DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
January, 2011

Probability (M. A. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10

b. Use a different booklet for each question. Write the problem number and your code number **(NOT YOUR NAME)** on the outside cover.

c. Keep scratch work on separate pages in the same booklet

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use

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1. Let X and Y be two independent random variables, both having geometric distribution with parameter p Set

$$S = \min(X, Y), \qquad T = |X - Y|$$

(i) Find the joint probability mass function of the random pair (S, T).
(ii) Find the marginal probability mass functions of S and T Are S and T independent?

2. Two players, A and B have their urns containing white and red balls. Let p_A and p_B be the proportions of white balls in the urns, respectively. They take turns in drawing a ball at random with replacement from their urns The winner is the first player who gets a white ball.
(i) Find the probability that the player who plays first wins.
(ii) For what values of p_A and p_B the game is fair, i. e. $P(A \text{ wins}) = P(B \text{ wins})$?

3. Let X be a random variable having symmetric distribution (i. e, $-X$ has the same distribution as X), such that $\mathbb{E}(X^2) < \infty$, and let Z be a Bernoulli random variable with parameter p, independent of X. Set

$$Y = \begin{cases} X, & \text{if } Z = 1, \\ -X & \text{if } Z = 0. \end{cases}$$

(i) Prove that Y and Z are independent and Y has the same distribution as X.
(ii) For what values of p are X and Y (a) uncorrelated, (b) independent?

4. Let $N, X_1, X_2, \ldots$ be a sequence of independent random variables with N having a geometric distribution with parameter p and $X_1, X_2, \ldots$ having the same exponential distribution with parameter λ. Set

$$Y = \min\{X_1, \ldots, X_N\}$$

(i) Find the distribution function of Y.
(ii) Calculate $\mathbb{E}(Y)$.

5. There are 10 coins in a bag. Six of them are normal coins, one coin has two 'heads' and three coins have two 'tails'. You draw at random a coin, look at one of its sides and see that it is a 'tail'. Find the probability that the drawn coin is normal

6. Let U and V be two independent random variables, both uniformly distributed on $(0,1)$. Set

$$T = 2\pi U, \qquad R = \sqrt{-2\log V}$$

(i) Find the marginal probability density function (pdf) of T and the joint pdf of (T, R).
(ii) Prove that the random variables

$$X = R\cos T, \qquad Y = R\sin T$$

are independent each having standard normal distribution.

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---

1. A person plays an infinite sequence of games He wins the n-th game with probability $1/\sqrt{n}$, independently of the other games.

(i) Prove that for any A, the probability is one that the player will accumulate A dollars if he gets a dollar each time he wins two games in a row.

(ii) Does the claim in part (i) hold true if the player gets a dollar only if he wins three games in a row? Prove or disprove it.

2. There are 10 coins in a bag. Five of them are normal coins, one coin has two 'heads' and four coins have two 'tails'. You pull one coin out, look at one of its sides and see that it is a 'tail'. What is the probability that it is a normal coin?

3. Let X_n be a Markov chain with the state space $\mathbb{N}$, with the transition probabilities $P(z, z^2) = P(z, z-1) = 1/2$ for $z \geq 2$, $P(z, z+1) = 1$ for $z = 1$

(i) Find a strictly monotonically decreasing non-negative function $f : \mathbb{N} \to \mathbb{R}^+$ such that $f(X_n)$ is a supermartingale.

(ii) Prove that for each initial distribution $\mathrm{P}(\lim_{n\to\infty} X_n = +\infty) = 1$.

4. Let ξ_n be independent identically distributed random variables with $P(\xi_n = -1) = P(\xi_n = 1) = 1/2$.

(i) Prove that the series $\sum_{n=1}^{\infty} e^{-n}\xi_n$ converges with probability one.

(ii) Prove that the distribution of $\xi = \sum_{n=1}^{\infty} e^{-n}\xi_n$ is singular, i. e., concentrated on a set of Lebesgue measure zero.

5. Let ξ_n be a sequence of independent random variables with ξ_n uniformly distributed on $[0, n^2]$. Find a_n and b_n such that $(\sum_{i=1}^{n} \xi_i - a_n)/b_n$ converges in distribution to a non-degenerate limit and identify the limit.

6. (i) Let X_t, $t > 0$, be random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Assuming that $E|X_t|^2 = E|X|^2 < \infty$ for all t, prove that $P(\lim_{t\to 0} X_t = X) = 1$ implies $\lim_{t\to 0} E|X_t - X|^2 = 0$, i. e., under the above assumption almost sure convergence implies convergence in mean square.

(ii) Let X_t, $t \in \mathbb{R}$, be a random process with the property that $\mathrm{E}X_t$ and $C(h) = \mathrm{E}(X_t X_{h+t})$ are finite and do not depend on t (such a process is called wide-sense stationary). Prove that the correlation function $C(h)$ is continuous if the trajectories of X_t are continuous.

DEPARTMENT OF MATHEMATICS
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---

1. Two persons, A and B, are playing a game. If A wins a round, he gets \$4 from B and also wins the next round with probability 0.7. If A loses the round, he pays \$5 to B and wins the next round with probability 0.5.
(i) Write down the transition matrix of the Markov chain with two states, $\{A$ won the current round, B won the current round$\}$ and find the stationary probabilities of the states.
(ii) Find $\lim_{n\to\infty} P(A$ has more money after n rounds than before the game$)$.

2. When voting, each of n persons acting independently of the others rolls a die and votes for candidate A if the die falls 1 or 2, votes for candidate B if the die falls 3 or 4, and votes for candidate C otherwise. Let $X(n)$, $Y(n)$ denote the number of votes received by A and B, respectively.
(i) Find the means, variances and covariance of $X(n)$, $Y(n)$.
(ii) For $n = 1,000,000$, use the Central Limit Theorem to approximate $P\{X(n) - Y(n) \geq 300\}$ (you may leave the answer in terms of the standard normal distribution function).

3. Let X, Y be independent random variables with $E(X) = E(Y) = 0$.
Show that $E(|X+Y|) \geq \max\{E(|X|),\ E(|Y|)\}$.
{**Hint**: Compute expectation by conditioning.}

4. Let $\xi_1,\ \xi_2, \ldots$ be a sequence of positive random variables,
$a_1 < a_2 < \ldots$ be a sequence of numbers with $\lim_{n\to\infty} a_n = \infty$, and
$A_n = \{\xi_n > a_n,\ \xi_{n+1} > a_{n+1}, \ldots\}$, $n = 1, 2, \ldots$ be a sequence of events.
(i) Show that if $\lim_{n\to\infty} P(A_n) = 1$, then $P(\lim_{n\to\infty} \xi_n = \infty) = 1$.
(ii) Show that there is a sequence of positive numbers $c_1,\ c_2, \ldots$ such that $P(\lim_{n\to\infty} c_n\xi_n = \infty) = 1$.

5. Using a relation between the moment generating functions and moments in (i) and another property in (ii), prove that
(i) the function $f(t) = e^{-ct^4}$, $-\infty < t < \infty$ where c is a constant, is not the moment generating function of a random variable X with finite second moment unless $c = 0$;
(ii) the function $f(t) = (1 + 2t)e^{-t^2/2}$, $-\infty < t < \infty$ is not a moment generating function.

6. Let X, Y be independent random variables and set $Z = X + Y$.
Prove that if the distribution function of Y (or of X) is continuous, so is the distribution function of Z.
{**Hint**: Compute probability by conditioning.}

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1. Two persons, A and B, are playing a game If A wins a round, he gets \$4 from B and also wins the next round with probability 0.7. If A loses the round, he pays \$5 to B and wins the next round with probability 0.5:
(i) Write down the transition matrix of the Markov chain with two states, $\{A$ won the current round, B won the current round$\}$ and find the stationary probabilities of the states.
(ii) Find $\lim_{n\to\infty} P(A$ has more money after n rounds than before the game$)$.

2. (i) Let X be a random variable with zero mean and finite variance σ^2. Show that for any $c > 0$

$$P(X > c) \leq \frac{\sigma^2}{\sigma^2 + c^2}.$$

Hint: consider $X + a$ and use the Chebyshev's inequality type arguments for an appropriate a.

(ii) Let $\{X_n,\, n \geq 1\}$ be a square-integrable martingale with $E(X_1) = 0$. Show that for any $c > 0$

$$P(\max_{1 \leq i \leq n} X_i \geq c) \leq \frac{\mathrm{var}(X_n)}{\mathrm{var}(X_n) + c^2}.$$

Hint: Use Doob's inequality for submartingales.

