } \partial\Omega \end{aligned}$$

where  $\Omega$  is a smooth, bounded domain in  $\mathbb{R}^n$  and  $\alpha(x) \in C(\bar{\Omega})$ .

a/ State the definition of a *weak solution* for the above boundary value problem. Make sure that you list the regularity requirements for the data  $f$  and  $g$  as well as the regularity of the (defined) weak solution  $u$ .

b/ Consider the case  $\alpha(x) > 0$  on  $\partial\Omega$ . Let  $u, v$  be two functions that satisfy your definition in a/., with the same data  $f$  and  $g$ . Prove or give a counterexample that  $u = v$  a.e in  $\Omega$ .

c/ Consider the case  $\alpha(x) = 0$  in  $\partial\Omega$ , and let  $u, v$  be as in b/. Prove or give a counterexample that  $u = v$  a.e. in  $\Omega$ .

DEPARTMENT OF MATHEMATICS  
QUALIFYING PDE EXAM, AUGUST 2007

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Instructions

- 1 . Answer all six questions. Each will be assigned a grade from zero to ten.
- 2 . Begin your answer to each question on a separate answer sheet. Write your code number on each page of your answers sheets. Do not use your name.
- 3 . Keep scratch work on separate sheets, which should not be submitted.
- 4 . Carefully explain your steps. If you invoke a "well known" theorem, it is your responsibility to make clear which theorem you are using, and to justify its use.

1 : It is given that the eigenvalues and eigenfunctions of the boundary value problem

$$y'' - 2\xi y' = \lambda y, \quad -\infty < \xi < +\infty$$

are  $\lambda_n = -2n$  and  $y_n(\xi) = H_n(\xi)$ ,  $n = 0, 1, \dots$ . The functions  $H_n(\xi)$  are called Hermite polynomials of degree  $n$  generated by the relation

$$H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n}{d\xi^n} (e^{-\xi^2}).$$

a) Show that the Hermite polynomials satisfy the orthogonality conditions,

$$\int_{-\infty}^{+\infty} H_m(\xi) H_n(\xi) e^{-\xi^2} d\xi = 0 \quad \text{for } n \neq m.$$

b) Consider the initial value problem

$$\begin{aligned} 4(t+1)\psi_t &= \psi_{\xi\xi} + 2\xi\psi_\xi & \xi \in \mathbf{R}, t > 0 \\ \psi(\xi, 0) &= \psi_0(\xi) & \xi \in \mathbf{R}. \end{aligned}$$

Solve this problem via the method of separation of variables.

Note : For part (b), you may use the fact that  $\{H_n e^{-\xi^2/2}\}_{n=0}^{\infty}$  is a complete orthogonal system in  $L^2(\mathbf{R})$ .

Hint : In part (b) consider eigenfunctions of the form  $H_n(\xi)e^{-a\xi^2}$  with the parameter  $a$  suitably chosen.

2 : For the canonical quasi-linear PDE  $u_t + uu_x = 0$  in  $-\infty < x < +\infty$  and  $t > 0$  :

a) Find (more than one)  $x$ -integrals that are conserved during the time evolution of this PDE (assuming that  $u \rightarrow 0$  sufficiently fast as  $|x| \rightarrow \infty$ ).

b) Find the (generalized) solution to that PDE when

$$u(x, 0) = \left\{ \begin{array}{ll} x & \text{if } 0 \leq x < a \\ a \frac{b-x}{b-a} & \text{if } a \leq x \leq b \end{array} \right\}$$

with  $u(x, 0) = 0$  for  $x < 0$  or  $x > b$ . Draw three sketches of the solution: (1) a characteristic sketch in the  $x, t$  plane, (2) two representative sketches in the  $x, u$  plane.

c) Show which of the conserved integrals are indeed preserved by your generalized solution.

3 : Let  $f \in W^{1,p}(\mathbf{R}^n)$  and assume that  $\Psi : \mathbf{R} \mapsto \mathbf{R}$  is a continuously differentiable function that satisfies  $\Psi(0) = 0$  and  $\sup_{s \in \mathbf{R}} |\Psi'(s)| \leq M$ , for some  $M > 0$ . Show that the composition  $g(x) := \Psi(f(x))$  belongs to  $W^{1,p}(\mathbf{R}^n)$ , and that the chain rule holds i.e.

$$D_i \Psi(f(x)) = \Psi'(f(x)) D_i f(x) .$$

4 : Assume that  $u \in H^1(\Omega)$  is a bounded weak solution of

$$-\sum_{i,j=1}^n (a^{ij} u_{x_i})_{x_j} = 0 \quad \text{in } \Omega$$

with  $a^{ij} \in C^1(\Omega)$  a symmetric, strictly positive matrix. Let  $\phi : \mathbf{R} \mapsto \mathbf{R}$  be convex and smooth, and set  $w := \phi(u)$ . Show that  $w$  is a weak subsolution; that is, it satisfies

$$B[w, u] := \int_{\Omega} \sum_{i,j=1}^n a^{ij} w_{x_i} v_{x_j} \, dx \leq 0$$

for all  $v \in H_0^1(\Omega)$  with  $v \geq 0$ .

5 : Suppose that you want to solve the one dimensional wave equation in the region  $x > at$ , where  $a$  is some constant such that  $0 \leq a < c$ , with zero initial data and Neumann type boundary condition i.e.

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 & \text{for } t > 0 \quad x > at \\ u(x, 0) = u_t(x, 0) &= 0 & \text{for } x > 0 \\ -u_x(at, t) + au_t(at, t) &= g(t) & \text{for } t \geq 0 . \end{aligned}$$

You may use any method you wish, or you may use the following suggestions. Recall that the Riemann-Green function for the one dimensional wave equation is

$$R(x, t) = \frac{1}{2c} \theta(t) \chi_{[-ct, ct]}(x)$$

where,

$$\theta(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}.$$

Make a guess that the solution is of the form

$$u(x, t) = \int_0^t R(x - as, t - s) h(s) ds$$

check that this is indeed a solution and find what function  $h(s)$  will satisfy the boundary condition  $u_x(at, t) - au_t(at, t) = g(t)$ .

**6 :** Suppose that  $u(x, t)$  is a smooth solution of the parabolic PDE,

$$\begin{aligned} u_t - \Delta u + c(x)u &= 0 & (x, t) \in \Omega \times (0, +\infty) \\ u &= 0 & (x, t) \in \partial\Omega \times (0, +\infty) \\ u(x, 0) &= f & x \in \Omega \end{aligned}$$

where  $\Omega$  is a bounded domain in  $\mathbf{R}^n$  with smooth boundary. Assume that  $c(x)$  satisfies the lower bound  $c(x) \geq c_0 > 0$  for some  $c_0$  positive constant. Prove that the solution  $u(x, t)$  satisfies the bound,

$$|u(x, t)| \leq C e^{-c_0 t} \quad (x, t) \in \Omega \times [T_1, T_2]$$

where  $0 < T_1 < T_2$  are two arbitrary positive times.

DEPARTMENT OF MATHEMATICS  
PDE EXAM JANUARY 12 2007

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Instructions

- 1 . Answer all six questions. Each will be assigned a grade from zero to ten.
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- 3 . Keep scratch work on separate sheets, which should not be submitted.
- 4 . Carefully explain your steps. If you invoke a "well known" theorem, it is your responsibility to make clear which theorem you are using, and to justify its use.

1 : a) Show that the function  $G(x) = (1/2)e^{-|x|}$  is the solution in the distribution sense of the equation,

$$-G'' + G = \delta(x) \quad ; \quad -\infty < x < +\infty .$$

b) Use part (a) to write a solution of

$$-u'' + u = f(x) \quad ; \quad -\infty < x < +\infty .$$

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2 : Consider the diffusion equation

$$\begin{aligned} \partial_t u - \partial_x(d(x)\partial_x u) &= 0 \quad : \quad t > 0 \quad 0 < x < 1 \\ u(0, x) &= f(x) \quad ; \quad u_x(0) = u_x(1) = 0 . \end{aligned}$$

where  $d(x)$  is a strictly positive continuously differentiable function.

a) Assume that you know all the eigenfunctions and eigenvalues of the operator

$$Lu := \partial_x(d(x)\partial_x u) \quad u_x(0) = u_x(1) = 0$$

Write a formula for the solution  $u(t, x)$  of the initial- boundary value problem above.

b) Does the solution  $u(t, x)$  tend to a limiting value  $u_\infty(x)$  as  $t \rightarrow \infty$ ? Carefully justify your answer.

c) If the limit in part b) exists, what is the limiting function  $u_\infty(x)$ ?

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3 : Let  $B$  be the unit ball (disc) in  $R^2$ . Prove that there exists some constant  $C$  such that for any function  $u \in H^1(B)$  the following inequality is true,

$$\|u\|_{L^2(B)} \leq C \left( \|\nabla u\|_{L^2(B)} + \|u\|_{L^2(\partial B)} \right) .$$

4 : a) Let  $B$  be the unit ball (disc) in  $R^2$  and  $f, g$  two continuous functions in  $\overline{B}$ . Find the weak formulation of the system

$$\begin{aligned} -\Delta u + u + v_x &= f \\ -\Delta v + v - u_x &= g , \end{aligned}$$

where  $(x, y) \in B$  with Dirichlet boundary conditions  $u = v = 0$  for  $(x, y) \in \partial B$ .

b) Show that the weak problem has a unique solution.

5 : The solution of the wave equation in three space dimensions, namely the problem

$$\begin{aligned} u_{tt} - \Delta_3 u &= f \quad ; \quad t > 0 \quad x \in R^3 \\ u(0, x) &= u_t(0, x) = 0 \end{aligned}$$

is given by the formula,

$$u(t, x) = \int_0^t \frac{1}{4\pi s} \int_{|y-x|=s} f(s, y) dS(s, y) ds ,$$

where  $dS(s, y)$  is surface measure on the sphere of radius  $s$ . Use this formula in order to solve the same problem in two space dimensions,

$$\begin{aligned} u_{tt} - \Delta_2 u &= f \quad ; \quad t > 0 \quad x \in R^2 \\ u(0, x) &= u_t(0, x) = 0 . \end{aligned}$$

6 : Let  $B$  be the unit ball in  $R^3$ . Consider the eigenvalue problem,

$$\begin{aligned} -\Delta u &= \lambda u \quad x \in B \\ \partial_\nu u + u &= 0 \quad x \in \partial B , \end{aligned}$$

where  $\partial_\nu$  denotes the normal derivative on the boundary  $\partial B$ . Show that all eigenvalues are positive and that eigenfunctions corresponding to different eigenvalues are orthogonal to each other.

Department of Mathematics  
Graduate Written Examination in Partial Differential Equations  
August 2006

Instructions

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- 3 . Keep scratch work on separate sheets, which should not be submitted.
- 4 . Carefully explain your steps. If you invoke a "well known" theorem, it is your responsibility to make clear which theorem you are using, and to justify its use.

1 : Assume  $f \in C^1(\mathbb{R})$ . Solve the initial value problem, for  $x \in \mathbb{R}$  and  $t \in \mathbb{R}$ . For what values of  $t$  is the solution defined?

$$u_t + xu_x - u^2 = 0$$
$$u(x, 0) = f(x) .$$

2 : Let  $u(\mathbf{x})$  be harmonic for  $\mathbf{x} \in \mathbb{R}^3$ . Suppose that  $\nabla u \in L^2(\mathbb{R}^3)$ . Prove that  $u$  is a constant.

3 : Let  $\psi(s)$  be a  $C^k$  function with  $k \geq 2$  and compact support in  $[1, 2]$ . For  $\mathbf{x} \in \mathbb{R}^3$  with  $\mathbf{x} \neq 0$  and  $t \in \mathbb{R}$  set

$$u(\mathbf{x}, t) = \frac{\psi(t+r)}{r}, \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

- a) Verify that  $u$  is a solution of the wave equation  $u_{tt} - \Delta u = 0$  for  $t < 1$ .
- b) How is the solution continued beyond  $t = 1$ ? Provide a formula.
- c) Does  $u$  continue to be of class  $C^k$  in  $\mathbf{x}$  and  $t$ ? In particular, check the regularity in the interval  $1 < t < 2$  at  $r = 0$ .
- d) What is the energy of  $u$ ? Verify directly that the energy is conserved.

4 : Assume  $f : \mathbb{R} \mapsto \mathbb{R}$  is  $C^1$  and  $f'$  is bounded. Suppose  $U$  is a bounded open subset of  $\mathbb{R}^n$  and  $u \in W^{1,p}(U)$  for some  $p \in (1, +\infty)$ . Show that  $f(u) \in W^{1,p}(U)$ .

5 : Assume  $U$  is an open bounded subset of  $R^n$  with  $C^1$  boundary. Suppose that the boundary is the disjoint union of two smooth subsets of positive measure, i.e.,  $\partial U = C_1 \cup C_2$  with  $C_1 \cap C_2 = \phi$ , and  $|C_1| \neq 0$  and  $|C_2| \neq 0$ . Consider the following problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } U \\ u &= 0 \quad \text{on } C_1 \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } C_2 \end{aligned}$$

Find the weak formulation of this problem in an appropriate Sobolev space. Show that the problem above has a unique weak solution for a given function  $f \in L^2(U)$ . Show in what sense the weak solution satisfies the boundary condition  $\partial u / \partial \nu = 0$  on the part of the boundary  $C_2$ .

6 : Let  $U$  be an open bounded subset of  $R^n$  with  $C^\infty$  boundary  $\partial U$ . For  $f \in L^2(U)$  consider the solution of

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } U \times [0, +\infty) \\ u(\mathbf{x}, 0) &= f(\mathbf{x}) \quad \text{and } u = 0 \quad \text{on } \partial U \end{aligned}$$

a) Show that for  $t > 0$  the function  $\mathbf{x} \mapsto u(\mathbf{x}, t)$  is  $C^\infty$

b) Show that  $\|u(\cdot, t)\|_\infty \rightarrow 0$  as  $t \rightarrow +\infty$ .

Hint : Use the eigenfunction expansion and the fact that the eigenfunctions are  $C^\infty$ . State clearly what estimates you need on the eigenfunctions.



Department of Mathematics  
Graduate Written Examination in Partial Differential Equations  
January 13 2006

Instruction

- 1 . Answer all six questions. Each will be assigned a grade from zero to ten.
- 2 . Begin your answer to each question on a separate answer sheet. Write your code number on each page of your answers sheets. Do not use your name.
- 3 . Keep scratch work on separate sheets, which should not be submitted.
- 4 . Carefully explain your steps. If you invoke a "well known" theorem, it is your responsibility to make clear which theorem you are using, and to justify its use.

**Problem 1.** Consider the heat conduction problem with Dirichlet type boundary condition,

$$\begin{aligned}u_t - \Delta u &= 0 \quad , \quad \mathbf{x} \in U \quad , \quad t > 0 \\u(0, \mathbf{x}) &= f(\mathbf{x}) \quad , \quad \mathbf{x} \in U \\u(t, \mathbf{x}) &= 0 \quad , \quad \mathbf{x} \in \partial U \quad , \quad t \geq 0 \quad ,\end{aligned}$$

where  $U$  is a bounded subset of  $R^n$  with smooth boundary. Let  $\Gamma \subset \partial U$  be a portion of the boundary of the domain  $U$ . The total heat flow through  $\Gamma$  is given by the integral

$$Q := \int_0^{+\infty} \left( \int_{\Gamma} \frac{\partial u}{\partial \nu} dS \right) dt$$

where  $\partial u / \partial \nu$  is the derivative of  $u$  in the direction normal to the boundary. Show that the total heat flow through  $\Gamma$  is also given by the integral,

$$Q = - \int_U f(\mathbf{x}) h(\mathbf{x}) d\mathbf{x}$$

where  $h$  is the harmonic function,

$$\begin{aligned}\Delta h &= 0 \quad , \quad \mathbf{x} \in U \\h(\mathbf{x}) &= \begin{cases} 1 & \text{if } \mathbf{x} \in \Gamma \\ 0 & \text{if } \mathbf{x} \in \partial U \setminus \Gamma \end{cases} .\end{aligned}$$

Hint : Apply Green's identity to the pair of functions  $u$  and  $h$ .

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**Problem 2.** Find the solution in the first quadrant  $x > 0$  and  $t > 0$  of the Wave equation with boundary conditions at  $x = 0$ ,

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \quad , \quad x > 0 \quad , \quad t > 0 \\ u(0, x) &= f(x) \quad , \quad u_t(0, x) = g(x) \\ u_t(t, 0) &= au_x(t, 0) \quad , \quad a \neq -c \end{aligned}$$

where  $f(x)$  and  $g(x)$  are  $C^2$  functions which vanish near  $x = 0$ . Show that no solution exists in general if  $a = -c$ . Hint : The solution can be written in general as  $u = F(x + ct) + G(x - ct)$ .

**Problem 3.** Let  $n \geq 2$  and  $0 < \lambda < 1$ . Consider the following equation

$$Lu := \Delta u + \left( -1 + \frac{n-1}{1-\lambda} \right) \frac{x_i x_j}{|\mathbf{x}|^2} D_{ij} u = 0 \quad ,$$

where  $D_{ij} = \partial^2 / \partial x_i \partial x_j$ .

a) Check that the equation above is uniformly elliptic.

b) Show that  $u = 1$  and  $w = |\mathbf{x}|^\lambda$  are both solutions of the equation i.e.  $Lu = 0$  and  $Lw = 0$  in  $B := B(0, 1)$  which are both in  $W^{2,2}(B)$  and have the same boundary data. (Make sure that  $w$  is indeed in  $W^{2,2}(B)$ .)

**Problem 4.** A weak solution of the biharmonic equation,

$$\begin{aligned} \Delta^2 u &= f \quad , \quad \mathbf{x} \in U \\ u &= \frac{\partial u}{\partial \nu} = 0 \quad , \quad \mathbf{x} \in \partial U \quad , \end{aligned}$$

is a function  $u \in H_0^2(U)$  such that

$$\int_U \Delta u \Delta v \, dx = \int_U f v \, dx \quad \text{for all } v \in H_0^2(U) \quad .$$

Assume that  $U$  is a bounded subset of  $R^n$  with smooth boundary and use the weak formulation of the problem to prove the existence of a unique weak solution.

**Problem 5.** Assume that  $f \in C^1(R)$ . Starting from the identity

$$f^2(x) = \int_x^{x+1} \left( f(y) - \int_x^y f'(s) \, ds \right)^2$$

and using the inequality  $2ab \leq a^2 + b^2$  show that

$$\sup |f(x)| \leq \sqrt{2} \left( \int |f|^2 + \int |f'|^2 \right)^{1/2}.$$

b) Use induction to show that if  $\mathbf{x} := (x_1, x_2, \dots, x_n)$  then

$$\sup |f(\mathbf{x})| \leq 2^{n/2} \left( \sum_{|\alpha| \leq n} \int |D^\alpha f|^2 \right)^{1/2}.$$

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**Problem 6.** Consider the initial value problem

$$u_t + u(u-1)u_x = 0 \quad ; \quad u(0, x) = f(x),$$

where  $f(x) = 2$  for  $x \leq 0$ ,  $f(x) = 2 - x$  for  $0 \leq x \leq 1$ , and  $f(x) = 0$  for  $x \geq 1$ .

a) Sketch the characteristics emanating from the interval  $[0, 1]$ .

b) What is the first time, say  $t_*$ , such that the solution is no longer single valued for  $t > t_*$ ?

What is the  $x$  coordinate of the point where this occurs?

c) Find a formula for the solution which is valid in the region  $\{2t < x < 2\} \cap U$  where  $U$  is a suitable neighborhood of  $\{(x, 0) : 0 \leq x \leq 1\}$ .

d) Find a formula for the curve of discontinuity that issues from the point  $(t_*, x_*)$ .

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Department of Mathematics  
 Graduate Written Examination in Partial Differential Equations  
 August, 2005

- Answer all six questions. Each will be assigned a grade from 0 to 10.
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**1:** a) Let  $c > 0$  be constant. State what is meant by a *weak* solution of the PDE  $u_t + cu_x = 0$  on  $\mathbb{R}^2$ .

b) Show that if  $f(\cdot)$  is a continuous function of one real variable, then  $f(x - ct)$  is a weak solution of the PDE  $u_t + cu_x = 0$  on  $\mathbb{R}^2$ .

Hint : In the weak formulation use new coordinates  $s = x - ct$  and  $y = x$ .

**2:** Suppose that you want to solve the following Neumann problem in two space dimensions,

$$\begin{aligned} -\Delta u &= f \quad ; \quad x \in D \\ \frac{\partial u}{\partial \nu} &= g \quad ; \quad x \in \partial D \end{aligned}$$

where  $D \subset \mathbb{R}^2$  is a bounded open set with smooth boundary  $\partial D$ . We let  $\partial u / \partial \nu$  denote the derivative of  $u$  in the outward normal direction on the boundary of  $D$ .

a) What compatibility condition must the data  $f$  and  $g$  satisfy to ensure the existence of a solution?

b) Derive a weak formulation of the problem above using an appropriate space of functions.

c) Use functional analysis (Lax-Milgram) to prove that the weak problem has a solution, provided the compatibility condition of part a) is satisfied. Show that the solution is unique up to a constant.

You may use the fact that a Poincare type inequality holds for functions in  $H^1(D)$  that have average value zero.

**3:** Let  $D \subset \mathbb{R}^3$  be a bounded open set, and let  $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$  be a  $C^1$  function such that  $\phi(x) > 0$  for  $x \in D$ , and  $\phi(x) = 0$  for  $x \in \partial D$ . Let  $S_\lambda = \{(x, t) : t = \lambda\phi(x)\}$  denote a hypersurface in  $(x, t)$  space with  $\lambda \geq 0$  and denote

$$Q_\lambda := \frac{1}{2} (u_t^2 + c^2 |\nabla u|^2) + \lambda c^2 u_t \nabla u \cdot \nabla \phi .$$

a) Show that the integral

$$E_\lambda := \int_{S_\lambda} Q_\lambda dS$$

is independent of  $\lambda$ . Hint : Integrate  $u_t(u_{tt} - c^2\Delta u)$  over the space-time domain  $\{(x, t) : x \in D, 0 < t < \lambda\phi(x)\}$ .

b) A smooth hypersurface  $S$  in  $\mathbb{R}^4$  is *space-like* for the wave equation  $u_{tt} - c^2\Delta u = 0$  if the unit normal  $\nu = (\nu_x, \nu_t)$  satisfies  $c|\nu_x| < |\nu_t|$  at each point of  $S$ . Show that  $Q_\lambda$  is positive definite as a quadratic form in  $(u_t, \nabla u)$  on  $S_\lambda$  when  $S_\lambda$  is space-like.

c) Show that the initial data on  $S_0$  uniquely determine the solution on  $S_\lambda$  for all sufficiently small  $\lambda$ .

**4:** Let  $U \subset \mathbb{R}^n$  be a bounded open set with smooth boundary  $\partial U$ . Let  $x \rightarrow P(x) : \bar{U} \rightarrow \mathbb{R}^{n \times n}$  be a smooth family of symmetric  $n \times n$  real matrices that is uniformly positive definite. Let  $c(x) \geq 0$  and  $f(x)$  be smooth functions on  $\bar{U}$ . Define the functional

$$I[u] = \int_U \left( \frac{1}{2} \langle \nabla u, P(x) \nabla u \rangle + \frac{1}{2} c(x) u^2 - f(x) u \right) dx$$

where  $\langle \cdot \rangle$  is the scalar product on  $\mathbb{R}^n$ . Suppose that  $u \in C^2(U) \cap C^0(\bar{U})$  is a minimizer of this functional subject to the Dirichlet condition

$$u = g \text{ on } \partial U, \text{ with } g \text{ continuous.}$$

a) Show that  $u$  satisfies the variational equation

$$\int_U \langle \nabla u, P(x) \nabla v \rangle + c(x) uv = \int_U f v$$

for any

$$v \in V = \{v \text{ Lipschitz} : v = 0 \text{ on } \partial U\}$$

b) What is the PDE satisfied by  $u$ ?

c) Suppose  $f, g \geq 0$ . Show that  $v = \min(u, 0)$  is an admissible test function, and use it to conclude that  $u \geq 0$ .

*Hint:* You may use the fact that  $\nabla v = \chi_{\{u < 0\}} \nabla u$  a.e.

d) Show that there can be only one minimizer of  $I[u]$  or, equivalently, only one solution  $u \in C^2(U) \cap C^0(\bar{U})$  of the corresponding PDE.

5: Suppose that you want to solve the Dirichlet type problem for the unit disc in two space dimensions,

$$\begin{aligned} -\Delta u &= 0 & \text{for } x \in \{x : |x| < 1\} \\ u &= f & \text{on } |x| = 1 . \end{aligned}$$

a) Show that the solution in polar coordinates  $(r, \theta)$  is given by an infinite sum of the form

$$u(r, \theta) = \sum_{n=0}^{+\infty} r^n (A_n \cos(n\theta) + B_n \sin(n\theta))$$

and give formulas for the coefficients  $A_n$  and  $B_n$  in terms of  $f(\theta)$ .

b) Compute the “energy” i.e. show that

$$E := \int_{\{|x|<1\}} |\nabla u|^2 dx = \pi \sum_{n=1}^{\infty} n(A_n^2 + B_n^2) .$$

6: Let  $\Omega \subset \mathbb{R}^n$  be an open bounded set with smooth boundary  $\partial\Omega$ . Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a  $C^1$  function with  $|f'|$  bounded. Show that  $C^2$  solutions of the initial-boundary value problem

$$\begin{aligned} u_t(x, t) - \Delta u + f(u) &= 0 & \text{in } \Omega \times (0, \infty), \\ u &= 0 & \text{on } \partial\Omega \\ u(x, t) &= u_0(x) & \text{in } \Omega \end{aligned}$$

are unique.

**Department of Mathematics**  
**Graduate Written Examination in Partial Differential Equations**  
**January, 2005**

Instructions

- Answer all six questions. Each will be assigned a grade from 0 to 10.
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- Keep scratch work on separate sheets, which should not be submitted.
- Carefully explain your steps. If you invoke a “well known” theorem, it is your responsibility to make clear exactly which theorem you are using, and to justify its use.

1. Let  $u(x, y)$  be a harmonic function on  $\mathbb{R}^2$  and suppose that

$$\int \int_{\mathbb{R}^2} |\nabla u|^2(x, y) dx dy < \infty.$$

Show that  $u$  is a constant function.

2. Let  $u(x, t)$  be a piecewise smooth weak solution of the conservation law

$$u_t + f(u)_x = 0 \quad -\infty < x < \infty, \quad t > 0.$$

- a) Derive the Rankine-Hugoniot conditions at a discontinuity of the solution.  
 b) Find a piecewise smooth solution of the IVP

$$u_t + (u^2 + u)_x = 0, \quad -\infty < x < \infty, \quad t > 0$$

$$u(x, 0) = \begin{cases} 1, & x < 0 \\ -2, & x > 0. \end{cases}$$

3. Consider the evolution equation with initial data

$$u_{tt} - u_{xx} + u_t = (u_{xt}^3)_x \quad \text{for } 0 < x < 1, \quad t > 0.$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad u(0, t) = u(1, t) = 0.$$

- a) What energy quantity is appropriate for this equation? Is it conserved, or is it dissipated?  
 b) Show that  $C^3$  solutions of this problem are unique.

4. Let  $\Omega \subset \mathbb{R}^n$  be a bounded open set with smooth boundary  $\partial\Omega$ . Consider the initial boundary value problem for  $u(x, t)$ ,

$$u_t - \nabla \cdot (a(x)\nabla u) + b(x)u = q, \quad x \in \Omega, \quad t > 0$$

$$u(x, 0) = f(x), \quad x \in \Omega$$

$$u_t + \partial u / \partial n + u = 0, \quad x \in \partial\Omega, \quad t > 0.$$

The function  $\partial u / \partial n$  is the exterior normal derivative. Assume that  $a, b \in C^1(\bar{\Omega})$  and that  $a(x) \geq 0$  for  $x \in \Omega$ . Show that smooth solutions of this problem are unique.

5. Let  $\Omega \subset \mathbb{R}^n$  be a bounded open set with smooth boundary. Let  $K = \{u \in H_0^1(\Omega) : u \geq 0 \text{ in } \Omega\}$ .

Let  $f \in L_2(\Omega)$  and define the functional

$$I(u) = (1/2) \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} f u dx.$$

Show that  $u$  is a minimizer of  $I$  over  $K$  if and only if  $u$  satisfies the variational inequality

$$\int_{\Omega} \nabla u \cdot \nabla(u - v) dx \leq \int_{\Omega} f(v - u) dx \quad \text{for all } v \in K.$$

Hint: If  $u, v \in K$ , then  $g(t) \equiv (1 - t)u + tv \in K$  for  $0 \leq t \leq 1$  and if  $u$  is a minimizer of  $I$  over  $K$ , then  $I(g(t)) \geq I(g(0)) = I(u)$  for  $0 \leq t \leq 1$ .

6. We shall write points in  $\mathbb{R}^n$  as  $(x, y)$  where  $x = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ , and  $y \in \mathbb{R}$ .

a) Let  $\Omega = \{(x, y) : |x| \leq 1 \text{ and } 0 \leq y \leq 1\}$ . Let  $\Gamma = \{x \in \mathbb{R}^{n-1} : |x| \leq 1\}$ . Prove there is a constant  $C > 0$ , independent of  $u$ , such that for all  $u \in H^1(\Omega)$ ,

$$\|u(\cdot, 0)\|_{L_2(\Gamma)} \leq C \|u\|_{H^1(\Omega)}.$$

b) Refine the argument to show that the same inequality (with perhaps a different constant  $C$ ) holds when  $\Omega = \{(x, y) : |x|^2 + y^2 \leq 1, y \geq 0\}$



**Department of Mathematics**  
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**August, 2004**

Instructions

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1. Let  $f \in C^1(\mathbb{R}^n)$  and suppose that for each open ball  $B$  that there exists a solution of the boundary value problem

$$-\Delta u = f \quad \text{in } B, \quad \partial u / \partial n = 0 \quad \text{on } \partial B.$$

Show that  $f \equiv 0$ .