3. When voting, each of n persons acting independently of the others rolls a die and votes for candidate A if the die falls 1 or 2, votes for candidate B if the die falls 3 or 4, and votes for candidate C otherwise. Let $X(n)$, $Y(n)$ denote the number of votes received by A and B, respectively.
(i) For $n = 1{,}000{,}000$, use the Central Limit Theorem to approximate $P\{X(n) - Y(n) \geq 300\}$ (you may leave the answer in terms of the standard normal distribution function).
(ii) Find $a(n)$ and $b(n) > 0$ such that limiting distribution as $n \to \infty$ of $\{X(n)/Y(n) - a(n)\}/b(n)$ is **non-degenerate**

4. Let X, Y be random variables with finite expectations.
(i) Show that $E(X|Y) = 0$ implies $E(|X + Y|) \geq E(|Y|)$.
(ii) Show that if (X, Y) is identically distributed with (Y, X), then

$$E(|3X - Y|) \geq E(|X + Y|).$$

5. Let ξ_1, ξ_2 be independent random variables and set $\xi = \xi_1 + \xi_2$.
(i) Prove that if ξ_1 is absolutely continuous, i. e., its distribution is given by a density function, then so is ξ.
(ii) Prove that if the distribution function of ξ_1 is continuous, so is the distribution function of ξ.

6. Using a relation between the characteristic functions and moments in (i) and another property in (ii), prove that
(i) the function $f(t) = e^{-ct^4}$ where c is a constant cannot be a characteristic function unless $c = 0$;
(ii) the function $(1 + 2t)e^{-t^2/2}$ is not a characteristic function.

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1. Let N have a Poisson distribution with parameter λ and let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with $P[X_i = j] = p_j$, $j = 0, 1, \ldots, k$. Define N_j as the number of occurrences of $\{X_m = j\}$ for $m = 1, \ldots, N$. Prove that $N_0, N_1, \ldots, N_k$ are independent and that N_j has a Poisson distribution with parameter λ_j.

2. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with uniform distribution on $[0, 1]$. Prove that

$$(X_1 X_2 \cdots X_n)^{1/n}$$

converges in probability as $n \to \infty$ and compute its limit.

3. Let $\{X_k \mid k = 0, 1, 2, \ldots\}$ be a sequence of independent random variables with $E[X_k] = 0$ and $\mathrm{Var}[X_k] = \sigma_k^2$. Suppose that

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} \sigma_k^2 \to 0$$

Prove that for any $\varepsilon > 0$, as $n \to \infty$,

$$P\left[\left|\frac{1}{n}\sum_{k=1}^{n} X_k\right| > \varepsilon\right] \to 0.$$

4. Suppose that the transition probability matrix $\mathbf{P}$ of a Markov chain is doubly stochastic. That is, the entries of $\mathbf{P}$ satisfy $\sum_{i=1}^{m} p_{ij} = \sum_{j=1}^{m} p_{ij} = 1$. Suppose in addition that $p_{ij} > 0$ for all i, j. Find the stationary distribution of the chain and justify the claim that this stationary distribution is also the limiting distribution.

5. Consider a sequence of random variables $X_1, X_2, \ldots$ such that $X_n = 1$ or 0. Assume $P[X_1 = 1] \geq \alpha$ and

$$P[X_n = 1 \mid X_1, \ldots, X_{n-1}] \geq \alpha > 0 \quad \text{for} \quad n = 2, 3, \ldots$$

Prove that $P[X_n = 1 \text{ for some } n] = 1$.

6. Let $\{N(t) : t \geq 0\}$ be a *nonhomogeneous Poisson process*. That is, $N(0) = 0$ a.s., $N(t)$ has independent increments and $N(t) - N(s)$ has a Poisson distribution with parameter

$$\int_s^t \lambda(u)du$$

where $0 \leq s \leq t$ and the rate function $\lambda(u)$ is a continuous positive function.

(a) Find a continuous strictly increasing function $h(t)$ such that the time-transformed process $\tilde{N}(t) = N(h(t))$ is a homogeneous Poisson process with rate parameter 1.

(b) Let T be the time until the first event in the nonhomogeneous process $N(t)$. Compute $P[T > t]$ and $P[T > t \mid N(s) = n]$ where $s > t$.

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1. Let $\{X_{nk}\}$, $k = 1, \ldots, r_n$, $n = 1, 2, \ldots$ be a triangular array of Bernoulli random variables with $p_{nk} = P[X_{nk} = 1]$. Suppose that

$$\sum_{k=1}^{r_n} p_{nk} \to \lambda \quad \text{and} \quad \max_{k \le r_n} p_{nk} \to 0.$$

Find the limiting distribution of $\sum_{k=1}^{r_n} X_{nk}$

2. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with uniform distribution on $[0, 1]$. Prove that

$$\lim_{n\to\infty} (X_1 X_2 \cdots X_n)^{1/n}$$

exists with probability one and compute its value.

3. Let $\{X_n \mid n = 0, 1, 2, \ldots\}$ be a square integrable martingale with respect to a nested sequence of σ-fields $\{\mathcal{F}_n\}$. Assume $E[X_n] = 0$. Prove that

$$P\left[\max_{1\le k\le n} |X_k| > \varepsilon\right] \le E[X_n^2]/\varepsilon^2.$$

4. The random variable X is defined on a probability space $\Omega, \mathcal{F}, P)$. Let $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{F}$ and assume X has finite variance. Prove that

$$E\left[(X - E(X \mid \mathcal{G}_2))^2\right] \leq E\left[(X - E(X \mid \mathcal{G}_1))^2\right].$$

In words, the dispersion of X about its conditional mean becomes smaller as the σ-field grows.

5. Consider a sequence of random variables $X_1, X_2, \ldots$ such that $X_n = 1$ or 0. Assume $P[X_1 = 1] \geq \alpha$ and

$$P[X_n = 1 \mid X_1, \ldots, X_{n-1}] \geq \alpha > 0 \quad \text{for } n = 2, 3, \ldots$$

Prove that

(a) $P[X_n = 1 \text{ for some } n] = 1$.

(b) $P[X_n = 1 \text{ infinitely often}] = 1$

6. Let $\{N(t) : t \geq 0\}$ be a *nonhomogeneous Poisson process*. That is, $N(0) = 0$ a.s., $N(t)$ has independent increments and $N(t) - N(s)$ has a Poisson distribution with parameter

$$\int_s^t \lambda(u) du$$

where $0 \leq s \leq t$ and the rate function $\lambda(u)$ is a continuous positive function.

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1. Let $N, X_1, X_2, \ldots$ be independent, Let $P[N = k] = q^{k-1}p$, $k \geq 1$, $p+q = 1$, and assume the X_i are i.i.d. with common exponential density

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Show that $X_1 + \cdots + X_N$ has the exponential density $f(x; \lambda p)$.

2. Let $X_1, X_2, \ldots$ be i.i.d. $N(0,1)$ variables and let $S_k = X_1 + \cdots + X_k$. If $m < n$, find the joint density of (S_m, S_n) and the conditional density of S_m given $S_n = t$.

3. Let $N_1(t)$ and $N_2(t)$ be independent homogeneous Poisson processes with rates λ_1 and λ_2, respectively. Let Z be the time of the first jump for the process $N_1(t) + N_2(t)$ and let J be the (random) index of the component process that made the first jump. Find the joint distribution of (J, Z). In particular, establish that J and Z are independent and that Z is exponentially distributed.

4. Let $X_1, \ldots, X_n$ be i.i.d. with moment generating function $M(t) = E[\exp(tX_1)]$, which is finite for all real t.

(a) Prove that if $t \geq 0$, $P[X_1 > a] \leq e^{-at}M(at)$. Extend this result to prove that $P[X_1 > a] \leq \exp[-h(a)]$ where

$$h(a) = \sup_{t \geq 0}[at - \psi(t)] \quad \text{and} \quad \psi(t) = \log M(t).$$

(b) Show that if $\bar{X} = (X_1 + \cdots + X_n)/n$, $P[\bar{X} > a] \leq \exp[-nh(a)]$

5. Let X, Y be i.i.d. $N(0,1)$. If (X, Y) are rectangular co-ordinates of a point in the plane, let (R, Θ) be the polar co-ordinates of that point.

(a) Show that the density of Θ is

$$g(t) = \frac{\sqrt{1-\rho^2}}{2\pi(1 - 2\rho \cos t \sin t)}$$

for $0 < t < 2\pi$ and that Θ is uniformly distributed iff $\rho = 0$.

(b) If $\rho = 0$, show also that R and Θ are independent and find the distribution of R^2.