2. Suppose that  $u(x, t)$  is a smooth solution of

$$\begin{aligned} u_t + uu_x &= 0 \quad \text{for } x \in \mathbb{R}, t > 0 \\ u(x, 0) &= f(x) \quad \text{for } x \in \mathbb{R}. \end{aligned}$$

Assume that  $f$  is a  $C^1$  function such that

$$f(x) = \begin{cases} 0 & \text{for } x < -1 \\ 1 & \text{for } x > 1 \end{cases} \quad \text{and} \quad f'(x) > 0 \quad \text{for } |x| < 1.$$

Show that for  $t > 0$ ,

$$\lim_{r \rightarrow \infty} u(rx, rt) = \begin{cases} 0 & \text{for } x < 0 \\ x/t & \text{for } 0 < x < t \\ 1 & \text{for } x > t \end{cases}$$

3. Let  $D \subset \mathbb{R}^2$  be the open disk  $D = \{x^2 + y^2 < 1\}$  and let  $C = \{x^2 + y^2 = 1\}$ . Using polar coordinates, let  $u_r = \partial u / \partial r$  and  $u_\theta = \partial u / \partial \theta$ .

a) Find the weak formulation of the oblique derivative problem

$$-\Delta u + \lambda u = f \quad \text{in } D, \quad u_r + u_\theta = 0 \quad \text{on } C. \tag{1}$$

In other words, find a bilinear form  $a_\lambda(u, v)$  continuous on  $H^1(D) \times H^1(D)$  such that

$$a_\lambda(u, v) = \int_D f v \, dx dy \quad \text{for all } v \in H^1(D) \quad (2)$$

is equivalent to (1) if  $u \in C^2(\bar{D})$ . You may assume the trace theorem

$$\int_C uv_\theta \, d\theta \leq C \|u\|_{H^1(D)} \|v\|_{H^1(D)} \quad \text{for } u, v \in H^1(D).$$

b) Show that for  $f \in L_2(D)$  and for  $\lambda > 0$ , there is unique solution  $u \in H^1(D)$  of (2).

4. For  $x \in \mathbb{R}$ ,  $t \geq 0$ , solve the problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \quad \text{for } |x| < t \\ u(-t, t) &= \alpha(t) \quad \text{for } t \geq 0 \\ u_t(t, t) &= \beta(t) \quad \text{for } t \geq 0, \end{aligned}$$

where  $\alpha$  and  $\beta$  are smooth functions.

5. Let  $\Omega$  be a bounded open set of  $\mathbb{R}^n$  with smooth boundary  $\partial\Omega$ . Let  $u(x, t) \in C^2(\Omega \times (0, \infty))$ , with  $u$  continuous on  $\bar{\Omega} \times [0, \infty)$ , be a solution of the initial-boundary value problem

$$\begin{aligned} u_t - \Delta u &= \sin(u) \quad \text{for } x \in \Omega, t > 0 \\ u(x, t) &= 0 \quad \text{for } x \in \partial\Omega \\ u(x, 0) &= f(x) \quad \text{for } x \in \Omega. \end{aligned}$$

Show that if  $f(x) \leq 1$ , then  $u(x, t) \leq e^t$  for all  $x \in \Omega, t > 0$ . Hint: Note that  $\sin x = x(\sin x)/x$  and that  $(\sin x)/x$  is bounded. Look for a way to use the maximum principle.

6. Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  have two continuous derivatives, with  $F(0) = 0$  and suppose that there is constant  $M$  such that  $|F(x)| + |F'(x)| + |F''(x)| \leq M$  for all  $x \in \mathbb{R}$ . Let  $u \in C^2(\mathbb{R})$  with compact support and set  $v(x) = F(u(x))$ .

Show that there is a constant  $C$ , independent of  $u$  such that

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$$\|v\|_{H^2(\mathbb{R})} \leq C \|u\|_{H^2(\mathbb{R})} (1 + \|u\|_{H^1(\mathbb{R})}).$$

Department of Mathematics  
 Graduate Written Examination in Partial Differential Equations  
 January, 2004

Instructions

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1. Let  $\Omega \subset \mathbb{R}^n$  be a connected, bounded, open set with smooth boundary  $\partial\Omega$ . Let  $u(x)$  be a smooth solution of

$$\Delta u = u^3 - u, \quad \text{in } \Omega \quad u = 0 \quad \text{on } \partial\Omega.$$

Show that  $|u(x)| \leq 1$  for all  $x \in \Omega$ .

2. Consider the initial value problem (IVP)

$$\begin{aligned} u_t + uu_x &= 0 \quad \text{for } x \in \mathbb{R}, 0 \leq t < 1. \\ u(x, 0) &= f(x) \quad \text{for } x \in \mathbb{R}. \end{aligned}$$

Find functions  $\alpha = \alpha(t)$  and  $\beta = \beta(t)$  so that whenever  $u$  is a solution of the above IVP, the function

$$v(x, t) = \alpha(t)[x + u(\alpha(t)x, \beta(t))]$$

satisfies

$$\begin{aligned} v_t + vv_x &= 0 \quad \text{for } x \in \mathbb{R}, t \geq 0 \\ v(x, 0) &= x + f(x) \quad \text{for } x \in \mathbb{R}. \end{aligned}$$

3. Consider the initial value problem (IVP)

$$\begin{aligned} u_t &= \partial_x(u_x + (\sin x)u), & u(x, 0) &= f(x) \\ u(x + 2\pi, t) &= u(x, t) \quad \text{for all } x \in \mathbb{R}, t \geq 0. \end{aligned}$$

a) Using Fourier series, show that if  $g \in C^1(\mathbb{R})$  and has period  $2\pi$ , and  $\int_0^{2\pi} g dx = 0$ , then the Poincare inequality is

$$\int_0^{2\pi} |g(x)|^2 dx \leq \int_0^{2\pi} |g'(x)|^2 dx.$$

b) Show that if  $u$  is a  $C^2$  solution of the IVP, then for all  $t \geq 0$ ,

$$\int_0^{2\pi} u(x, t) dx = \int_0^{2\pi} f(x) dx.$$

c) Show that if  $w$  is a  $C^2$  solution of the IVP with  $\int_0^{2\pi} w(x, 0) dx = 0$ , then

$$\lim_{t \rightarrow \infty} \int_0^{2\pi} |w(x, t)|^2 dx = 0.$$

d) Let

$$v(x) = \left[ \frac{\int_0^{2\pi} f(x) dx}{\int_0^{2\pi} e^{\cos x} dx} \right] e^{\cos x}.$$

Show that if  $u$  is a solution of the IVP, then

$$\lim_{t \rightarrow \infty} \int_0^{2\pi} |u(x, t) - v(x)|^2 dx = 0.$$

4. Let  $\Omega \subset \mathbb{R}^d$  be a bounded open set with smooth boundary  $\partial\Omega$ . Let  $u \in C^2(\bar{\Omega})$  with  $u = 0$  on  $\partial\Omega$  and let

$$f = -\Delta u + u.$$

Let  $v \in C^2(\bar{\Omega})$  satisfy

$$-\Delta v + v = f \quad \text{for } x \in \Omega, \quad \partial u / \partial n = 0 \quad \text{on } \partial\Omega.$$

Let  $Q$  be the quadratic form

$$Q(w) = (1/2) \int_{\Omega} [w^2 + |\nabla w|^2] dx - \int_{\Omega} f w dx$$

a) Show by a direct calculation that for an appropriate Sobolev space  $H$ ,  $Q(w) \geq Q(v)$  for all  $w \in H$ .

b) Using this result, show that

$$\int_{\Omega} [u^2 + |\nabla u|^2] dx \leq \int_{\Omega} [v^2 + |\nabla v|^2] dx.$$


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5. Let  $f(x)$  be continuous on  $\mathbb{R}$  and suppose that

$$\int f(x)\varphi''(x) dx = 0$$

for all  $\varphi \in C_0^\infty(\mathbb{R})$ . Show that  $f(x) = ax + b$  for some constants  $a$  and  $b$ .

6. Let  $u(x, t) \in C^3(\mathbb{R}^2)$ . Show that  $u$  satisfies the wave equation  $u_{tt} - u_{xx} = 0$  if and only if  $u$  satisfies the difference equation

$$u(x - k, t - h) + u(x + k, t + h) = u(x + h, t + k) + u(x - h, t - k)$$

for all  $h, k \geq 0$ . (Hint: For the 'if' part set  $h = 0$  and use a Taylor expansion.)

Department of Mathematics  
 Graduate Written Examination in Partial Differential Equations  
 August, 2003

Instructions

- Answer all six questions. Each will be assigned a grade from 0 to 10.
- Begin your answer to each question on a separate answer sheet. Write your code number on each page of your answers. Do not use your name.
- Keep scratch work on separate sheets, which should not be submitted.
- Carefully explain your steps. If you invoke a “well known” theorem, it is your responsibility to make clear exactly which theorem you are using, and to justify its use.

1. Let  $\Omega$  be a bounded, open set in  $\mathbb{R}^3$  with smooth boundary  $\partial\Omega$ . Let  $G(x, y)$  be the Green’s function for the Laplacian with zero boundary conditions. Show that  $G(x, y) > 0$  for  $x, y \in \Omega$ ,  $x \neq y$ .

2. Consider the initial value problem for  $(x, t) \in \mathbb{R}^2$ ,

$$u_t + u^2 u_x = 0, \quad u(x, 0) = x.$$

a) Sketch the characteristics emanating from  $(x_0, 0)$  for several values of  $x_0 < 0$  and for several values of  $x_0 > 0$ .

b) Find the solution in closed form. Verify that the initial condition is satisfied. What is the set of  $(x, t) \in \mathbb{R}^2$  where the solution exists ?

c) Let  $x(t, x_0)$  be a characteristic emanating from  $(x_0, 0)$  with  $x_0 < 0$ . Verify that the solution of part b) is constant on this characteristic for  $0 \leq t < -1/(2x_0)$ . Explain why the solution is not constant on this characteristic for  $t > -1/(2x_0)$ . Explain why this solution does not become discontinuous.

3. Let  $u(x, t)$  be a  $C^2$  solution of the initial-boundary value problem

$$u_t - u_{xx} = 0, \quad 0 < x < L, \quad t > 0$$

$$u(0, t) = u(L, t) = 0, \quad t \geq 0$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq L.$$

Show that if there is a constant  $C$  such that for  $t \geq 0$ ,

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$$|u(x, t)| \leq C e^{-t^2},$$

then  $f(x) = 0$  for  $0 < x < L$ .

4. Let  $\Omega$  be the square.  $\Omega = \{0 < x, y < 1\}$ . Show that  $C^3$  solutions of the following initial-boundary value problem are unique:

$$\begin{aligned} u_{tt} - u_{xx} - u_{yy} &= 0, & (x, y) \in \Omega, & t > 0 \\ u(x, y, 0) &= f(x, y), & u_t(x, y, 0) &= g(x, y), \\ u(0, y, t) &= u(1, y, t) = 0, & u_y(x, 0, t) &= u_y(x, 1, t) = 0. \end{aligned}$$

5. Let  $u(\mathbf{x}, t)$ ,  $\mathbf{x} \in \mathbb{R}^3$ ,  $t \in \mathbb{R}$  be a  $C^2$  solution of

$$u_{tt} - \Delta u = 0, \quad u(\mathbf{x}, 0) = 0, \quad u_t(\mathbf{x}, 0) = g(\mathbf{x})$$

with  $\int_{\mathbb{R}^3} g^2(\mathbf{x}) \, d\mathbf{x} < \infty$ . Show that

$$\int_0^\infty u^2(0, t) \, dt \leq \frac{1}{4\pi} \int_{\mathbb{R}^3} g^2(\mathbf{x}) \, d\mathbf{x}.$$

6. Let  $\Omega$  be the open unit ball in  $\mathbb{R}^n$  and let  $\Gamma$  be the boundary of  $\Omega$ .

For  $k = 1, 2, 3, \dots$ , let  $\Gamma_k = \Gamma \cap \{x_n \leq 1 - 1/k\}$ .

Let  $V_k = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_k\}$ . Let  $f \in L^2(\Omega)$ .

a) Give the weak formulation of the boundary value problem

$$(*) \quad \begin{aligned} -\Delta u &= f & \text{in } \Omega \\ u &= 0 & \text{on } \Gamma_k, \quad \partial u / \partial n = 0 & \text{on } \Gamma \setminus \Gamma_k. \end{aligned}$$

b) Show that for each  $k \geq 1$ , there is a unique weak solution  $u_k \in V_k$  of the boundary value problem (\*). You may use without proof the following inequality

$$\|v\|_{H^1(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega)}$$

for all  $v \in V_k$ , with the constant  $C$  independent of  $k$ .

c) Let  $u \in H_0^1(\Omega)$  be the unique weak solution of  $-\Delta u = f$ ,  $u = 0$  on  $\Gamma$ . Show that there is a subsequence  $k_j$ ,  $k_j \rightarrow \infty$ , such that  $u_{k_j}$  converges to  $u$  weakly in  $H^1(\Omega)$  and strongly in  $L^2(\Omega)$  as  $k_j \rightarrow \infty$ .

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Department of Mathematics  
Graduate Written Examination on Partial Differential Equations  
January 2003

Instructions

- Answer all six questions. Each will be assigned a grade from 0 to 10.
- Begin your answer to each question on a separate answer sheet. Complete the top of the first page of each answer sheet. Write your **code number** on each page of each answer sheet; **do not use your name**.
- Keep scratch work on separate sheets, which should not be submitted.
- Carefully explain your steps. If you invoke a “well-known” theorem, it is your responsibility to make clear exactly which theorem you are using, and to justify its use.

1. Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a bounded and increasing  $C^1$  function and that

$$\max_{x \in \mathbb{R}} f'(x) = K > 1 \quad \text{with } K < \infty.$$

Consider the initial value problem

$$\begin{aligned} u_t - uu_x + u &= 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x, 0) &= f(x) & \text{for } x \in \mathbb{R}. \end{aligned}$$

Show that there is no classical  $C^1$  solution defined globally for all  $t > 0$ . Find the time  $T$  (in terms of  $K$ ) at which the classical solution breaks down.

2. Consider the operator  $A : \mathcal{D}(A) \rightarrow L^2(\mathbb{R}^n)$ , where  $\mathcal{D}(A) = H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$  and

$$Au = \Delta u \quad \text{for } u \in \mathcal{D}(A).$$

Show that

a) the resolvent set  $\rho(A)$  contains the interval  $(0, \infty)$ , i.e.,  $(0, \infty) \subset \rho(A)$ ;

b) the estimate

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda} \quad \text{for } \lambda > 0$$

holds.

3. Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with smooth boundary.

a) For  $f \in L^2(\Omega)$  show that there exists a  $u_f \in H_0^1(\Omega)$  such that

$$\|f\|_{H^{-1}(\Omega)}^2 = \|u_f\|_{H_0^1(\Omega)}^2 = (f, u_f)_{L^2(\Omega)}.$$

b) Show that  $L^2(\Omega)$  is compactly embedded in  $H^{-1}(\Omega)$ . You may use the fact that  $H_0^1(\Omega)$  is compactly embedded in  $L^2(\Omega)$ .

4. Let  $f$  be continuous on  $\mathbb{R}$  with  $f(x) = 0$  for  $|x| \geq a$ . Let  $u(x, t)$  be the unique bounded solution of the heat equation

$$\begin{aligned} u_t - ku_{xx} &= 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x, 0) &= f(x) & \text{for } x \in \mathbb{R}. \end{aligned}$$



In the following, justify all steps in your calculation.

a) Show that

$$Q = \int_{\mathbb{R}} u(x, t) dx$$

is a constant independent of  $t$ .

b) Show that

$$m = \int_{\mathbb{R}} xu(x, t) dx$$

is independent of  $t$ .

c) Let

$$p(t) = \int_{\mathbb{R}} x^2 u(x, t) dx.$$

Show that  $p(t) = p(0) + 2kQt$ .

5. Let  $\Omega = \{x \in \mathbb{R}^n : |x| > 1\}$ . Let  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  be a bounded solution of  $\Delta u = 0$  with  $u(x) \geq 0$  for all  $x \in \Omega$ .

a) Show for  $n = 2$  the following maximum principle:

$$(1) \quad \sup_{x \in \Omega} u(x) = \max_{|x|=1} u(x).$$

Hint: consider  $u \pm \epsilon \ln r$ ,  $r = |x|$ .

b) Show by a simple example that (1) does not hold for  $n \geq 3$ .

6. Let  $u$  be a  $C^2$  solution of

$$\begin{aligned} u_{tt} - \Delta u &= 0 && \text{in } \mathbb{R}^2 \times \mathbb{R}, \\ u(x, 0) &= 0 && \text{for } x \in \mathbb{R}^2, \\ u_t(x, 0) &= g(x) && \text{for } x \in \mathbb{R}^2, \end{aligned}$$

where  $g(x) = 0$  for  $|x| \geq a > 0$ .

a) Use the method of descent to derive Poisson's formula

$$u(x, t) = \frac{1}{2\pi} \int_{|x-y| \leq t} \frac{g(y)}{\sqrt{t^2 - |x-y|^2}} dy.$$

b) Show that there exists a constant  $C$  such that

$$|u(x, t)| \leq \frac{C}{t}, \quad \text{for } t \geq 2(|x| + a).$$

c) Show that for each  $x \in \mathbb{R}^2$ ,

$$\lim_{t \rightarrow \infty} tu(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} g(y) dy.$$

Department of Mathematics  
Graduate Written Examination on Partial Differential Equations  
August, 2002

Instructions

- Answer all six questions. Each will be assigned a grade from 0 to 10
- Begin your answer to each question on a separate answer sheet. Complete the top of the first page of each answer sheet. Write your **code number** on each page of each answer sheet; **do not use your name**.
- Keep scratch work on separate sheets, which should not be submitted.
- Carefully explain your steps. If you invoke a “well-known” theorem, it is your responsibility to make clear exactly which theorem you are using, and to justify its use.

1. Let  $\Omega$  be an open set in  $R^2$ . Let  $H = \{u \in L_2(\Omega) : u \text{ is harmonic in } \Omega\}$ . Show that  $H$  is a closed subspace of  $L_2(\Omega)$ .

2. Let  $\Omega$  be a bounded, open set of  $R^n$  with smooth boundary  $\Gamma$ . Let  $q(x), f(x) \in L_2(\Omega)$  and suppose  $\int_{\Omega} q(x) dx = 0$ . Let  $u(x, t)$  be the solution of the initial-boundary value problem

$$\begin{aligned}u_t - \Delta u &= q, & x \in \Omega, t > 0 \\u(x, 0) &= f(x), & x \in \Omega \\ \partial u / \partial n &= 0, & \text{on } \Gamma.\end{aligned}$$

Show that  $\|u(\cdot, t) - v\|_{H^1(\Omega)} \rightarrow 0$  exponentially as  $t \rightarrow \infty$ , where  $v$  is the solution of the boundary value problem

$$\begin{aligned}-\Delta v &= q, & x \in \Omega \\ \partial v / \partial n &= 0, & \text{on } \Gamma \\ \int_{\Omega} v(x) dx &= \int_{\Omega} f(x) dx.\end{aligned}$$

3. Let  $\Omega$  be a bounded, connected, open set in  $R^n$  with smooth boundary. Let  $V$  be a closed subspace of  $H^1(\Omega)$  that does *not* contain nonzero constant functions. Using the compactness of the embedding of  $H^1(\Omega)$  into  $L_2(\Omega)$ , show that there is a constant  $C$ , independent of  $u$ , such that for  $u \in V$ ,

$$\int_{\Omega} |u|^2 dx \leq C \int_{\Omega} |\nabla u|^2 dx.$$

4. Let  $\Omega$  be a bounded, open set in  $R^n$  with smooth boundary  $\Gamma$ . Suppose  $\Gamma = \Gamma_0 \cup \Gamma_1$ , where  $\Gamma_0$  and  $\Gamma_1$  are nonempty, disjoint, smooth  $n - 1$  dimensional surfaces in  $R^n$ .

a) Find the weak formulation of the boundary value problem

$$-\Delta u = f \in L_2(\Omega), \quad \text{in } \Omega$$

$$\begin{aligned} \partial u / \partial n &= 0 \quad \text{on } \Gamma_1 \\ u &= 0 \quad \text{on } \Gamma_0. \end{aligned}$$

b) Using the result of Problem 3, prove that for  $f \in L_2(\Omega)$ , there is a unique weak solution  $u \in H^1(\Omega)$ .

5. Let  $u(x, t)$ ,  $x \in R^3$ ,  $t \in (-\infty, \infty)$ , be the solution of the initial value problem for the wave equation

$$u_{tt} - \Delta u = 0$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x)$$

Assume that  $f, g \in C^\infty$  with

$$(1/2) \int_{R^3} [|\nabla f(x)|^2 + |g(x)|^2] dx < \infty.$$

a) Show that the energy

$$E(t) \equiv (1/2) \int_{R^3} [|\nabla u(x, t)|^2 + |u_t(x, t)|^2] dx$$

is finite for all  $t \geq 0$  by showing  $E(t) \leq E(0)$ .

b) Show that  $E(t) = E(0)$  for all  $t$ .

6. Let  $f(x)$  be continuously differentiable and bounded on  $R$ . Let  $u(x, t)$  be the solution of the initial value problem

$$u_t + uu_x = 0, \quad u(x, 0) = f(x).$$

Suppose that  $m = \inf_R f'(x)$  is finite with  $m < 0$ . Find the maximal interval  $[0, T)$  such that  $u$  exists as a  $C^1$  function for  $0 \leq t < T$ . Show how  $T$  depends on  $m$ .

Department of Mathematics  
University of Maryland  
Graduate Written Examination on Partial Differential Equations  
January 2002

**Instructions**

- Answer all six questions. Each will be assigned a grade from 0 to 10.
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- Carefully explain all your steps. If you invoke a “well-known” theorem, it is your responsibility to make clear exactly which theorem you are using and to justify its use.

1. Let  $v$  satisfy

$$(1) \quad v_{tt} + av = c^2 v_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$

$$(2) \quad v(x, 0) = 0 = v_t(x, 0), \quad -\infty < x < \infty,$$

$$(3) \quad \text{for each } t, \quad v(\cdot, t) \text{ has compact support,}$$

where  $a$  and  $c$  are constants. By using the energy functional

$$E(t) := \int_{-\infty}^{\infty} [v_t(x, t)^2 + c^2 v_x(x, t)^2 + v(x, t)^2] dx,$$

show that  $v(x, t) = 0$  for  $-\infty < x < \infty$ ,  $t > 0$ .

2. Let  $B$  be the unit ball in  $\mathbb{R}^2$ . Prove that there is a number  $C$  such that

$$\|u\|_{L^2(B)} \leq C(\|\nabla u\|_{L^2(B)} + \|u\|_{L^2(\partial B)}) \quad \forall u \in C^1(\bar{B}).$$

3. Let  $u$  be a harmonic function on  $\mathbb{R}^n$  and let there be a number  $M$  such that  $u(\mathbf{x}) \leq M$  for all  $\mathbf{x} \in \mathbb{R}^n$ . Prove that  $u = \text{constant}$  on  $\mathbb{R}^n$ .

4. Consider Burgers' equation

$$u_t + uu_x = 0, \quad -\infty < x < \infty, \quad t > 0.$$

(a) Formulate the Rankine-Hugoniot condition for weak solutions of this equation with shock discontinuities.

(b) Construct the entropy solution of the initial-value problem for this equation subject to initial data

$$u(x, 0) = g(x) := \begin{cases} 1 & \text{for } x \leq 0, \\ \frac{1}{2} & \text{for } 0 < x < 1, \\ 0 & \text{for } x \geq 1. \end{cases}$$

Sketch the characteristic curves of the solution. Make sure that your sketch shows the characteristics of the solution for *all* times.

5. Let  $B$  be the unit ball in  $\mathbb{R}^n$  and let  $u$  be a smooth function satisfying

$$u_t - \Delta u + u^{1/2} = 0, \quad 0 \leq u \leq M \quad \text{on } B \times (0, \infty),$$

$$u_r = 0 \quad \text{on } \partial B \times (0, \infty).$$

Here  $M$  is a positive number and  $u_r$  is the radial derivative on  $\partial B$ . Prove that there is a number  $T$  depending only on  $M$  such that  $u = 0$  on  $B \times (T, \infty)$ . (Hint: Let  $v$  be the solution of the initial-value problem

$$\frac{dv}{dt} + v^{1/2} = 0, \quad v(0) = M,$$

and consider the function  $w = v - u$ .)

6. Let  $B$  be the unit ball in  $\mathbb{R}^2$ . (a) Find the weak formulation of the boundary-value problem

$$-\Delta u = f \quad \text{in } B, \quad u \pm u_r = 0 \quad \text{on } \partial B.$$

Here  $u_r$  is the radial derivative of  $u$  on  $\partial B$ .

(b) Decide for which choice of  $\pm$  this problem has a unique weak solution  $u \in H^1(B)$  for all  $f \in L^2(B)$ . For this purpose you may use without proof the estimate

$$\|u\|_{L^2(B)} \leq C(\|\nabla u\|_{L^2(B)} + \|u\|_{L^2(\partial B)}) \quad \forall u \in C^1(\bar{B}).$$

(c) For the other sign and for  $f = 0$ , find two smooth solutions of this boundary-value problem.

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF MARYLAND  
GRADUATE WRITTEN EXAMINATION: PARTIAL DIFFERENTIAL EQUATIONS  
15 AUGUST 2001  
Instructions to the student

- a. Answer all six questions. Each will be assigned a grade from 0 to 10.
- b. Begin each question on a separate answer sheet. Complete the top of the first page of each answer sheet. Write your **code number** on each page of the answer sheet. **DO NOT USE YOUR NAME.**
- c. Keep scratch work on separate sheets; these should not be submitted.
- d. Carefully explain all your steps. If you use a “well-known” theorem in your solution to any problem, it is your responsibility to make clear exactly which theorem you are using and to justify its use.

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1. Let  $g \in C(\mathbb{R}^3)$  with  $g \geq 0$  and  $g = 0$  for  $|x| \geq a > 0$ . Let  $u(x, t)$  be the solution of the initial-value problem

$$u_{tt} - \Delta u = 0, \quad u(x, 0) = 0, \quad u_t(x, 0) = g(x).$$

- a) Show that  $u(x, t) = 0$  in the set  $\{(x, t) : |x| \leq t - a, t \geq a\}$ .
- b) Suppose that there is a point  $x_0$  such that  $u(x_0, t) = 0$  for all  $t \geq 0$ . Show that  $u(x, t) \equiv 0$ .

2. Suppose that  $u \in H^1(\mathbb{R})$  and for simplicity suppose that  $u$  is continuously differentiable. Prove that  $\sum_{n=-\infty}^{\infty} |u(n)|^2 < \infty$ .

3. Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with a smooth boundary. Use a sharp version of the Maximum Principle and an energy method to show that the only smooth solutions of the boundary-value problem

$$\Delta u = 0 \quad \text{in } \Omega, \quad \mathbf{n} \cdot \nabla u = 0 \quad \text{on } \partial\Omega$$

(where  $\mathbf{n}$  is the unit outer normal to  $\partial\Omega$ ) have the form  $u = \text{const}$ .

4. Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with a smooth boundary. Consider the initial-boundary-value problem

$$\begin{aligned} u_{tt} &= \Delta u + \nabla \cdot \mathbf{f}(\nabla u_t), & \mathbf{x} \in \Omega, \\ u(\mathbf{x}, t) &= 0, & \mathbf{x} \in \partial\Omega, \\ u(\mathbf{x}, 0) &= \varphi(\mathbf{x}), & u_t(\mathbf{x}, 0) = \psi(\mathbf{x}) \end{aligned} \tag{1}$$

where  $\varphi$  and  $\psi$  are given well-behaved functions, and where  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a given continuously differentiable function for which

$$[\mathbf{f}(\mathbf{z}) - \mathbf{f}(\mathbf{y})] \cdot (\mathbf{z} - \mathbf{y}) > 0 \quad \text{for } \mathbf{z} \neq \mathbf{y}.$$

Prove that this problem has at most one classical solution.

5. Let  $\Omega$  be a bounded domain in the plane with smooth boundary. Let  $f$  be a positive, strictly convex, smooth function over the reals. Let  $w \in L^2(\Omega)$ . Show that there exists a unique  $u \in H^1(\Omega)$  that is a weak solution to the equation

$$-\Delta u + f'(u) = w, \quad u \Big|_{\partial\Omega} = 0.$$

6. Let

$$f(x) = \begin{cases} 0, & x \leq 0 \\ 1, & 0 < x \end{cases}.$$

Consider the initial-value problem

$$u_t + uu_x = 0, \quad u(x, 0) = f(x).$$

a) Show that this initial value problem has more than one weak solution. Make sketches in the  $(x, t)$ -plane showing characteristics and lines of discontinuity to illustrate your solutions.

b) Find the unique weak solution satisfying the entropy condition.

c) Let

$$f_\varepsilon(x) = \frac{1}{1 + e^{-x/\varepsilon}}.$$

Using the entropy condition, show that for  $\varepsilon > 0$ , there is a unique smooth solution  $u_\varepsilon(x, t)$  of the initial value problem

$$u_t + uu_x = 0, \quad u(x, 0) = f_\varepsilon(x).$$

d) Show that for each  $(x, t)$ ,  $t > 0$ ,  $u_\varepsilon$  converges as  $\varepsilon \rightarrow 0$  to  $v(x, t)$  where  $v$  is a weak solution of the initial value problem of part a). Does  $v$  satisfy the entropy condition?