6. Consider the following process $\{X_n\}_{n \geq 0}$ taking values in $\{0, 1, \ldots\}$. Assume U_n, $n = 1, 2, \ldots$, is an i.i.d. sequence of positive integer valued random variables and let X_0 be independent of the U_n. Then

$$X_n = \begin{cases} X_{n-1} - 1 & \text{if } X_{n-1} \neq 0 \\ U_k - 1 & \text{if } X_{n-1} = 0 \text{ for the } k\text{th time} \end{cases}$$

(a) Prove that this process is an irreducible homogeneous Markov chain and give its transition probability matrix $\mathbf{P}$.

(b) Show that the chain admits a unique stationary distribution π iff $E[U_1] < \infty$ and give an expression for π in terms of the distribution of U_1.

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1. Let $X_1, \ldots, X_n$ be i.i.d. random variables with moment generating function $M(t) = E[\exp(tX_1)]$ which is finite for all t. Let $\bar{X}_n = (X_1 + \cdots + X_n)/n$.

(a) Prove that

$$P[X_1 > a] \leq \exp[-h(a)]$$

where

$$h(a) = \sup_{t \geq 0}[at - \psi(t)] \quad \text{and} \quad \psi(t) = \log M(t)$$

(b) Prove that

$$P[\bar{X}_n \geq a] \leq \exp[-nh(a)].$$

(c) Assume $E[X_1] = 0$. Use the result of (b) to establish that $\bar{X}_n \to 0$ almost surely.

2. Let $(\Omega, \mathcal{F}, P)$ be a probability space, let X be a random variable with finite second moment and let $\mathcal{G}_1 \subset \mathcal{G}_2$ be sub-σ-fields. Prove that

$$E[(X - E(X \mid \mathcal{G}_2))^2] \leq E[(X - E(X \mid \mathcal{G}_1))^2].$$

3. Let $N_1(t)$ and $N_2(t)$ be independent homogeneous Poisson processes with rates λ_1 and λ_2, respectively. Let Z be the time of the first jump for the process $N_1(t) + N_2(t)$ and let J be the (random) index of the component process that made the first jump. Find the joint distribution of (J, Z). In particular, establish that J and Z are independent and that Z is exponentially distributed.

4. Let $(X_n, \mathcal{F}_n)$ be a martingale sequence and for each n let ε_n be an $\mathcal{F}_{n-1}$-measurable random variable. Define

$$Y_n = \sum_{i=1}^{n} \varepsilon_i (X_i - X_{i-1}), \quad Y_0 = 0$$

Assuming that Y_n is integrable for each n, show that Y_n is a martingale.

5. Let $X_1, \ldots, X_n, \ldots$ be an i.i.d. sequence with $E[X_i] = 0$ and $\mathrm{Var}[X_i] = \sigma^2 < \infty$. Prove that for any $\gamma > 1/2$, the series $\sum_{k=1}^{\infty} X_k / k^{\gamma}$ converges almost surely.

6. Consider the following process $\{X_n\}_{n \geq 0}$ taking values in $\{0, 1, \ldots\}$. Assume U_n, $n = 1, 2, \ldots$, is an i.i.d. sequence of positive integer valued random variables and let X_0 be independent of the U_n. Then

$$X_n = \begin{cases} X_{n-1} - 1 & \text{if } X_{n-1} \neq 0 \\ U_k - 1 & \text{if } X_{n-1} = 0 \text{ for the } k\text{th time} \end{cases}$$

(a) Prove that this is an irreducible homogeneous Markov chain and give its transition probability matrix $\mathbf{P}$.

(b) Show that it admits a unique stationary distribution π iff $E[U_1] < \infty$ and give an expression for π in terms of the distribution of U_1.

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1. When voting, each person rolls a die and votes for candidate A if he/she gets 1 or 2, for candidate B if he/she gets 3 or 4, and for candidate C otherwise. Use the central limit theorem to approximate the probability that A gets at least 300 more votes than B if 1 million people voted.

2. Let U and V be independent and uniformly distributed on $[0,1]$. Define $X = UV$ and $Y = U(1-V)$. Find the density of X.

3. Let $\{N_t \mid t \geq 0\}$ be a Poisson process with rate λ.

(a) Let S_n be the time at which the nth event in the process occurs. Show that the S_n has a gamma distribution

$$f(x) = \frac{\lambda^n x^{n-1}}{\Gamma(n)} \exp(-\lambda x).$$

(b) For $s < t$, show that

$$P[N_s = k \mid N_t = n] = \binom{n}{k} (s/t)^k (1 - s/t)^{n-k}.$$

4. Let (X, Y) have the bivariate normal density

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho)^2}\right].$$

(a) Show that ρ is the correlation coefficient of X and Y

(b) Show that ρ^2 is the correlation coefficient of X^2 and Y^2.

5. Suppose you toss a coin n times. Let X_n be the number of heads and let $p_n = P[X_n \text{ is divisible by } 3]$.

(a) Compute p_n in terms of a power of a matrix.

(b) Compute $\lim_{n\to\infty} p_n$.

6. A large number $N = mk$ of people are subject to a blood test. This can be administered in two ways:

(i) Each person can be tested separately. In this case N tests are required.

(ii) The blood samples of k people can be pooled and analyzed together. If the test is negative, this *one* test suffices for k people. If the test is positive,k each of the k persons must be tested separately, and altogether $k+1$ tests are required for the k people.

Assume that $p = P[\text{test is positive}]$ is the same for all people and that the test results for different people are independent.

(a) What is the probability that the test for a pooled sample of k people will be positive?

(b) Let X be the number of blood tests necesary under plan (ii). Find $E[X]$.

(c) Suppose it is known that p is close to zero. Which of plans (i) and (ii) has the smaller value of $E[X]$? Justify your answer using the expression found in part (b).

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
JANUARY, 2009

Probability (Ph. D. Version)

Instructions to the Student

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---

1. Suppose ξ_n, $n \geq 1$, are independent random variables such that ξ_n is uniformly distributed on $[-1-1/n, 1+1/n]$. Prove that $(\sum_{i=1}^n \xi_i)/\sqrt{n}$ converges in distribution and identify the limiting distribution.

2. Suppose that ξ_n are defined on the same probability space and $\mathrm{E}(\xi_n - \eta)^2 \leq 1/n^2$, where $\eta > 0$ almost surely. Prove that $n\xi_n \to \infty$ almost surely.

3. A process X_n, $n = 0, 1, 2, \ldots$ satisfies $P[0 \leq X_n \leq 1] = 1$ and

$$P(X_{n+1} = X_n^2 \,|\, X_0, X_1, \ldots, X_n) = 1/2 = P(X_{n+1} = 2X_n - X_n^2 \,|\, X_0, X_1, \ldots, X_n).$$

(That is, from x the process jumps either to x^2 or to $2x - x^2$, each with probability 1/2).

(a) Show that X_n is a martingale.

(b) Prove that if the process starts at $X_0 = x_0$, a constant, then $P(\lim_{n\to\infty} X_n = 0) = 1 - x_0$ and $P(\lim_{n\to\infty} X_n = 1) = x_0$.

4. Let ξ_i, $i \geq 0$, be a sequence of random variables such that $\xi_0 = 1$ and $\xi_{i+1} - \xi_i = \eta_i \xi_i / 2$, where η_i is equal to 1 with probability 1/2, -1 with probability 1/2, and is independent of $\xi_1, \ldots, \xi_n, \eta_0, \ldots, \eta_{n-1}$. (Thus ξ_n models the amount of money an investor will have after n days if he wins or loses half of the money daily, both with probability 1/2). Prove that $\xi_n \to 0$ almost surely but $\mathrm{E}\xi_n$ does not converge to zero.

5. Suppose you toss a coin n times. Let X_n be the number of heads and let $p_n = P[X_n \text{ is divisible by } 3]$.

(a) Compute p_n in terms of a power of a matrix.

(b) Compute $\lim_{n\to\infty} p_n$.

6. Let X_n, $n \geq 1$, be independent exponential random variables with parameter 1. Let $M_n = \min_{i\leq n} X_i$. Prove that $\sum_{n=1}^{\infty} M_n^2$ converges almost surely.

DEPARTMENT OF MATHEMATICS
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AUGUST 2008

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---

1. Let (ξ_1, ξ_2) be a Gaussian vector with zero mean and covariance matrix $\mathbf{D}$ with entries $D_{11} = D_{22} = 1$, $D_{12} = D_{21} = 1/2$. Find $\mathrm{E}(\xi_1^2 \xi_2 \,|\, 2\xi_1 - \xi_2)$.

2. Let X_n, $n \geq 0$, be a Markov chain on the state space $\mathcal{X} = \{1, 2\}$ having transition matrix $\mathbf{P}$ with elements $P_{11} = 1/3$, $P_{12} = 2/3$, $P_{21} = P_{22} = 1/2$. Let $f : \mathcal{X} \to \mathbb{R}$ be the function with $f(1) = 1$ and $f(2) = 4$. Find a function $g : \mathcal{X} \to \mathbb{R}$ such that

$$Y_n = f(X_n) - f(X_0) - \sum_{i=0}^{n-1} g(X_i), \; n \geq 1,$$

is a martingale relative to the filtration $\mathcal{F}_n^X$ generated by the process X_n.

3. Let ξ_n be independent identically distributed random variables with uniform distribution on $[0, 1]$. For which values of $\alpha > 0$ does the series

$$\sum_{n=1}^{\infty} \left(\xi_n + n^{-\alpha}\right)^{(n^{\alpha+1})}$$

converge almost surely?

4. Let X_n have a Poisson distribution with parameter n. Find a_n such that

$$\sqrt{X_n} - a_n$$

converges in distribution to a nontrivial limiting distribution and compute the limiting distribution. (A distribution is called nontrivial if it is not concentrated at one point.)

5. Three players are playing a game. At each turn, each player tosses a fair die, and the one with the largest number of points takes one dollar from the one with the least number of points. However, if any two of the players have the same number of points, then nothing happens. The game stops when one of the players is out of money.

Initially the first player had two dollars, while the second and the third players had one dollar each. What is the probability that the first player has a positive amount of money at the end of the game?

6. Let ξ_n, $n \geq 1$, be a sequence of independent random variables with $\mathrm{P}(\xi_n = -1) = \mathrm{P}(\xi_n = 1) = 1/2$. Find

$$\lim_{n\to\infty} \mathrm{P}(\xi_1 + \ldots + \xi_{2n} \geq 0 \text{ and } \xi_{n+1} + \ldots + \xi_{3n} \geq 0).$$

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---

1. Let (ξ_1, ξ_2) be a Gaussian vector with zero mean and covariance matrix D with entries $D_{11} = D_{22} = 1$, $D_{12} = D_{21} = 1/2$.

(a) Find a linear function $\eta_2 = a\xi_1 + b\xi_2$ which is independent of $\eta_1 = 2\xi_1 - \xi_2$.

(b) Find $\mathrm{E}(\xi_1^2 \xi_2 \,|\, 2\xi_1 - \xi_2)$.

2. A post office has two clerks. Three customers, A, B and C, enter simultaneously. Customers A and B go directly to the clerks and C waits until either A or B leaves before he begins service. What is the probability that A is still in the post office after B and C have left, assuming

(a) the service times are i.i.d. and equal to i with probability $1/3$ for $i = 1, 2, 3$?

(b) the service times are i.i.d. exponential variables with mean μ?

3. Suppose that N balls are independently distributed into r boxes with equal probabilities, where N has a Poisson distribution with mean λ. Let X_i be the number of balls in box i, $i = 1, \dots, r$, and let Y be the number of empty boxes.

(a) Find the joint distribution of the X_i.

(b) Find the distribution of Y.

4. Let X_n have a Poisson distribution with parameter n. Find a_n such that

$$\sqrt{X_n} - a_n$$

converges in distribution to a nontrivial limiting distribution and compute the limiting distribution. (A distribution is called nontrivial if it is not concentrated at one point.)

5. Three players are playing a game. At every turn, each player tosses a fair die, and the player with the largest number of points takes one dollar from the player with the lowest number of points. However, if any two of the players have the same number of points, then nothing happens. The game stops when one of the players is out of money. Initially the first player has two dollars, while the second and the third players have one dollar each.

(a) Model this game as a Markov chain. Describe the states and write the transition probability matrix.

(b) What is the probability that the game ends after one turn?

(c) What is the probability that the first player has a positive amount of money at the end of the game?

6. Let ξ_n, $n \geq 1$, be a sequence of independent random variables with $\mathrm{P}(\xi_n = -1) = \mathrm{P}(\xi_n = 1) = 1/2$. Find

$$\lim_{n \to \infty} \mathrm{P}(\xi_1 + \dots + \xi_{2n} \geq 0 \;\text{ and }\; \xi_{n+1} + \dots + \xi_{3n} \geq 0).$$

DEPARTMENT OF MATHEMATICS
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---

1. Let $A = \{(x,y) \in R^2 : |x-y| < a,\ |x+y| < b\}$, where $0 < a < b$. Assume that the random vector (ξ_1, ξ_2) is uniformly distributed on A.

(a) Find the density of ξ_2.

(b) Calculate $E(\xi_1 \mid \xi_2)$ as a function of ξ_2.

(c) Find the distribution of $E(\xi_1 \mid \xi_2)$.

2. Consider a discrete time Markov chain X_k with state space $S = \{(a,b) \mid a = 0, 1;\ b = 0, 1, 2, \ldots\}$ and transition probabilities

$$P[X_{k+1} = (0, j+1) \mid X_k = (0,j)] = p_j = 1 - P[X_{k+1} = (1,0) \mid X_k = (0,j)]$$

$$P[X_{k+1} = (1, j+1) \mid X_k = (1,j)] = q_j = 1 - P[X_{k+1} = (0,0) \mid X_k = (1,j)]$$

where all p_j, q_j are positive and less than 1. You may use the fact that the chain is irreducible and aperiodic without proof. Give necessary and sufficient conditions for the chain to be (a) recurrent, and (b) positive-recurrent. *Hint:* Draw a picture.

3. Let P be any probability measure on $([0,1], \mathcal{B}([0,1]))$.

(a) Show that there exists a sequence of probability measures P_n such that $P_n \to_w P$ and such that each P_n is atomic with finitely many atoms. (The symbol $\to_w$ denotes weak convergence of probability measures.)

(b) Show that there exists a sequence of probability measures Q_n such that $Q_n \to_w P$ and such that each Q_n is absolutely continuous with respect to Lebesgue measure.

4. Let $\xi_1, \xi_2, \ldots$ be independent random variables with uniform distribution on $[0,1]$. Let η_n be the median of $\xi_1, \ldots, \xi_{2n+1}$. That is, the random variables ξ_1, $\xi_2, \ldots, \xi_{2n+1}$ are arranged in increasing order, and η_n is the $(n+1)$-st element of this sequence. Prove that η_n converges to $1/2$ in probability and almost surely.

5. Consider a sequence of independent random variables $\xi_1, \xi_2, \ldots$, where for each $k \in N$, ξ_{3k} has a uniform distribution on $[0,1]$, ξ_{3k+1} is Gaussian with mean 0 and variance 3, and ξ_{3k+2} is Poisson with parameter 4. Find a such that

$$\frac{(|\xi_1| + \cdots + |\xi_n| - an)}{\sqrt{n}}$$

converges in distribution to a nontrivial limit as $n \to \infty$, and identify the limiting distribution.

6. Suppose that $\{Y_n\}_{n \geq 0}$ is a sequence of nonnegative integer-valued integrable random variables such that for all $n \geq 0$ and some constant $\varepsilon > 0$,

$$E(Y_{n+1} - Y_n \mid Y_0, Y_1, \ldots, Y_n) \leq -\varepsilon I_{[Y_n > 0]}.$$

(a) Prove that with probability 1, $Y_n = 0$ infinitely often.

(b) Use your result in (a) to prove recurrence for the homogeneous Markov chain X_k with states $\{0, 1, 2, \ldots\}$ and

$$P(X_{k+1} = X_k + 2 \mid X_k) = \frac{3}{X_k + 3} = 1 - P(X_{k+1} = X_k - 1 \mid X_k).$$

Hint: Apply (a) to the random sequence $Y_n = \max(X_n - 7, 0)$.

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---

1. Consider a sequence of Bernoulli trials with success probability $p \in (0,1)$. Define X_n as the number of failures before the n-th success.

(a) Find the probability mass function and the moment generating function of X_n.

(b) Now let $p = p_n = 1 - \alpha/n$, where $\alpha > 0$. Calculate the limiting distribution of X_n as $n \to \infty$.

2. Let (X, Y) be a Gaussian random vector with zero mean and covariance matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Compute $E[(X+Y)^2 \,|\, X - Y = z]$.

3. Consider a sequence of independent random variables $\xi_1, \xi_2, \ldots$, where, for each $k \in N$, ξ_{3k} has a uniform distribution on $[0,1]$, ξ_{3k+1} is $N(0,1)$, and ξ_{3k+2} is Poisson with parameter 4. Find a such that

$$W_n = \frac{(\xi_1 + \cdots + \xi_n - an)}{\sqrt{n}}$$

converges in distribution to a nontrivial limit as $n \to \infty$, and identify the limiting distribution.