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF MARYLAND  
GRADUATE WRITTEN EXAMINATION  
JANUARY 2001

MATH/MAPL 673-674

Instructions to the student

- a. Answer all six questions. Each will be assigned a grade from 0 to 10.
- b. Use a different booklet for each question. Complete the top of the first page of each booklet. Write your **code number** on each page of the booklet. **DO NOT USE YOUR NAME.**
- c. Keep scratch work on separate pages of the booklet.
- d. If you use a “well-known” theorem in your solution to any problem, it is your responsibility to make clear exactly which theorem you are using and to justify its use.

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1. Let  $A$  be a compact self-adjoint operator in a Hilbert space  $H$ . Let  $\{u_k\}_{k=1}^\infty$  be an orthonormal basis of  $H$  consisting of eigenvectors of  $A$  with associated eigenvalues  $\{\lambda_k\}_{k=1}^\infty$ . Prove that if  $\lambda \neq 0$  and  $\lambda$  is not an eigenvalue of  $A$ , then  $A - \lambda I$  has a bounded inverse and

$$\|(A - \lambda I)^{-1}\| \leq \frac{1}{\inf_k |\lambda_k - \lambda|} < \infty.$$

2. Let  $U$  be a bounded domain in  $\mathbf{R}^n$  with smooth boundary and suppose  $g: \bar{U} \rightarrow \mathbf{R}$  is smooth and positive. Given  $f_1, f_2: U \rightarrow \mathbf{R}$ , show that there can be at most one  $C^2$  solution  $u: \bar{U} \times [0, \infty) \rightarrow \mathbf{R}$  to the initial value problem

$$\begin{aligned} u_{tt} &= g(x)\Delta u && \text{for } x \in U, t > 0, \\ u(x, 0) &= f_1(x), \quad u_t(x, 0) = f_2(x) && \text{for } x \in U, \\ u(x, t) &= 0 && \text{for } x \in \partial U, t \geq 0. \end{aligned}$$

(Hint: Find an energy  $E$  for which  $dE/dt \leq CE$ .)

3. Let  $\Omega$  be a bounded domain in  $\mathbf{R}^n$  and let  $f \in L^2(\Omega)$  be fixed. Let  $H$  denote the Hilbert space  $H_0^1(\Omega)$  with norm  $\|v\| = (\int_\Omega |\nabla v|^2)^{1/2}$  for  $v \in H$ . Let

$$\lambda = \inf \{ \|v\|^2 : v \in H \text{ and } \int_\Omega f v = 1 \}.$$

- (a) Show that the infimum is attained, i.e., that there exists  $u \in H_0^1(\Omega)$  with  $\int_\Omega f u = 1$  such that  $\lambda = \int_\Omega |\nabla u|^2$ .
- (b) Show that this minimizing  $u$  is a weak solution of the boundary value problem

$$\begin{aligned} -\Delta u &= \lambda f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$



4. Let  $\Omega$  be the unit ball in  $\mathbf{R}^2$ . Suppose that  $u, v: \overline{\Omega} \rightarrow \mathbf{R}$  are smooth and satisfy

$$\begin{aligned}\Delta u &= \Delta v && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega.\end{aligned}$$

(a) Prove that there is a constant  $C$  independent of  $u$  and  $v$  such that

$$\|u\|_{H^1(\Omega)} \leq C\|v\|_{H^1(\Omega)}.$$

(b) Show there is *no* constant  $C$  independent of  $u$  and  $v$  such that

$$\|u\|_{L^2(\Omega)} \leq C\|v\|_{L^2(\Omega)}.$$

(Hint: Consider sequences satisfying  $u_n - v_n = 1$ .)

5. (Maximum principle in a punctured disk.) Let  $U = \{x : 0 < |x| \leq 1\} \subset \mathbf{R}^2$ . Suppose that  $u: U \rightarrow \mathbf{R}$  is continuous and is harmonic in the interior of  $U$ . Prove that if  $u$  is bounded then  $u(x) \leq \max_{|y|=1} u(y)$  for all  $x \in U$ . Give a counterexample to show that the conclusion can be false without the assumption that  $u$  is bounded.

6. Find a classical solution to the initial value problem

$$u_t - (u_x)^2 = 0, \quad u(x, 0) = x^2.$$

For what  $(x, t)$  is the solution defined?

PDE

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF MARYLAND  
GRADUATE WRITTEN EXAMINATION  
AUGUST 2000

MATH/MAPL 673-674

Instructions to the student

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- Let  $\Omega = B(0, 1)$  be the unit ball in  $\mathbf{R}^3$ . For any function  $u: \bar{\Omega} \rightarrow \mathbf{R}$  define the trace  $Tu$  by restricting  $u$  to  $\partial\Omega$ , i.e.,  $Tu = u|_{\partial\Omega}$ . Show that  $T: L^2(\Omega) \rightarrow L^2(\partial\Omega)$  is not bounded.
- Let  $U \subset \mathbf{R}^n$  be open and suppose that the operator  $L$ , formally defined by

$$Lu = - \sum_{i,j=1}^n a^{ij} u_{x_i x_j}$$

has smooth coefficients  $a^{ij}: U \rightarrow \mathbf{R}$ . Suppose that  $L$  is uniformly elliptic, i.e., for some  $c > 0$ ,  $\sum_{i,j=1}^n a^{ij} \xi_i \xi_j \geq c|\xi|^2$  for all  $\xi \in \mathbf{R}^n$  uniformly in  $U$ . Show that if  $\lambda > 0$  is sufficiently large then whenever  $Lu = 0$  we have

$$L(|Du|^2 + \lambda|u|^2) \leq 0.$$

- Let  $U$  be a bounded domain in  $\mathbf{R}^n$  with smooth boundary and let

$$\lambda_1 = \inf \left\{ \int_U |Du|^2 : u \in H_0^1(U) \text{ and } \int_U u^2 = 1 \right\}.$$

Using an energy method, show that if  $u: \bar{U} \times [0, \infty) \rightarrow \mathbf{R}$  is  $C^2$  and satisfies

$$u_t = \Delta u \text{ in } U, \quad u = 0 \text{ on } \partial U,$$

then  $\|u(\cdot, t)\| \leq e^{-\lambda_1 t} \|u(\cdot, 0)\|$ , where  $\|\cdot\|$  denotes the  $L^2$  norm on  $U$ .

- Suppose  $u: \mathbf{R} \times [0, T] \rightarrow \mathbf{R}$  is a smooth solution of  $u_t + uu_x = 0$  that is periodic in  $x$  with period  $L$ , i.e.,  $u(x + L, t) = u(x, t)$ . Show that

$$\max_x u(x, 0) - \min_x u(x, 0) \leq \frac{L}{T}.$$

5. Suppose that  $q: \mathbf{R}^3 \rightarrow \mathbf{R}$  is continuous with  $q(x) = 0$  for  $|x| > a > 0$ . Let  $u(x, t)$  be the solution of

$$u_{tt} - \Delta u = e^{i\omega t} q(x), \quad u(x, 0) = u_t(x, 0) = 0.$$

a) Show that  $e^{-i\omega t} u(x, t) \rightarrow v(x)$  as  $t \rightarrow \infty$  uniformly on compact sets, where

$$\Delta v + \omega^2 v = q.$$

b) Show that  $v$  satisfies the “outgoing” radiation condition

$$\left| \left( \frac{\partial}{\partial r} + i\omega \right) v(x) \right| \leq \frac{C}{|x|^2}$$

as  $r = |x| \rightarrow \infty$ .

6. Find a solution of

$$(u_x - y)^2 + u_y^2 = 1$$

which is smooth and positive in the first quadrant of the unit disk in  $\mathbf{R}^2$  and which vanishes for  $y = 0$ . Show that the (projected) characteristics are circular arcs.

PDE

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF MARYLAND  
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JANUARY 2000

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- Let  $\Omega = B(0, 1)$  be the unit ball in  $\mathbb{R}^n$ . Using iteration and the maximum principle (and other results – state them clearly and justify their use), prove that there exists some solution to the boundary value problem

$$\begin{aligned} -\Delta u &= \arctan(u + 1) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

- Writing the PDE  $u_t + u^2 u_x = 0$  as a conservation law, find a weak solution in the quarter plane  $x > 0$ ,  $t > 0$  that contains a rarefaction wave and a shock wave satisfying the entropy condition, having the form

$$u(t, x) = \begin{cases} f(x/t) & 0 < x < X(t), \\ 0 & x > X(t), \end{cases}$$

such that  $\int_0^\infty u(t, x) dx = 1$  for all  $t > 0$ .

- Suppose  $g : \mathbb{R}^3 \rightarrow \mathbb{R}$  is smooth with compact support and  $u : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$  satisfies

$$\begin{aligned} u_{tt} - \Delta u &= 0, \quad t > 0, \quad x \in \mathbb{R}^3, \\ u(0, x) &= 0, \quad u_t(0, x) = g(x), \quad x \in \mathbb{R}^3. \end{aligned}$$

For every  $x \in \mathbb{R}^3$ , show that  $h(x) = \lim_{t \rightarrow \infty} tu(t, (1+t)x)$  exists, and find  $h(x)$ .

4. For an arbitrary continuous periodic function  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  with period 1, the unique 1-periodic solution of the heat equation

$$\begin{aligned} u_t &= u_{xx} & -\infty < x < \infty, t > 0, \\ u(0, x) &= \psi(x) & -\infty < x < \infty, t = 0, \end{aligned}$$

can be expressed in two ways: (i) as a Fourier series of the form

$$u(t, x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos 2\pi kx + b_k \sin 2\pi kx) e^{-c_k t}$$

and (ii) using the heat kernel  $\Phi(t, x) = e^{-x^2/4t}/\sqrt{4\pi t}$ . Using both of these methods to describe  $u(t, 0)$ , prove that

$$\sum_{k=-\infty}^{\infty} e^{-\pi^2 k^2 t} \cos 2\pi kx = \frac{1}{\sqrt{\pi t}} \sum_{k=-\infty}^{\infty} e^{-(x+k)^2/t}, \quad t > 0, x \in [0, 1].$$

5. (a) Show that the closed unit ball in the Hilbert space  $H = L^2([0, 1], \mathbb{R})$  is not compact.  
 (b) Suppose that  $g : [0, 1] \rightarrow \mathbb{R}$  is a continuous function and define the operator  $T : H \rightarrow H$  by

$$(Tu)(x) = g(x)u(x) \quad \text{for } x \in [0, 1].$$

Prove that  $T$  is a compact operator if and only if the function  $g$  is identically zero.

6. Let  $H = H^1([0, 1])$  and let  $Tu = u(1)$ .

(a) Explain precisely how  $T$  is defined for all  $u \in H$ , and show that  $T$  is a bounded linear functional on  $H$ .

(b) Let  $(\cdot, \cdot)_H$  denote the standard inner product in  $H$ . By the Riesz representation theorem, there exists a unique  $v \in H$  such that  $Tu = (u, v)_H$  for all  $u \in H$ . Find  $v$ .

PDE

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF MARYLAND  
GRADUATE WRITTEN EXAMINATION  
AUGUST 1999

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1. Prove that if  $\Omega$  is an open ball in  $\mathbf{R}^n$  and  $u \in H^1(\Omega)$  with  $Du = 0$  in the sense of distributions, then  $u$  is a constant.

2. Let  $\Omega \subset \mathbf{R}^2$  be a smooth domain and let  $Lu = -u_{xx} - u_{yy}$ . A barrier at a point  $x_0 \in \partial\Omega$  is a  $C^2$  function  $w$  such that  $Lw \geq 0$  in  $\Omega$ ,  $w \geq 0$  on  $\partial\Omega$  and  $w(x_0) = 0$ .

(a) Suppose that  $f$  and  $u : \mathbf{R}^2 \rightarrow \mathbf{R}$  are smooth and  $Lu = \sin f$  in  $\Omega$  with  $u = 0$  on  $\partial\Omega$ . Prove that

$$|Du(x_0)| \leq \left| \frac{\partial w}{\partial \nu}(x_0) \right|$$

for any barrier at  $x_0$ .

(b) Show that a barrier exists at any point  $x_0 \in \partial\Omega$ .

3. Suppose  $u : \mathbf{R} \times \mathbf{R}^3 \rightarrow \mathbf{R}$  is  $C^2$  and satisfies

$$u_{tt} = \Delta u, \quad u(0, x) = 0, \quad u_t(0, x) = h(x) \quad (x \in \mathbf{R}^3).$$

Prove or give a counterexample: If  $\lim_{|x| \rightarrow \infty} h(x) = 0$ , then  $\lim_{t \rightarrow \infty} u(t, x) = 0$  for all  $x$ .

4. Let  $\Omega \subset \mathbf{R}^n$  be a bounded open set and  $a^{ij} \in C(\bar{\Omega})$  for  $i, j = 1, \dots, n$ , with  $\sum_{i,j} a^{ij}(x) \xi_i \xi_j \geq |\xi|^2$  for all  $\xi \in \mathbf{R}^n$ ,  $x \in \Omega$ . Suppose  $u \in H^1(\Omega) \cap C(\bar{\Omega})$  is a weak solution of

$$-\sum_{i,j} \partial_j(a^{ij} \partial_i u) = 0 \quad \text{in } \Omega.$$

Set  $w = u^2$ . Show that  $w \in H^1(\Omega)$ , and show that  $w$  is a weak subsolution; i.e., prove that

$$B(w, v) := \int_{\Omega} \sum_{i,j} a^{ij} \partial_j w \partial_i v \, dx \leq 0$$

for all  $v \in H_0^1(\Omega)$  such that  $v \geq 0$ .

5. Let  $\varepsilon > 0$  and consider the equation

$$u_t + (u^2)_x - \varepsilon u_{xx} = 0.$$

Let  $v_\ell, v_r$  be distinct real numbers. Show that there is a smooth solution of the form  $u(x, t) = v(x - ct)$  given implicitly by  $(s = x - ct)$

$$s = \int_a^{v(s)} \frac{\varepsilon dv}{v^2 - cv - b} \quad (a, b \text{ constant})$$

with the property that

$$v(s) \rightarrow \begin{cases} v_\ell & \text{as } s \rightarrow -\infty \\ v_r & \text{as } s \rightarrow +\infty, \end{cases}$$

*if and only if*

$$v_\ell > v_r \quad \text{and} \quad c = \frac{v_\ell^2 - v_r^2}{v_\ell - v_r}.$$

6. Let  $T$  be a bounded linear operator in a Hilbert space  $H$ . Prove that  $T$  is compact if and only if the Hilbert adjoint  $T^*$  is compact.