4. Let $X_1, X_2, \ldots X_n$ be independent random variables uniformly distributed on $[0,1]$. Let

$$Y_{1,n} \leq Y_{2,n} \leq \cdots \leq Y_{n,n}$$

be the order statistics, that is, the same random variables ordered from the smallest to the largest.

(a) Derive the density of $Y_{1,n}$.

(b) Derive the joint density of $Y_{1,n}$ and $Y_{2,n}$.

(c) Find c_n such that $c_n Y_{1,n}$ has a nontrivial limiting distribution. Derive this limiting distribution.

5. Let X_1 and X_2 be independent exponentially distributed random variables with parameters λ_1 and λ_2, so that

$$P[X_i > t] = \exp(-\lambda_i t) \quad \text{for} \quad t \geq 0.$$

Define

$$\begin{aligned} N &= \begin{cases} 1 & \text{if } X_1 < X_2, \\ 2 & \text{if } X_1 \geq X_2; \end{cases} \\ U &= \min\{X_1, X_2\}; \\ V &= \max\{X_1, X_2\}; \\ W &= V - U \; = \; |X_1 - X_2|. \end{aligned}$$

Prove the following statements.

(a) $P[N = 1] = \lambda_1/(\lambda_1 + \lambda_2)$.

(b) $P[U > t] = \exp[-(\lambda_1 + \lambda_2)t]$.

(c) N and U are independent random variables.

(d) $P[W > t \,|\, N = 1] = \exp(-\lambda_2 t)$ for $t \geq 0$.

(e) U and $W = V - U$ are independent random variables.

6. A Markov chain on states $\{0, 1, 2, 3\}$ has transition probability matrix

$$\mathbf{P} = \begin{bmatrix} q & p & 0 & 0 \\ 0 & 0 & q & p \\ q & p & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

where $0 < p < 1$ and $p + q = 1$. Compute the stationary distribution in terms of p and q. Is this stationary distribution also a limiting distribution? Explain your answer.

DEPARTMENT OF MATHEMATICS
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AUGUST, 2007

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---

1. Let $X_1, X_2, \ldots$ be independent random variables with

$$P(X_n = 0) = p_n, \qquad P(X_n = -n) = P(X_n = n) = (1 - p_n)/2$$

for $n = 1, 2, \ldots$. Prove that the sequence $\{X_n\}$ diverges with probability 1 iff $\sum p_n < \infty$.

2. Let X and Y be random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$ and let $\mathcal{G}$ be a σ-field contained in $\mathcal{F}$.

(a) Show that if $E(Y \mid \mathcal{G}) = X$ and $E(Y^2) = E(X^2) < \infty$, then $X = Y$ a.s.

(b) Prove that if $E(|Y|) < \infty$ and $E(Y \mid \mathcal{G})$ has the same distribution as Y, then $E(Y \mid \mathcal{G}) = Y$ a.s. Part (a) establishes the same result under the stronger assumption that $E(Y^2) < \infty$. [*Hint:* Show that $\mathrm{sgn} X = \mathrm{sgn} E[X \mid \mathcal{G}]$ a.s. and then take $X = Y - c$ to get the desired result.]

r is a function

3. Let $X_1, X_2, \ldots$ be ~~i.i.d.~~ be random variables with means zero and $E[X_m X_n] = r(n-m)$ whenever $n \geq m$. Assume that $r(k) \to 0$ as $k \to \infty$. Prove that

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \to_p 0 \quad \text{as } n \to \infty.$$

4. Suppose that $\{X_k,\ k \geq 0\}$ is a discrete-time homogeneous Markov chain with state space the set of all integers and with $X_0 = 0$. Assume the chain is irreducible. Using any theorems you know about Markov chains, carefully stated and with hypotheses stated and verified clearly, prove the following statements.

(a) If X_k is null recurrent, then $E(|X_n|) \to \infty$ as $n \to \infty$.

(b) if X_k is positively recurrent and aperiodic, and g is a fixed bounded function defined on the integers, then $E[g(X_n)]$ has a limit which is uniquely determined by the stationary probability distribution for the Markov chain.

5. Suppose that every day a stock either gains 25 percent or loses 20 percent. Each of these events happens with probability 1/2, independently of what happened in the past. Assume that the current price of the stock is 100 dollars.

(a) Find the expected gain after 10 days.

(b) What is the probability that the price of the stock becomes less than 50 dollars before it becomes more than 150 dollars?

6. Let $Z_1, Z_2, , \ldots$ be a sequence of i.i.d. random variables with $E[Z_i] = 0$, $\mathrm{Var}[Z_i] = 1$. Let a_{nk}, $k = 1, \ldots, n$, $n = 1, 2, \ldots$, be a double array of constants satisfying

$$\lim_{n\to\infty}\left[\frac{\max_k a_{nk}^2}{\sum_{k=1}^{n} a_{nk}^2}\right] = 0$$

State and prove a central limit theorem for the sequence

$$S_n = \sum_{k=1}^{n} a_{nk} Z_k.$$

DEPARTMENT OF MATHEMATICS
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AUGUST, 2007

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---

1. Let X be a random variable whose moment generating function $M(t) = E[\exp(tX)]$ is finite for all t.

(a) Prove that

$$P[X \geq x] \leq e^{-tx} M(t), \quad t \geq 0.$$

(b) Let X have a gamma density with density

$$f(x) = \frac{\lambda^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x}$$

for $x > 0$. Prove that $P[X \geq 2\alpha/\lambda] \leq (2/e)^\alpha$. Compare this bound to Chebyshev's inequality.

2. Let $X_1, \ldots, X_n$ be i.i.d. with c.d.f $F(x)$. Assume that for some $\alpha > 0$

$$\lim_{x \to \infty} x^\alpha (1 - F(x)) = b > 0.$$

Define $\zeta_n = \max\{X_1, \ldots, X_n\}$. Prove that

$$P\left[\frac{\zeta_n}{(bn)^{1/\alpha}} \leq x\right] \to \tilde{F}(x) = \begin{cases} \exp(-1/x^\alpha) & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

3. Let $Z_1, \ldots, Z_n$ be i.i.d. $N(0,1)$ variables and define $S_k = Z_1 + \cdots + Z_k$ for $k = 1, \ldots, n$.

(a) If $m < n$, find the joint density of (S_m, S_n) and the conditional density of S_m given that $S_n = c$.

(b) If $m < n$, find the conditional density of $V = Z_1^2 + \cdots + Z_m^2$ given $W = Z_1^2 + \cdots + Z_n^2 = c$

4. (a) Suppose that X and Y are independent nonnegative random variables, that X is Poisson with mean α and that $X+Y$ is Poisson with mean β. Prove that either $\alpha = \beta$ and $P[Y = 0] = 1$ or that $\alpha < \beta$ and that Y has a Poisson distribution with mean $\beta - \alpha$.

(b) Let $\{N_t, t \geq 0\}$ be a Poisson process with rate λ and let $(a, b]$ be a bounded interval contained in $[0, \infty)$. Let $N(a, b)$ be the number of events occurring at times lying in $(a, b]$. Show that $N(a, b)$ has a Poisson distribution with mean $\lambda(b - a)$.

5. A gambler make a series of \$1 bets. He decides to quit betting as soon as his net winnings reach \$25 or his net losses reach \$50. Suppose the probabilities of winning or losing each bet are both equal to 1/2.

(a) Find the probability that when he quits he will have lost \$50.

(b) Find his expected loss.

(c) Find the expected number of bets he will make before quitting.

6. Consider a renewal process with interarrival times distributed with c.d.f. F. Let W denote the smallest time (measured from the origin) such that the waiting time for the next renewal exceeds a given constant s.

(a) Find $E[W]$.

(b) Determine an integral equation satisfied by $H(t) = P[W \leq t]$.

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---

1. Let ξ_i be a sequence of i.i.d. random variables with $P[\xi_i > 0] = 1$, $E[\xi_i] = a$, and $\mathrm{Var}\,[\xi_i] = \sigma^2 < \infty$. Define $S_n = \sum_{i=1}^n \xi_i$. Prove that

$$\sqrt{S_n} - \sqrt{na} \to N[0, \sigma^2/(4a)]$$

in distribution as $n \to \infty$.

2. Let $X_1, X_2, \ldots$ be independent identically distributed random variables with $P(X_i = 0) = P(X_i = 1) = 1/2$.

(a) Prove that the series $Y = \sum_1^\infty X_i/2^i$ converges with probability 1.