PARTIAL DIFFERENTIAL  
EQUATIONS

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF MARYLAND  
GRADUATE WRITTEN EXAMINATION  
JANUARY 1999

MATH/MAPL 673-674

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1 : Suppose  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  is harmonic in  $B(0, R)$ . For  $0 < a < c < R$  let

$$F(a, c) = \int_{|y|=1} u(ay)u(cy) dS(y).$$

Show that  $F(a, c) = F(b, b)$  if  $b^2 = ac$ . (Hint: Show that  $F(a, b^2/a)$  is independent of  $a$  using Green's identity.)

2 : A smooth function  $u$  on the first quadrant of the  $x_1x_2$  plane satisfies

$$x_1 \frac{\partial u}{\partial x_2} - x_2 \frac{\partial u}{\partial x_1} + 2u = 0, \quad u(x_1, 0) = x_1.$$

Determine  $u(0, x_2)$ .

3 : Let  $\Omega \subset B(0, R_0)$  be a compact set in  $\mathbb{R}^3$  with smooth boundary. Suppose that for all  $R > R_0$  there is a function  $u_R : \mathbb{R}^3 \rightarrow \mathbb{R}$  that is continuous and satisfies

$$\Delta u = 0 \text{ in } B(0, R) \setminus \Omega, \quad u(x) = 0 \text{ for } |x| \geq R, \quad u(x) = 1 \text{ for } x \in \Omega.$$

(a) Show that for all  $x$ , if  $R_0 < R_1 \leq R_2$  then

$$0 \leq u_{R_1}(x) \leq u_{R_2}(x) \leq 1.$$

(b) Show that  $u(x) = \lim_{R \rightarrow \infty} u_R(x)$  is harmonic and that  $\lim_{|x| \rightarrow \infty} u(x) = 0$ . (Note:  $1/|x|$  is harmonic.)

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4 : Assume  $u : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$  is smooth and satisfies

$$u_{tt} = \Delta u, \quad u(0, x) = 0, \quad u_t(0, x) = h(x)$$

for  $x \in \mathbb{R}^3$ . Fix  $t > 0$  and let

$$\psi(x) = u(t - |x|, x), \quad \phi(x) = u_t(t - |x|, x).$$

(a) Show that

$$\operatorname{div} \left( \frac{1}{|x|} \nabla \psi + \frac{x}{|x|^3} \psi + \frac{2x}{|x|^2} \phi \right) = 0.$$

(b) Integrating the identity in (a) over  $\varepsilon \leq |x| \leq t$ , derive the Kirchhoff formula

$$u(t, 0) = \frac{1}{4\pi t} \int_{|x|=t} h(x) dS(x).$$

5 : Assume  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous and its first-order distribution derivatives  $Du$  are bounded functions, with  $|Du(x)| \leq L$  for all  $x$ . By approximating  $u$  by mollification, prove that  $u$  is Lipschitz, with

$$|u(x) - u(y)| \leq L|x - y| \quad \text{for all } x, y \in \mathbb{R}^n.$$

6 : On the Hilbert space  $H^1(0, 1)$  consider the functional

$$I(u) = u(0)^2 + \int_0^1 (u'(x)^2 - 2f(x)u(x)) dx,$$

where  $f \in L^2(0, 1)$  is given.

(a) Suppose  $u^*$  is a function in  $C^2([0, 1])$  that minimizes  $I$  over all of  $H^1(0, 1)$ . Derive a classical boundary value problem satisfied by  $u^*$ .

(b) Show that for all  $u \in H^1(0, 1)$ ,

$$\int_0^1 u(x)^2 dx \leq 2u(0)^2 + 2 \int_0^1 u'(x)^2 dx.$$

(c) Find the weak form of the boundary value problem satisfied by  $u^*$  in part (a), and prove that this problem has a unique solution.

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# PDE

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF MARYLAND  
GRADUATE WRITTEN EXAMINATION  
AUGUST 1998

MATH/MAPL 673-674

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1 : Let  $u, v \in C^1(\bar{\Omega})$  be conjugate harmonic functions, i.e.,  $u_x = v_y$  and  $u_y = -v_x$ , in a simply connected domain  $\Omega$  with  $C^1$  boundary. Show that on the boundary curve  $\partial\Omega$ ,

$$\frac{du}{dn} = \frac{dv}{ds}, \quad \frac{dv}{dn} = -\frac{du}{ds},$$

where  $d/dn$  denotes differentiation in the direction of the exterior normal and  $d/ds$  differentiation in the counter-clockwise tangential direction. Show how these relations can be used to reduce the Neumann problem for  $u$  to the Dirichlet problem for  $v$ , and conversely.

2 : Let  $n = 2$  and  $\Omega$  be the half-plane  $x_2 > 0$ . Prove the maximum principle

$$\sup_{\bar{\Omega}} u = \sup_{\partial\Omega} u$$

for  $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$  harmonic in  $\Omega$ , under the additional assumption that  $u$  is bounded in  $\bar{\Omega}$ .

Hint: Take for  $\epsilon > 0$  the harmonic function

$$u(x_1, x_2) - \epsilon \log(x_1^2 + (x_2 + 1)^2).$$

3 : Let  $\Omega \subset \mathbb{R}^n$  be open (perhaps not bounded, nor with smooth boundary). Show that if there exists a function  $u \in C^2(\bar{\Omega})$  vanishing on  $\partial\Omega$  for which the quotient

$$\frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} u^2}$$

reaches its infimum  $\lambda$ , then  $u$  is an eigenfunction for the eigenvalue  $\lambda$ , so that  $\Delta u + \lambda u = 0$  in  $\Omega$ .

# PDE

August 1998

4 : Let  $S$  denote a space-like hyperplane with equation  $t = \gamma x_1$  in  $x - t$  space. Show that the Cauchy problem  $u_{tt} - c^2 \Delta u = 0$  with data on  $S$  can be reduced to the initial-value problem for the same equation by introducing new independent variables by the Lorentz transformation

$$x'_1 = \frac{x_1 - \gamma c^2 t}{\sqrt{1 - \gamma^2 c^2}}, \quad x'_2 = x_2, \quad x'_3 = x_3, \quad t' = \frac{t - \gamma x_1}{\sqrt{1 - \gamma^2 c^2}}.$$

5 : Let  $u$  be a  $C^1$  solution of  $u_t + uu_x = 0$  in each of two regions separated by a  $C^1$  curve  $x = \xi(t)$ . Let  $u$  be continuous but  $u_x$  have a jump discontinuity on the curve. Prove that

$$\frac{d\xi}{dt} = u$$

on the curve and hence that the curve is a characteristic.

Hint: From the equation show

$$u_t^+ - u_t^- + u(u_x^+ - u_x^-) = 0$$

and moreover  $u(\xi(t), t)$  and  $d/dt(u(\xi(t), t))$  are continuous on the curve.

6 : Consider the discretization of the problem

$$\begin{aligned} u_t - u_{xx} &= 0 & \text{for } 0 < t \leq T, -\infty < x < \infty, \\ u(x, 0) &= f(x) & \text{for } -\infty < x < \infty. \end{aligned}$$

Let  $\Sigma$  denote the lattice points  $(x, t)$  with  $x, t$  of the form  $x = nh$  and  $t = mk$ ,  $n, m$  integers, and  $v$  the solution of

$$\frac{v(x, t+k) - v(x, t)}{k} - \frac{v(x+h, t) - 2v(x, t) + v(x-h, t)}{h^2} = 0$$

with  $v(x, 0) = f(x)$ . Show that for  $\lambda := k/h^2 = 1/2$ ,

$$v(nh, mk) = 2^{-m} \sum_{j=0}^m \binom{m}{j} f((n-m+2j)h)$$

and then show that

$$\sup_{\Sigma} |v| \leq \sup |f|.$$

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF MARYLAND  
GRADUATE WRITTEN EXAMINATION  
JANUARY 1998

PDE Written Examination - Ph.D. Level

Instructions to the student

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  3. Keep scratch work on separate pages of the booklet.
  4. All SIX booklets must be returned at the end of the exam.
-

1995  
 PDE EXAM (FALL 1997)

1. Suppose you want to solve Laplace's equation in the unit disc  $D \in \mathbb{R}^2$ .

$$-\Delta u = 0 \quad \text{in } x \in D := \{|x| < 1\} \quad , \quad (1.a)$$

with data continuous on the boundary  $\partial D$ .

$$u = f \quad \text{for } x \in \partial D := \{|x| = 1\} \quad (1.b)$$

Below is an outline of the solution, you are asked to fill the details.

a) Check that the function

$$K(x, y) := -\frac{((x-y), n_y)}{|x-y|^2}$$

where  $n_y$  is the unit outward normal on the boundary  $\partial D$  and  $x \in D$ ,  $y \in \partial D$ , satisfies

$$-\Delta_x K(x, y) = 0 \quad \text{for } x \in D \quad .$$

Hint: Start with  $\Delta_x \log(|x-y|) = 0$  for  $x \neq y$ .

b) Therefore it makes sense to try to find a solution  $u$  by superposition

$$u(x) := \int_{\partial D} K(x, y) g(y) dS(y) \quad .$$

Where  $dS(y)$  is the measure on the surface  $\partial D$  and  $g$  is some function to be computed. Write  $K(r, \theta - \phi)$ , using polar coordinates,  $x = re^{i\theta}$ ,  $y = e^{i\phi}$  and verify that  $K(r, \theta) \geq 1/2$  and

$$\left| K(r, \theta) - \frac{1}{2} \right| \leq \frac{1-r}{2r(1-\cos(\theta))} \quad .$$

c) Assume the following

$$\int_0^{2\pi} K(r, \theta) d\theta = 2\pi$$

and show that if  $g(\theta)$  is continuous then

$$\lim_{r \rightarrow 1} \int_0^{2\pi} K(r, \theta - \phi) g(\phi) d\phi = \pi g(\theta) + \frac{1}{2} \int_0^{2\pi} g(\phi) d\phi \quad .$$

Hint: Start with the integral

$$\int_0^{2\pi} (K(r, \theta - \phi) - 1/2) (g(\theta) - g(\phi)) d\phi \quad ,$$

and show that it converges to zero as  $r \rightarrow 1$ .

d) Solve the integral equation

$$f(\theta_0) = \pi g(\theta_0) + \frac{1}{2} \int_0^{2\pi} g(\theta) d\theta \quad ,$$

and conclude that you can always solve problem (1) if  $f(\theta)$  is continuous.

PDE EXAM (SUMMER 1997)

2. Consider the elliptic equation in divergence form

$$L(u) := - \sum_{i,j=1}^n (a_{ij}(x)u_{x_i})_{x_j},$$

over a domain  $\Omega \in \mathbb{R}^n$ . A barrier at  $x_0 \in \partial\Omega$  is a  $C^1$  function  $w$  satisfying in a weak sense

$$\begin{aligned} L(w) &\geq 1 \quad \text{in } \Omega \\ w(x_0) &= 0 \quad \text{and } w \geq 0 \quad \text{on } \partial\Omega \end{aligned}$$

a) Let  $u$  be a  $C^2(\bar{\Omega})$  weak solution of  $L(u) = f \in C(\Omega)$  with  $u = 0$  on  $\partial\Omega$ . Show that

$$\left| \frac{\partial u}{\partial \nu}(x_0) \right| \leq C \left| \frac{\partial w}{\partial \nu}(x_0) \right|$$

for all directions  $\nu$  pointing inwards with respect to  $\Omega$  at  $x = x_0$ . 8 pts.

Hint: Use the weak maximum principle.

b) Let  $\Omega$  be convex. Show that there exists a barrier  $w$  for all  $x_0 \in \partial\Omega$ . 7 pts.

Hint: Use that  $\Omega$  is on one side of a hyperplane passing through  $x_0 \in \partial\Omega$ .

PDE EXAM (SUMMER 1997)

3. Consider the equation  $u_t + uu_x = 0$ . Verify that the function

$$\begin{aligned} u(t, x) &= -\frac{2}{3} \left( t + \sqrt{3x + t^2} \right) \quad \text{if } t^2 + 4x > 0 \\ u &= 0 \quad \text{if } t^2 + 4x < 0 \end{aligned}$$

is a weak solution of  $u_t + uu_x = 0$ . 10 pts.

PDE EXAM (SUMMER 1997)

4. Show that a solution of  $\Delta u + u = w(x)$  in  $B := \{|x| < \pi\}$  with  $x \in \mathbb{R}^3$  and boundary data  $u(x) = 0$  if  $|x| = \pi$  has a solution only if

$$\int_B w(x) \frac{\sin(|x|)}{|x|} dx = 0$$

10 pts.

PDE EXAM (SUMMER 1997)

5. Let  $u$  and  $v$  be smooth functions. Let  $D$  a subset of  $\mathbb{R}^n$  and call  $\Omega = D \times [0, T]$  a cylindrical domain in  $\mathbb{R}^n \times \mathbb{R}^+$ . Integrate by parts the following

$$\int_{\Omega} v(u_t - \Delta u) dx dt$$

to obtain a Green's identity.

Use the above to solve the following problem

$$u_t - u_{xx} = 0 \quad \text{in } t > 0 ; \quad x > 0$$

$$u(0, x) = 0 \quad \text{for } x > 0$$

$$u(t, 0) = h(t) \quad \text{for } t > 0$$

15 pts.

Hint: Try  $v(t, x) = K(x, y, T - t + \epsilon) - K(x, -y, T - t + \epsilon)$  where  $\epsilon$  is small positive and

$$K(x, y, t) := \frac{1}{\sqrt{4\pi t}} \exp\left(\frac{-(x-y)^2}{4t}\right)$$

and let  $\epsilon \rightarrow 0$ .

PDE EXAM (SUMMER 1997)

6. a) Show the Sobolev type inequality for a smooth function  $g(x)$  with  $x \in \mathbb{R}$ ,

$$g^2(x) \leq 2 \left( \int_{\mathbb{R}} g^2(y) dy + \int_{\mathbb{R}} (Dg(y))^2 dy \right) , \quad (4)$$

starting from the identity (5), which you have to verify.

$$g^2(x) = \int_x^{x-1} \left( g(y) - \int_x^y g'(z) dz \right)^2 dy . \quad (5)$$

6 pts.

b) Prove by induction over the dimension  $n$  that if  $x = (x_1, x_2, \dots, x_n)$  then

$$g^2(x) \leq 2^n \sum_{|\alpha| \leq n} \int (D^\alpha g(y))^2 dy .$$

4 pts.

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF MARYLAND  
GRADUATE WRITTEN EXAMINATION  
AUGUST 1997

PDE Written Examination - Ph.D. Level

Instructions to the student

1. Answer all six questions. Each will be assigned a grade from 1 to 10.
2. Use a different booklet for each question. Complete the top of the first page of each booklet. Put your CODE NUMBER on each page of the booklet. DO NOT USE YOUR NAME.
3. Keep scratch work on separate pages of the booklet.
4. All SIX booklets must be returned at the end of the exam.

PDE EXAM (SUMMER 1997)

1. Consider the nonlinear equation

$$u_{tt} - u_{xx} = 2e^u, \quad (2)$$

and the linear equation

$$w_{tt} - w_{xx} = 0. \quad (3)$$

- a) Set

$$u := \log \left[ \frac{w_t^2 - w_x^2}{w^2} \right]$$

and show that if  $w$  satisfies the linear equation (3) then  $u$  defined above satisfies the nonlinear equation (2).  
7 pts.

Hint: It is convenient to use the coordinates  $\tau := t+x$  and  $\xi := t-x$  and factor  $\log(ab) = \log(a) + \log(b)$ .

- b) Now the general solution of equation (3),  $w$  can be written as

$$w := f(t+x) + g(t-x)$$

where  $f$  and  $g$  are arbitrary functions. Find a choice of  $f$  and  $g$  such that  $u(t,x)$  is bounded when  $t=0$  but  $u$  blows-up in finite time. What is the blow-up profile? This means the set in  $R^2$  where  $u = \infty$ . 8 pts.

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2. Consider again the same problem

$$-\Delta u = 0 \quad \text{in } z \in D = \{|z| < 1\} \quad , \quad (1.2)$$

$$u(z) = f(z) \quad \text{for } z \in \partial D = \{|z| = 1\} \quad (1.3)$$

Here  $D$  is the unit disc in  $\mathbb{R}^2$  and you can use polar coordinates  $(r, \theta)$  with  $r \leq 1$ . Assume that  $f(\theta)$  is continuous and suppose that you want to solve the above problem by Hilbert space methods.

a) Show that the solution must be of the form (for  $r < 1$ ),

$$u(r, \theta) = \sum_{k=0}^{\infty} (a_k \cos(k\theta) + b_k \sin(k\theta)) r^k \quad \text{where} \quad f(\theta) = \sum_{k=0}^{\infty} a_k \cos(k\theta) + b_k \sin(k\theta)$$

b) Show that the Dirichlet integral is given by

$$\int_D |\nabla u|^2 dz = \pi \sum_{k=1}^{\infty} k (a_k^2 + b_k^2) \quad .$$

c) Show that there is continuous  $f(\theta)$  for which the Dirichlet integral is infinite.

Hint: Find a sequence  $(a_k, b_k)$  which are absolutely summable

$$\sum_k (|a_k| + |b_k|) < \infty$$

while the Dirichlet integral diverges.

d) Conclude that you cannot solve the problem (1) by Hilbert space methods for arbitrary continuous data  $f$ .

3 : Let  $f$  be a bounded and locally Hölder continuous function (with exponent  $\alpha \leq 1$ ). Assume that  $f$  has compact support. Show that the solution of  $\Delta u = f$  is a  $C^2$  function and moreover

$$D_{i,j}u(x) = \int_{\Omega} D_{i,j}\Gamma(x-y)(f(y) - f(x))dy - f(x) \int_{\partial\Omega} D_i\Gamma(x-y)\nu_j(y)dS(y)$$

where  $\Omega$  is any smooth domain and  $\Gamma(x)$  is the Newton potential.

$$\Gamma(x) := \frac{1}{c_n} \frac{1}{|x|^{n-2}} \quad \text{with } c_n = \int_{S^{n-1}} 1 dS$$

$c_n$  the area of the unit sphere in  $R^n$ .

Hint : Consider the function

$$v_\epsilon(x) = \int_{\Omega} D_i\Gamma(x-y)\eta_\epsilon(|x-y|)f(y)dy$$

where  $\eta(t)$  is  $C^1$  with  $0 \leq \eta \leq 1$ ,  $0 \leq \eta' \leq 2$ ,  $\eta(t) = 0$  if  $t \leq 1$  and  $\eta(t) = 1$  if  $t \geq 2$ . Set  $\eta_\epsilon(|x|) := \eta(|x|/\epsilon)$ .

4 : Let  $\alpha$  be a constant  $\neq -c$ . Find the solution of  $u_{tt} - c^2u_{xx} = 0$  in the quadrant  $x > 0, t > 0$  for which

$$u = f, \quad u_t = g \quad \text{for } t=0, x > 0$$

and

$$u_t = \alpha u_x \quad \text{for } x=0, t > 0$$

where  $f, g$  are  $C^2$  for  $x > 0$  and vanish near  $x=0$ . Show that in general no solution exists when  $\alpha = -c$ .

5 : Let the operators  $L_1, L_2$  be defined by

$$L_1u = au_x + bu_y + cu, \quad L_2u = du_x + eu_y + fu$$

where  $a, b, c, d, e, f$  are constant and  $ae - bd \neq 0$ . Prove that

- the equations  $L_1u = w_1, L_2u = w_2$  have common solution  $u$  if  $L_1w_2 = L_2w_1$ .
- The general solution of  $L_1L_2u = 0$  has the form  $u = u_1 + u_2$  where  $L_1u_1 = 0$  and  $L_2u_2 = 0$ .

6 : Let  $\Delta u = f$  in  $\Omega$  a subset of  $R^n$ . Show that the Kelvin transform of  $u$  defined by

$$v(x) := |x|^{n-2}u(x/|x|^2) \quad \text{for } x/|x| \in \Omega,$$

satisfies

$$\Delta v(x) = |x|^{-n-2}f(x/|x|^2)$$

# PDE

DEPARTMENT OF MATHEMATICS — UNIVERSITY OF MARYLAND  
GRADUATE WRITTEN EXAMINATION  
JANUARY 1997

MATH/MAPL 673-674

Instructions to the student

- Answer all six questions. Each will be assigned a grade from 0 to 10.
- Use a different booklet for each question. Complete the top of the first page of each booklet. Write your code number on each page of the booklet. DO NOT USE YOUR NAME.
- Keep scratch work on separate pages of the booklet.
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1. Consider the initial value problem

$$u_t + uu_x + u = 0 \quad (-\infty < x < \infty, t > 0)$$
$$u(x, 0) = a \sin x$$

Find the characteristic curves explicitly. Show that if  $a > 1$ , then a global smooth solution does not exist. Also, find the time of breakdown, i.e., the maximal time of existence of the smooth solution.

2. Prove that the initial boundary value problem

$$u_{tt} = u_{xx} + u_{xxt} \quad (0 < x < 1, t > 0) \tag{1}$$

$$u(x, 0) = g(x), \quad u_t(x, 0) = h(x) \quad (0 < x < 1) \tag{2}$$

$$u(0, t) = 0, \quad u_{tt}(1, t) + u_x(1, t) + u_{xt}(1, t) = 0 \quad (t > 0) \tag{3}$$

has at most one classical solution, assuming  $g$  and  $h$  are well-behaved functions.

3. In proving uniqueness for an initial boundary value problem for some nonlinear PDE, one encounters the following problem: Suppose  $w(x, t)$  is a well-behaved function which satisfies

$$\int_0^t \int_0^1 w_t(x, \tau)^2 dx d\tau \leq \int_0^t \int_0^1 |w(x, \tau)| |w_t(x, \tau)| dx d\tau$$

for all  $t > 0$ , and  $w(x, 0) = 0$  for  $0 < x < 1$ . Prove that  $w(x, t) = 0$  for  $0 < x < 1, t > 0$ .

4. Let  $q(x, t)$  be a smooth function for  $0 < x < 1$  which is periodic in  $t$  with period  $T$ . For any  $\varepsilon > 0$ , take for granted that there is a smooth solution  $u = u^\varepsilon(x, t)$  of the boundary value problem

$$-\varepsilon u_{tt}^\varepsilon - u_{xx}^\varepsilon + u_t^\varepsilon = q(x, t) \quad \text{for } 0 < x < 1, -\infty < t < \infty,$$
$$u^\varepsilon(0, t) = u^\varepsilon(1, t) = 0 \quad \text{for all } t,$$

which is periodic in  $t$  with period  $T$ .

- (a) Let  $\Omega_T = (0, 1) \times (0, T)$ . Using energy methods (multiply by  $u^\varepsilon$  and by  $u_t^\varepsilon$ ), show that  $u_x^\varepsilon$  and  $u_t^\varepsilon$  are uniformly bounded in  $L^2(\Omega_T)$  independent of  $\varepsilon > 0$ .
- (b) Describe the sense in which the bounds obtained in part (a) justify passing to the limit  $\varepsilon \downarrow 0$ . Describe precisely in what weak sense the limiting function that you obtain satisfies the boundary value problem above with  $\varepsilon = 0$ .

5. Suppose  $f : \mathbf{R}^n \mapsto \mathbf{R}$  is continuous, nonnegative and has compact support. Consider the initial value problem

$$\begin{aligned} u_t - \Delta u &= u^2, \quad \text{for } x \in \mathbf{R}^n, 0 < t < T, \\ u(x, 0) &= f(x) \quad \text{for } x \in \mathbf{R}^n \end{aligned}$$

Using Duhamel's principle, reformulate the initial value problem as a fixed point problem for an integral equation. Prove that if  $T > 0$  is sufficiently small, the integral equation has a *nonnegative* solution in  $C(\mathbf{R}^n \times [0, T])$ .

6. Let  $\Omega \subset \mathbf{R}^m$  be a bounded domain with smooth boundary, and let  $u(x, t)$  be the solution of the initial-boundary value problem

$$\begin{aligned} u_{tt} - c^2 \Delta u &= q(x, t) \quad \text{for } x \in \Omega, t > 0, \\ u &= 0 \quad \text{for } x \in \partial\Omega, t > 0 \\ u(x, 0) &= u_t(x, 0) = 0 \quad \text{for } x \in \Omega. \end{aligned}$$

Let  $\{\phi_n(x)\}_{n=1}^\infty$  be a complete orthonormal set of eigenfunctions for the Laplace operator  $\Delta$  in  $\Omega$  with Dirichlet boundary conditions, so  $-\Delta\phi_n = \lambda_n\phi_n$ . Set  $\omega_n = c\sqrt{\lambda_n}$ . Suppose

$$q(x, t) = \sum_{n=1}^{\infty} q_n(t)\phi_n(x)$$

where each  $q_n(t)$  is continuous,  $\sum q_n(t)^2 < \infty$  and  $|q_n(t)| \leq Ce^{-nt}$ . Show that there are sequences  $\{a_n\}$  and  $\{b_n\}$  such that  $\sum(a_n^2 + b_n^2) < \infty$  and if

$$v(x, t) = \sum_{n=1}^{\infty} [a_n \cos(\omega_n t) + b_n \sin(\omega_n t)]\phi_n(x),$$

then

$$\|u(\cdot, t) - v(\cdot, t)\| \rightarrow 0$$

at an exponential rate as  $t \rightarrow \infty$ .  $\|\cdot\|$  is the  $L^2$  norm in  $\Omega$ .

# PDE

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF MARYLAND  
GRADUATE WRITTEN EXAMINATION  
AUGUST 1996

MATH/MAPL 673-674

Instructions to the student

- Answer all six questions. Each will be assigned a grade from 1 to 10.
- Use a different booklet for each question. Complete the top of the first page of each booklet. Write your code number on each page of the booklet. DO NOT USE YOUR NAME.
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- Let  $\Omega \subset \mathbf{R}^n$  be a bounded domain with smooth boundary, and let  $a : \mathbf{R}^n \mapsto \mathbf{R}^n$  and  $f : \overline{\Omega} \mapsto \mathbf{R}$  be smooth. Consider the boundary value problem

$$\Delta u + a(x) \cdot \nabla u = f \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \quad (\text{P})$$

and the associated homogeneous adjoint problem

$$\Delta v - \nabla \cdot (a(x)v) = 0, \quad \frac{\partial v}{\partial n} - n \cdot a(x)v = 0 \quad \text{on } \partial\Omega \quad (\text{Q})$$

- Assume that  $u \in C^2(\overline{\Omega})$  satisfies (P) and that  $f \geq 0$ . Show that  $u$  is constant and  $f = 0$ .
- It is possible to show that the solution space of (Q) is one-dimensional, and furthermore that (P) has a solution if and only if

$$\int_{\Omega} f v = 0$$

for every solution of (Q). Taking these facts for granted, show that any  $C^2$  solution of (Q) is either non-negative or non-positive.

- Suppose  $u \in C^2(\mathbf{R}^2)$  satisfies  $\Delta u \geq 0$  and that  $u$  is not harmonic. Show that  $\sup u(x) = +\infty$ . (Hint: Consider means over circles centered at the origin.)
- Suppose  $\Omega \subset \mathbf{R}^n$  is a bounded domain with smooth boundary. Show that the initial-boundary value problem

$$u_t = \Delta u + \int_{\Omega} u^2 dx \quad \text{in } \Omega \text{ for } t > 0, \\ u = 0 \quad \text{on } \partial\Omega \text{ for } t > 0, \quad u(0, x) = u_0(x) \quad \text{for } x \in \Omega$$

may have *at most one* solution  $u \in C^2([0, T] \times \overline{\Omega})$ , for any  $T > 0$ .

4. Suppose  $g : \mathbb{R}^3 \mapsto \mathbb{R}$  is smooth, has compact support, and  $g(x) \geq 0$  for all  $x \in \mathbb{R}^3$ . Let  $u(x, t)$  be the solution of

$$\begin{aligned} u_{tt} &= \Delta u \quad \text{in } \mathbb{R}^3 \times \mathbb{R}, \\ u(x, 0) &= 0, \quad u_t(x, 0) = g(x) \quad \text{for } x \in \mathbb{R}^3. \end{aligned}$$

Show that if  $u(x_0, t) = 0$  for some  $x_0$  and for all  $t$ , then  $u \equiv 0$ .

5. Consider the scalar conservation law

$$u_t + (u^3 - u)_x = 0, \quad -\infty < x < \infty, \quad t > 0$$

with Riemann initial data  $u(x, 0) = u_0(x)$ .

- (a) For what values of  $a$  does the initial value problem have a centered rarefaction wave solution, if

$$u_0(x) = \begin{cases} a & x < 0 \\ -1 & x > 0. \end{cases}$$

- (b) For what values of  $a$  does the initial value problem have a simple shock solution

$$u(x, t) = \begin{cases} u_- & x < st \\ u_+ & x > st \end{cases}$$

satisfying the Lax entropy condition, if

$$u_0(x) = \begin{cases} 1 & x < 0 \\ a & x > 0. \end{cases}$$

- (c) Show that if

$$u_0(x) = \begin{cases} 1 & x < 0 \\ -1 & x > 0, \end{cases}$$

then there is a weak solution consisting of a centered rarefaction wave in combination with an admissible shock solution (which is marginally admissible in the sense that the Lax inequalities are not required to be strict.)