(b) Prove that Y is uniformly distributed on $(0, 1)$.

3. Ann and Bob are gambling at a casino. In each game the probability of winning one dollar is 0.48, and the probability of losing one dollar is 0.52. Ann decides to play 20 games, but will stop after 2 games if she wins them both. Bob decides to play 20 games, but will stop after 10 games if he wins at least 9 out of the first 10. Which is larger: the amount of money Ann is expected to lose or the amount of money Bob is expected to lose?

4. Let $X_1, X_2, \ldots$ be independent identically distributed $N(0,1)$ random variables and let $Y_1, Y_2, \ldots$ be independent identically distributed exponential random variables with mean one. Prove that

$$P[\max\{Y_1, \ldots, Y_n\} \geq \max\{X_1, \ldots, X_n\}] \to 1$$

5. Let $\{Y_n : n \geq 0\}$ be a branching process. That is, $Y_0 = 1$ and

$$Y_{n+1} = \sum_{i=1}^{Y_n} Z_{in}$$

where $\{Z_{in} : i = 1, \ldots, n,\ n = 1, 2, \ldots\}$ is a double array of i.i.d. nonnegative integer-valued random variables. Assume that $E[Z_{in}] = \mu > 1$ and $\operatorname{Var} Z_{in} = \sigma^2 < \infty$.

(a) Prove that $T_n = Y_n/\mu^n$ is a martingale.

(b) Find a recursive expression for $\operatorname{Var} T_n$ and prove that it remains bounded as $n \to \infty$.

(c) Prove that T_n converges a.s. to a nondegenerate random variable as $n \to \infty$.

6. Let $\{X_t : t \geq 0\}$ be a continuous time Markov chain with state space $\{0, 1, 2\}$, $X_0 = 0$ and intensity matrix

$$\mathbf{Q} = \begin{bmatrix} -4 & 2 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{bmatrix}.$$

(a) For $j = 1, 2$, find the expected time to return to state 0 given that $X_t = j$.

(b) Let $U_t = \inf\{s - t \mid s > t \text{ and } X_s = 0\}$ be the time until the next visit to state 0. (We define $U_t = 0$ if $X_t = 0$). Find $\lim_{t\to\infty} E[U_t]$.

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JANUARY, 2007

Probability (M.A. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (**NOT YOUR NAME**) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

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1. Let $X_1, \ldots, X_n$ be i.i.d. with c.d.f $F(x)$. Assume that for some $\alpha > 0$

$$\lim_{x\to\infty} x^\alpha[1-F(x)] = b > 0.$$

Define $\zeta_n = \max\{X_1, \ldots, X_n\}$. Prove that

$$P\left[\frac{1}{(bn)^{1/\alpha}}\zeta_n \le x\right] \to \tilde{F}(x) = \begin{cases} \exp(-1/x^\alpha) & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

2. Let $\xi_1, \xi_2, \ldots$ be a sequence of i.i.d. random variables with $P[\xi_i > 0] = 1$, $E[\xi_i] = a$, and $\mathrm{Var}\,[\xi_i] = \sigma^2 < \infty$. Define $S_n = \sum_{i=1}^n \xi_i$. Prove that

$$\sqrt{S_n} - \sqrt{na} \to N[0, \sigma^2/(4a)]$$

in distribution as $n \to \infty$.

3. Let X and Y be independent standard normal random variables. Calculate $E[X^3 \mid X+Y]$.

4. Let the random variable X have an exponential distribution with parameter λ. Prove that the integer and fractional parts of X are independent.

5. Let a Markov chain have state space $\{1, 2, 3, \ldots\}$ and transition probability matrix

$$\mathbf{P} = \begin{bmatrix} f_1 & f_2 & f_3 & \cdots \\ 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{bmatrix}$$

where $f_k \geq 0$, $\sum f_k = 1$ and $\sum k f_k = \mu < \infty$. Let π_k denote the limiting probabilities of this Markov chain. Show that $\pi_1 = 1/\mu$ and that $\pi_k = (1/\mu) \sum_k^\infty f_j$.

6. An auto inspection takes c minutes to complete. The inspection station opens at 7 a.m., and cars arrive according to a Poisson process with rate λ. Assume no cars are present at 7 a.m.

(a) Find the probability that the second car to arrive must wait for its inspection.

(b) Calculate the mean waiting time for the second car.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
AUGUST, 2006

Probability (Ph. D. Version)

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---

1. Consider a four state Markov chain with state space $\{1,2,3,4\}$, initial state $X_0 = 1$, and transition probability matrix

$$\begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/6 & 1/3 & 1/6 & 1/3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Compute $\lim_{n\to\infty} P(X_n = 3)$.

(b) Let $\tau = \inf\{n \geq 0 : \ X_n \in \{3,4\}\}$. Compute $E(\tau)$.

2. If $X_1, \ldots, X_n$ are independent uniformly distributed random variables on $[0,1]$, then let $X_{(2),n}$ be the second smallest among these numbers. Find a nonrandom sequence a_n such that $T_n = a_n - \log X_{(2),n}$ converges in distribution, and compute the limiting distribution function.

3. Suppose that the real-valued random variables ξ and η are independent, that ξ has a bounded density $p(x)$ (for $x \in \mathbf{R}$, with respect to Lebesgue measure), and that η is integer valued.

(a) Prove that $\zeta = \xi + \eta$ has a density.

(b) Calculate the density of ζ in the case where $\xi \sim \text{Uniform}[0, 1]$ and $\eta \sim \text{Poisson}(1)$.

4 Let $(N(t),\ t \geq 0)$ be a Poisson process with unit rate, and let

$$W_{m,n} = \sum_{k=1}^{n} I\left\{ N\left(\frac{mk}{n}\right) - N\left(\frac{m(k-1)}{n}\right) \geq 2 \right\}$$

where $I(A)$ is the indicator of the event A, equal to 1 for elements of the probability space within A and equal to 0 for elements of A^c.

(a) Find a formula for $E(W_{m,n})$ in terms of m, n.

(b) Show that if $m = n^{\alpha}$, with $\alpha > 1/2$ a fixed constant, then $W_{m,n} \to \infty$ in probability.

5. Let $X_0 = 0$ and for $n \geq 1$, $X_n = \sum_{j=1}^{n} \xi_j$, where the r.v.'s ξ_j are i.i.d. with $P(\xi = -2) = 1/4$, $P(\xi = 1) = 3/4$.

(a) Prove that there exist constants a, b such that $Y_n = X_n - an$ and $Z_n = \exp(bX_n)$ are martingales.

(b) If $\tau = \inf\{n \geq 1 : X_n = 3\}$, then prove that $\tau < \infty$ almost surely, and find $E(\tau)$.

(c) Prove that $\exp(bX_n)$ is **not** a uniformly integrable martingale.

6. For each $n = 1, 2, \ldots$, let X_{nk}, $k = 1, \ldots, n$, be independent. Let $b_n > 0$ with $b_n \to \infty$, let $Y_{nk} = X_{nk} I\{|X_{nk}| \le b_n\}$, and define $a_n = \sum_{k=1}^n E[Y_{nk}]$. Suppose that as $n \to \infty$,

$$\sum_{k=1}^n P[\,|X_{nk}| > b_n] \to 0 \quad \text{and} \quad \frac{1}{b_n^2} \sum_{k=1}^n E[Y_{nk}^2] \to 0.$$

(a) Let $T_n = \sum_{k=1}^n Y_{nk}$. Prove from first principles that $(T_n - a_n)/b_n \to 0$ in probability.

(b) Let $S_n = \sum_{k=1}^n X_{nk}$. Prove from first principles that $(S_n - a_n)/b_n \to 0$ in probability.

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---

1. If $X_1, \ldots, X_n$ are independent uniformly distributed random variables on $[0, 1]$, then let X_{2n} be the second smallest among these numbers.

(a) Calculate the density and c.d.f. of X_{2n}.

(b) Find a nonrandom sequence a_n such that $T_n = a_n - \log X_{2n}$ converges in distribution, and compute the limiting distribution function.

2. Suppose that the real-valued random variables ξ, η are independent, that ξ has a bounded density $f(x)$ (for $x \in \mathrm{R}$, with respect to Lebesgue measure), and that η is integer-valued.

(a) Prove that $\zeta = \xi + \eta$ has a density.

(b) Calculate the density of ζ in the case where $\xi \sim$ Uniform$[0, 1]$ and $\eta \sim$ Poisson(1).

3. Let the number of cars passing a certain point on a highway be a Poisson process $(N(t),\ t \geq 0)$ with rate λ cars per second. A deer needs s seconds to cross the highway.

(a) What is the probability that the deer crosses the highway without encountering a car?

(b) The deer can escape from a single car, but it will be injured if it encounters two or more cars while crossing the road. What is the probability that the deer is uninjured while trying to cross the road?

4. Suppose n independent trials are conducted. The trials are identical and may result in any of r outcomes with positive probabilities $p_1, \ldots, p_r$. Let X be the number of outcomes that never occurred during the n trials. Find $E[X]$ and show that among all probability vectors $(p_1, \ldots, p_r)$, $E[X]$ is minimized when $p_i = 1/r$.

5. Let R and Θ be independent random variables, let R^2 have an exponential density with mean 2, and let Θ be uniformly distributed on the interval $[0, 2\pi]$.

(a) Show that $X = R\cos\Theta$ and $Y = R\sin\Theta$ are i.i.d. $N(0,1)$.

(b) Use the result of (a) to suggest an algorithm to generate i.i.d. standard normal random variables from i.i.d. uniform variables on $[0,1]$.

6. Consider a four state Markov chain with state space $\{1,2,3,4\}$, initial state $X_0 = 1$, and transition probability matrix

$$\begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/6 & 1/3 & 1/6 & 1/3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Compute $\lim_{n\to\infty} P(X_n = 3)$.

(b) Let $\tau = \inf\{n \geq 0 :\ X_n \in \{3,4\}\}$. Compute $E(\tau)$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
JANUARY, 2006

Probability (Ph. D. Version)

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---

1. Let $X_1, X_2, \ldots, X_n, \ldots$ be independent identically distributed random variables such that $E[X_n] = 0$ and $|X_n| \leq 1$ a.s., and let $S_n = \sum_{k=1}^{n} X_k$.

(a) Find a number c such that $S_n^2 - cn$ is a martingale, and justify the martingale property.

(b) Define $\tau_m = \inf\{n \geq 1 : S_n > 2m \quad \text{or} \quad S_n < -m\}$. Compute $\lim_{m\to\infty} P(S_{\tau_m} > 2m)$.

(c) Compute $\lim_{m\to\infty} E[\tau_m]/m^2$.

2. Let $N_1(t)$, $N_2(t)$ be independent Poisson processes with respective parameters λ and λ^2, where λ is an unspecified positive real number. For each $r \geq 1$, let $\tau_r \equiv \inf\{t > 0 : N_1(t) \geq r\}$. Show that $\alpha_r = E[N_2(\tau_r^2)]$ does not depend upon λ, and find α_n explicitly.

3. Let X_{kn} for $1 \leq k \leq n$ be independent random variables such that

$$P(X_{kn} = 0) = 1 - \frac{1}{n} \quad \text{and} \quad P(X_{kn} = k^2) = \frac{1}{n}$$

(a) Find the characteristic function of $S_n \equiv \sum_{k=1}^n X_{kn}$.

(b) Show that S_n/n^2 converges in distribution to a non-degenerate random variable.

4. Let (X, Y) be a Gaussian random vector with zero mean vector and covariance matrix $\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$.

(a) Compute $E[(X+Y)^2 \,|\, X - Y = z]$.

(b) Find the probability density of $5X^2 - 8XY + 5Y^2$.

5. A continuous-time Markov chain $X(t)$ takes integer values and jumps $+2$ with rate 1 per unit time and jumps -1 with rate 2 per unit time.

(a) Show that regardless of where it starts, the Markov chain revisits 0 infinitely often with probability 1.

(b) Show that regardless of where the chain starts, $T^{-1} \int_0^T X(t)dt$ has a limit as $T \to \infty$, and find it.

6. Let $\{\varepsilon_k\}_{k\geq 1}$ be a sequence of i.i.d. random variables with $P(\varepsilon_k = 1) = P(\varepsilon_k = -1) = 1/2$, and let $\{U_k\}_{k\geq 1}$ be a sequence of *iid* Uniform$(0,1)$ random variables independent of the $\{\varepsilon_k\}$. Define

$$S_n = \sum_{k=1}^n U_k \varepsilon_k, \qquad T_n = \sum_{k=1}^n (U_k^2 - \frac{1}{2})\, \varepsilon_k.$$

Find $\lim_{n\to\infty} P(|S_n| > T_n\sqrt{20/7})$.

DEPARTMENT OF MATHEMATICS
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AUGUST 2005

Probability (Ph.D. Version)

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---

1. Let $\xi_1, \ldots, \xi_n$ be independent random variables and let η be a random variable with a finite variance and $E[\eta] = a$. Prove that

$$\frac{1}{n} \sum_{k=1}^{n} E[\eta \mid \xi_k] \to a$$

in probability.

2. Let the random variables $\xi(\omega)$ and $\eta(\omega)$, defined on a common probability space $(\Omega, \mathcal{F}, P)$, have characteristic functions $f(t)$ and $g(t)$, respectively. Prove that

$$\sup_t |f(t) - g(t)| \leq 2P[\xi \neq \eta].$$

3. Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of Gaussian random variables with zero means and $\mathrm{Cov}[\xi_i, \xi_j] = a_{ij}$. Let $\eta_n = \sum_{i=1}^{n} \xi_i$. Prove that $\{\eta_n\}$ is a martingale with respect to the sigma fields $\mathcal{F}_m = \sigma(\xi_1, \ldots, \xi_m)$ if and only if $a_{ij} = 0$ whenever $i \neq j$.

4. Consider a sequence of iid positive random variables $X = \{X_n : n = 1, 2, \ldots\}$, with $F(x) = P[X_1 \leq x]$. Put $S_n = \sum_{j=1}^{n} X_j$. Let $\{N_t : t \geq 0\}$ be the renewal process generated by X. That is,

$$N_t = \sup\{n \geq 1 : S_n \leq t\} \quad \text{for } t \geq 0.$$

Assume that X_n possesses mean μ, variance σ^2 and finite third moment. Let $\gamma_t = S_{N_t+1} - t$ be the residual lifetime, and set $A(t) = E\gamma_t^2$.

(a) Show that $A(t)$ satisfies the following renewal equation

$$A(t) = \int_t^{\infty} (x-t)^2 dF(x) + \int_0^t A(t-x) dF(x).$$

(b) Show that

$$A(t) = E\gamma_t^2 \to \frac{EX^3}{3\mu} \quad \text{as } t \to \infty.$$

5. For each $n = 1, 2, \ldots$ let $\{X_{ni}, 1 \leq i \leq n\}$ be iid random variables with

$$P[X_{ni} = k] = \left(1 - \frac{\lambda}{n}\right)\left(\frac{\lambda}{n}\right)^k, \quad k = 0, 1, 2, \ldots$$

where $\lambda > 0$ is fixed. Let $S_n = X_{n1} + \ldots + X_{nn}$. Show that S_n converges in distribution and find the limiting distribution.

6. Let $\{X_n, n = 0, 1, \ldots\}$ be a Markov chain starting from $X_0 = 0$, with stationary transition probabilities P_{ij}^n . Let

$$\begin{aligned} \tau &= \min\{n \geq 1 : X_n = 0\} \\ &= \infty, \quad \text{if no such } n \text{ exists.} \end{aligned}$$

Let

$$G(s) = \sum_{n=1}^{\infty} s^n P[\tau = n], \quad H(s) = \sum_{n=1}^{\infty} s^n P[X_n = 0].$$

(a) Find the relationship between $G(s)$ and $H(s)$.

(b) Let $\theta = \sum_{n=1}^{\infty} P[X_n = 0]$. Assuming that $\theta < \infty$, find a formula for $P[\tau < \infty]$ in terms of θ.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
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~~JANUARY~~ 2005
August

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---

1. Prove that a 2×2 stochastic matrix is a two step transition probability matrix for some stationary Markov chain if and only if the sum of its diagonal entries is greater than one.

2. Let $\xi_1, \ldots, \xi_n$ be positive, independent and identically distributed random variables. Prove that for $k \leq n$

$$E\left[\frac{\xi_1 + \cdots + \xi_k}{\xi_1 + \cdots + \xi_n}\right] = \frac{k}{n}.$$

3. Each of the random variables X and Y can take on only two possible values.

(a) Prove that if $E[XY] = E[X]E[Y]$ then X and Y are independent.

(b) Show that in general, the equality $E[XY] = E[X]E[Y]$ does not imply independence.

4. A particle moves on a circle through points labeled 0,1,2,3,4 (arranged clockwise). At each step it moves one step to the right (clockwise) with probability p and to the left (counterclockwise) with probability $1-p$. Let X_n denote the location of the particle after the nth step. The process $\{X_n \mid n \geq 0\}$ is a Markov chain.