6. Let  $\Omega \subset \mathbb{R}^2$  be a bounded set which is strictly convex and has smooth boundary. Define the inner product space

$$V(\Omega) = H_0^1(\Omega) \cap \{u \in L^2(\Omega) \mid u_{xx} \in L^2(\Omega)\},$$

with inner product

$$(u, v)_V = \int_{\Omega} u_x v_x + u_y v_y + u_{xx} v_{xx}.$$

Here points in  $\mathbb{R}^2$  are denoted  $(x, y)$ .

- (a) Show that  $V$  is a Hilbert space. (In particular, show that  $V$  is *complete*.)  
 (b) Given  $f \in L^2(\Omega)$ , show that there exists a unique  $u \in V$  such that

$$\int_{\Omega} u_x v_x + u_y v_y + u_{xx} v_{xx} = \int_{\Omega} f v \quad \text{for all } v \in V(\Omega). \quad (K)$$

- (c) Suppose  $f \in C(\overline{\Omega})$  and suppose  $u \in C^4(\overline{\Omega})$  satisfies (K). Then  $u$  satisfies a boundary value problem consisting of a PDE and two boundary conditions. Find this boundary value problem and show that  $u$  is a solution.

# PDE

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF MARYLAND  
GRADUATE WRITTEN EXAMINATION  
JANUARY 1996

MATH/MAPL 673-674

Instructions to the student

- Answer all six questions. Each will be assigned a grade from 1 to 10.
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- Let  $\Omega \subset \mathbb{R}^2$  be a bounded domain with smooth boundary. Suppose  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  satisfies

$$u_{xx} + u^2 u_{yy} + 1 + u - u^3 = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Prove that  $u(x, y) < 2$  for  $(x, y) \in \Omega$ .

- Consider the Cauchy problem

$$\begin{aligned} u_{tt} &= c^2 \Delta u - q(x)u, & x \in \mathbb{R}^3, t > 0, \\ u(x, 0) &= f(x), & u_t(x, 0) = g(x), \end{aligned}$$

where  $f$  and  $g$  are smooth with compact support and  $q(x) \geq 0$  and  $c > 0$  is a constant. Use an energy method to show that this problem has at most one  $C^2$  solution. (You may assume finite speed of propagation.)

- Consider the following boundary value problem for Laplace's equation in the infinite strip  $\Omega = \{(x, y) \in \mathbb{R}^2 \mid -\infty < x < \infty, 0 < y < 1\}$ ,

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \quad \text{in } \Omega, \\ u(x, 1) &= f(x), \quad \frac{\partial u}{\partial n}(x, 0) = 0, \quad (-\infty < x < \infty). \end{aligned}$$

Here  $n$  denotes the outward unit normal on the boundary  $\partial\Omega$ . The *Dirichlet-Neumann* map  $T$  is defined (when possible) by

$$(Tf)(x) = \frac{\partial u}{\partial n}(x, 1) \quad -\infty < x < \infty.$$

Show that if  $f \in \mathcal{S}$  is a function in the Schwartz class  $\mathcal{S}$  of smooth, rapidly decreasing functions on  $\mathbb{R}$ , then  $Tf \in \mathcal{S}$ . Show that  $T$  may be extended by continuity to be a bounded map  $T : H^1(\mathbb{R}) \mapsto L^2(\mathbb{R})$ . (Hint: Use the Fourier transform.)

4. Consider the conservation laws

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0, \quad (1)$$

$$v_t + \left(\frac{2}{3}v^{3/2}\right)_x = 0. \quad (2)$$

- (i) If  $u$  is a  $C^1$  positive solution of (1) for  $-\infty < x < \infty$ ,  $0 < t < T$ , show that  $v = u^2$  satisfies (2).  
(ii) Find global weak solutions of (1) and (2) which satisfy the Lax entropy condition, and which take the respective initial values

$$u(x, 0) = \begin{cases} 2 & x \leq 0 \\ 2 - x & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}, \quad v(x, 0) = \begin{cases} 4 & x \leq 0 \\ (2 - x)^2 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

Sketch the characteristics and any shock curves which arise. Is the following statement true or false? (Justify your answer.)

$$v(x, t) = u(x, t)^2 \quad \text{for } -\infty < x < \infty, 0 < t < \infty.$$

5. Let  $\Omega \subset \mathbb{R}^n$  be a connected domain with smooth boundary, and with volume equal to 1.

(a) Let  $\lambda > 0$ , and consider the boundary value problem

$$-\Delta u + \lambda \int_{\Omega} u = f \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \quad (1)$$

Describe the weak formulation of this problem. (Do not solve it.)

(b) Show that a weak solution of problem (1) is also a weak solution of the problem

$$-\Delta u = f - \int_{\Omega} f \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \quad (2)$$

(c) The solution of problem (1) depends on  $\lambda$ . Suppose  $u = u_1$  is a weak solution for  $\lambda = \lambda_1$ , and  $u = u_2$  is a weak solution for  $\lambda = \lambda_2$ . What is  $u_1 - u_2$ ? (Hint: solutions of problem (2) are not *quite* unique.)

6. Consider the first order equation

$$u_t + a(x)u_x = 0$$

where  $a$  is a smooth function on  $\mathbb{R}$ .

- (i) Define what a *weak solution* of this equation is.  
(ii) Suppose  $u$  is a piecewise smooth weak solution which is discontinuous along the curve  $\mathcal{C}$  described by  $x = s(t)$ . Show that  $\mathcal{C}$  must be a *characteristic* of the PDE.



# PDE

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF MARYLAND  
GRADUATE WRITTEN EXAMINATION  
AUGUST 1995

MATH/MAPL 673-674

Instructions to the student

- Answer all six questions. Each will be assigned a grade from 1 to 10.
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1. The equation

$$\phi_x^2 + \phi_y^2 = y^2 + 1$$

is the eikonal equation from geometrical optics in a certain stratified 2-dimensional medium. Consider this equation with initial conditions given for  $x = 0$ ,

$$\phi(0, y) = \frac{y^2}{2}, \quad \phi_x(0, y) > 0$$

for all  $y \in \mathbf{R}$ . Find  $\phi(x, y)$ . Sketch the characteristics. Do they focus or defocus as  $x \rightarrow \pm\infty$ ?

2. Recall that if  $f : \mathbf{R}^2 \mapsto \mathbf{R}$  is  $C^2$  with compact support, then  $u(x) = \int_{\mathbf{R}^2} G(y)f(x-y) dy$  satisfies  $\Delta u = f$ , where the Green's function  $G(y) = -(2\pi)^{-1} \ln |y|$ . Let  $v : \mathbf{R}^2 \mapsto \mathbf{R}^2$  be a  $C^3$  vector field with compact support,  $v = (v_1, v_2)$ . We seek to write  $v = w + \nabla\phi$  where  $\operatorname{div} w = 0$ . Hence we wish to solve  $\Delta\phi = \operatorname{div} v$  and compute  $\nabla\phi$ . Prove that

$$\frac{\partial\phi}{\partial x_i} = cv_i(x) + \lim_{\varepsilon \rightarrow 0} \sum_{j=1}^2 \int_{|y|>\varepsilon} B_{ij}(y)v_j(x-y) dy, \quad (1)$$

for some constant  $c$ , and find  $c$ . Here

$$B_{ij}(y) = \frac{\partial^2}{\partial y_i \partial y_j} G(y) = -\frac{1}{2\pi|y|^2} \left( \delta_{ij} - 2\frac{y_i y_j}{|y|^2} \right),$$

where  $\delta_{ij} = 1$  if  $i = j$ , 0 otherwise. Note that  $B_{ij}$  is not absolutely integrable in a neighborhood of  $y = 0$  in  $\mathbf{R}^2$ .

3. For  $\varepsilon \geq 0$ , let  $f_\varepsilon(u) = \sqrt{u^2 + \varepsilon}$ . Note  $f_0(u) = |u|$ . For  $\varepsilon > 0$ , consider the Riemann problem

$$u_t + f_\varepsilon(u)_x = 0, \quad u(x, 0) = \begin{cases} u_- & x < 0 \\ u_+ & x > 0. \end{cases}$$

Assume that  $u_- < 0 < u_+$ .

- Find a discontinuous weak solution of this problem, and determine whether the Lax shock condition holds.
- Show that a continuous solution  $u^\varepsilon(x, t)$  exists in the form of a centered rarefaction wave. Find  $u^0(x, t) = \lim_{\varepsilon \rightarrow 0} u^\varepsilon(x, t)$ , and prove that  $u^0$  is a weak solution of

$$u_t + |u|_x = 0.$$

4. Let  $\Omega$  be a bounded domain in  $\mathbf{R}^n$  with smooth boundary, and suppose  $u \in C^2(\overline{\Omega})$  satisfies

$$\Delta u + \sum_{j=1}^n b_j(x) \frac{\partial u}{\partial x_j} + c(x)u = \lambda u \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

where  $c$  and  $b_j$  ( $j = 1 \dots n$ ) are given smooth functions on  $\overline{\Omega}$ . Suppose that  $\lambda > \max_{\Omega} c(x)$ . Prove that  $u$  can have no positive maximum on  $\partial\Omega$ . (Take care,  $u$  is not defined outside  $\Omega$ . Consider  $u(x(t))$  for various curves  $x(t)$ .) Deduce that  $u = 0$  using the maximum principle.

5. Consider the initial value problem in  $\mathbf{R}^3 \times \mathbf{R}$ ,

$$u_{tt} - \Delta u = q(x, t), \quad u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad (P)$$

where  $q$  is smooth with  $q(x, t) = 0$  for  $|x| \geq \rho > 0$ . Show that if  $q$  has the form  $q(x, t) = e^{i\omega t} q_0(x)$  then there is a function  $v(x)$  such that for each  $x \in \mathbf{R}^3$ ,

$$u(x, t) - v(x)e^{i\omega t} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

6. Let  $\Omega$  be a bounded domain in  $\mathbf{R}^n$  and consider the elliptic system

$$\begin{aligned} -\Delta u_1 + \alpha(x)u_1 + \beta u_2 &= f_1, \\ -\Delta u_2 &= f_2, \end{aligned}$$

with the boundary conditions

$$u_1 = 0, \quad u_2 = 0 \quad \text{on } \partial\Omega.$$

- What is the weak formulation of this boundary value problem?
- Assuming that  $\alpha \geq 0$  is continuous on  $\overline{\Omega}$ , show that if  $|\beta|$  is sufficiently small, then for each  $f = (f_1, f_2) \in L^2(\Omega) \times L^2(\Omega)$ , there exists a unique weak solution to this problem.

# PDE

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF MARYLAND  
GRADUATE WRITTEN EXAMINATION  
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MATH/MAPL 673-674

Instructions to the student

- Answer all six questions. Each will be assigned a grade from 1 to 10.
- Use a different booklet for each question. Complete the top of the first page of each booklet. Write your code number on each page of the booklet. DO NOT USE YOUR NAME.
- Keep scratch work on separate pages of the booklet.

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- Let  $\varepsilon > 0$ ,  $c \in \mathbb{R}$ . Solve the initial value problem

$$u_t + cu_x = \varepsilon u_{xx}, \quad u(x, 0) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases} \quad (1)$$

The solution depends on  $\varepsilon$ , so we write  $u = u_\varepsilon$ . Find  $\lim_{\varepsilon \rightarrow 0^+} u_\varepsilon(x, t)$ . For each fixed  $t > 0$ , determine whether the convergence is uniform in  $x$ , and justify your answer. Discuss in what sense, if any, the limit is a solution to  $u_t + cu_x = 0$ . (Hint: Change to a moving coordinate system.)

- A simple model for traffic flow, in which one assumes that cars move more slowly in denser traffic, leads to this conservation law for  $u(x, t)$ , a normalized density of cars per unit length along a highway:

$$u_t + f(u)_x = 0, \quad \text{where } f(u) = (1 - u)u. \quad (2)$$

As a model for a traffic stream that has been partly stopped at a traffic light, and starts up again at time  $t = 0$ , consider the initial value problem for (2) with initial data

$$u(x, 0) = \begin{cases} 0 & \text{for } x > 1, \\ 1 & \text{for } 0 < x < 1, \\ a & \text{for } x < 0, \end{cases}$$

where  $a \in (0, 1)$  is the density of a uniform stream of cars approaching from the left.

- Find a solution to this problem, valid for  $0 < t < 1/(1 - a)$ , in terms of a shock wave and a rarefaction wave. Sketch the characteristics. Verify that any discontinuities satisfy the appropriate jump and entropy conditions.
- What happens at  $t = 1/(1 - a)$ ? Describe how to continue the solution beyond this time.

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3. Suppose  $f : \mathbf{R}^3 \mapsto \mathbf{R}$  is  $C^2$  and has compact support.

(a) Prove that if  $u$  is defined by

$$u(x) = \int_{\mathbf{R}^3} \frac{f(y)}{|x-y|} dy, \quad (3)$$

then  $u$  satisfies the Poisson equation  $\Delta u = cf$  for some constant  $c$ , and find  $c$ .

(b) Show that there is a function  $g(r)$  such that  $|u(x)| \leq g(|x|) \rightarrow 0$  as  $|x| \rightarrow \infty$ .

(c) Show that  $u$  as given by (3) is the unique solution of  $\Delta u = cf$  having the property in (b).

4. Let  $\Omega = \mathbf{R}^n \times (0, T]$ . Suppose  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  is a bounded solution of the heat equation  $u_t = \Delta u$ . Prove that

$$\sup_{(x,t) \in \Omega} u(x,t) \leq \sup_{x \in \mathbf{R}^n} u(x,0).$$

(Hint: Consider the function  $v = u - \varepsilon(2nt + |x|^2)$ .)

5. Suppose  $f \in C[0, 1]$ , and consider the ODE boundary value problem

$$y'' + p(x)y' + q(x)y = f(x), \quad y(0) = y(1) = 0.$$

Suppose  $p \in C^1[0, 1]$ ,  $q \in C[0, 1]$  and  $p' - 2q \geq 0$ . What is the weak formulation of this problem as an elliptic boundary value problem? Using the Lax-Milgram theorem, prove that a unique solution exists.

6. Consider the nonlinear PDE

$$u_{tt} - \Delta u + u^3 = 0, \quad x \in \mathbf{R}^3, \quad t \in \mathbf{R}.$$

(a) Assuming  $u$  is smooth and has compact support in  $x$  for each  $t$ , what is the energy expression

$$E(t) = \iiint_{\mathbf{R}^3} q(u, u_t, \nabla u) dx$$

which is conserved?

(b) Let

$$E_a(t) = \iiint_{|x| \leq a} q(u, u_t, \nabla u) dx$$

be the energy contained in the ball of radius  $a > 0$ . Show that for any  $T > 0$ ,

$$E_a(T) \leq E_{a+T}(0).$$

(c) Show that if  $u(x, 0) = u_t(x, 0) = 0$  for  $|x| \geq a$ , then  $u(x, t) = 0$  for  $|x| \geq a + t$ ,  $t \geq 0$ .