(a) Find the transition probability matrix.

(b) Argue that $\{X_n\}$ has a limiting distribution and find the limiting probabilities.

5. Let $X_1, \ldots, X_n$ be $N(0,1)$ random variables and let $E[X_i X_j] = \rho$ whenever $i \neq j$.

(a) Find the joint distribution of X_1 and $X_2 - \rho X_1$.

(b) Show that $U = X_1 + \ldots + X_n$ and $\sum_{i=1}^n c_i X_i$ are independent whenever $\sum_{i=1}^n c_i = 0$.

6. Consider a sequence of iid positive random variables $X = \{X_n : n = 1, 2, \ldots\}$, with $F(x) = P[X_1 \leq x]$. Put $S_n = \sum_{j=1}^n X_j$. Let $\{N_t : t \geq 0\}$ be the renewal process generated by X, i.e.,

$$N_t = \sup\{n \geq 1 : S_n \leq t\}, \quad \text{for } t \geq 0.$$

Assume that X_n possesses finite mean μ, variance σ^2 and third moment. Let $\gamma_t = S_{N_t+1} - t$ be the residual lifetime, and set $A(t) = E\gamma_t^2$.

(a) Show that $A(t)$ satisfies the following the renewal equation:

$$A(t) = \int_t^\infty (x-t)^2 dF(x) + \int_0^t A(t-x) dF(x).$$

(b) Show that

$$A(t) = E\gamma_t^2 \longrightarrow \frac{EX^3}{3\mu} \quad \text{as } t \longrightarrow \infty.$$

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
JANUARY 2005

Probability (Ph.D. Version)

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---

1. The random variables ξ and η are independent. The variable ξ is uniformly distributed on $[0, 1]$, and η takes on the value 1 with probability p and the value -1 with probability $1-p$. Let $\zeta = \xi + \eta$. Prove that ζ has a density and evaluate it.

2. Let ξ and η be independent $N(0, 1)$ variables. Find $E[\xi\eta \,|\, \xi - \eta]$.

3. Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be i.i.d. random variables with mean zero and finite positive variance. Prove that with probability one the random variables $X_n = \sum_{k=1}^{n} \xi_k$, $n = 1, 2, \ldots$ take positive values infinitely often and take negative values infinitely often.

Hint: Use the Zero-One Law and the Central Limit Theorem.

4. Assume the two sequences of random variables $\{X_n\}$ and $\{Y_n\}$ are equivalent; that is,

$$\sum_{n=1}^{\infty} P[X_n \neq Y_n] < \infty.$$

Prove the following statements.

(a) $\sum_{n=1}^{\infty}(X_n - Y_n)$ converges with probability 1.

(b) If $a_n \uparrow \infty$, then

$$\frac{1}{a_n}\sum_{j=1}^{n}(X_j - Y_j) \to 0 \quad \text{with probability 1.}$$

(c) If

$$\frac{1}{a_n}\sum_{j=1}^{n} X_j \xrightarrow{p} X$$

then so does

$$\frac{1}{a_n}\sum_{j=1}^{n} Y_j.$$

5. The transition matrix of a Markov chain with three states is

$$\mathbf{P} = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/3 & 1/3 & 1/3 \\ 1/6 & 5/12 & 5/12 \end{bmatrix}.$$

Let ν_N be the number of transitions from State 3 to State 2 in the first N steps. Calculate $\lim_{N\to\infty} \nu_N/N$.

6. Let $\{X_n \mid n \geq 0\}$ be a stationary Markov chain with state space $\{0, 1, 2, \ldots\}$ and transition probability matrix $\mathbf{P}$. Prove that

$$P[X_n = i \mid X_m,\ m \neq n] = P[X_n = i \mid X_{n-1}, X_{n+1}].$$

DEPARTMENT OF MATHEMATICS
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1. Let $\{\xi_i\}$ be a sequence of independent exponentially distributed random variables with common distribution

$$F(x) = P[\xi_i \le x] = \begin{cases} 1 - \exp(-x) & \text{if } x \ge 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Let $\eta_n = \max\{\xi_1, \ldots, \xi_n\}$. Prove that a nonrandom sequence $\{a_n\}$ can be chosen such that $\{\eta_n - a_n\}$ has a limiting distribution, and find this distribution.

2. Let $\{\xi_n\}$ and $\{\eta_n\}$ be random sequences such that $\{\ln \xi_n\}$ and $\{\ln \eta_n\}$ are submartingales with respect to the same family of σ-fields $\{\mathcal{F}_n\}$. Prove that $\zeta_n = \sqrt{\xi_n \eta_n}$ is also a submartingale with respect to $\{\mathcal{F}_n\}$.

3. Let X_n, $n = 1, 2, \ldots$, be independent random variables with $EX_n = 0$ and EX_n^4 bounded, say by b. Let $S_n = \sum_{i=1}^n X_i$. Prove successively that

(a) $ES_n^4 \le cn^2$, for some positive c.

(b) $S_n/n \to 0$ with probability 1.

4. Let X be a Poisson random variable with parameter λ.

(a) Show that $Y_\lambda = (X - \lambda)/\sqrt{\lambda}$ converges in distribution to $N(0,1)$ as $\lambda \to \infty$.

(b) Show that

$$e^{-n}\left(1 + n + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!}\right) \to \frac{1}{2}$$

as $n \to \infty$.

5. Let $N = \{N(t); t \ge 0\}$ be a renewal process with with interarrival times $X_1, X_2, \ldots$ and finite mean interarrival time $\mu = 2/\lambda$. Suppose that the excess or residual life of N, γ_t, has the distribution

$$P[\gamma_t > z] = e^{-\lambda z}\left[1 + \frac{\lambda z}{2}\left(1 + e^{-2\lambda t}\right)\right], \quad z > 0.$$

(a) Calculate the expected value of γ_t.

(b) Use Wald's identity to obtain the renewal function $M(t) = EN(t)$.

(c) Determine

$$\phi(s) = E[\exp(-sX)],$$

the Laplace transform of the interarrival time distribution F.

6. Consider a random walk $\{X_t \mid t = 0, 1, 2, \ldots\}$ whose state space is the lattice $\{0, \pm 1, \pm 2, \ldots, \pm n, \ldots\}$. From an even numbered lattice point $2n$ the particle moves one step to the right with probability p_e and to the left with probability $q_e = 1 - p_e$. From an odd numbered lattice point $2n+1$ it moves one step to the right with probability p_o and to the left with probability $q_o = 1 - p_o$. The random walk starts at the origin 0.

(a) Let $p_e p_o = q_e q_o$. Define $\tau_{2N} = \min\{t : |X_t| = 2N\}$. Calculate $E[\tau_{2N}]$.

(b) Prove that if $p_e p_o > q_e q_o$, then $\lim_{t\to\infty} X_t = \infty$. Estimate how fast X_t tends to infinity.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
January, 2004

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---

1. X, Y are independent exponentially distributed with $\lambda = 1$. Derive the distribution of

$$U = \exp\{-2\min(X, Y)\}.$$

2. X, Y are independent $\mathrm{N}(0, 1)$ random variables.
(a) Prove that $X + Y$ and $X - Y$ are independent.
(b) Calculate $\mathrm{E}(\mathrm{X} + 2\mathrm{Y} \mid \mathrm{X} + \mathrm{Y})$.
(c) Compute $\mathrm{E}(\mathrm{Y} \mid \mathrm{Y} > 0)$.

3. Suppose both $X_1, ..., X_m$ and $Y_1, ..., Y_n$ are iid sequences from the Exponential(1) distribution. Define $\overline{X} = \frac{1}{m}\sum_{i=1}^{m} X_i$, $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$. Let

$$B_{m,n} = \frac{m\overline{X}}{m\overline{X} + n\overline{Y}}$$

where $m/(m+n) \to \alpha$ as $m, n \to \infty$.
For $0 < \alpha < 1$, derive the asymptotic distribution as $m, n \to \infty$ of

$$\frac{\sqrt{m+n}\left(B_{m,n} - \frac{m}{m+n}\right)}{\sqrt{\alpha(1-\alpha)}}$$

4. Let S_n be the number of successes in n Bernoulli trials with success probability $p = 1/2$. Prove that

(i) $\lim_{n\to\infty} P\left\{\max_{1\le k\le n}\left|S_k - \frac{k}{2}\right| > \sqrt{n\ln n}\right\} = 0$

(ii) $\liminf_{n\to\infty} P\left\{\max_{1\le k\le n}\left|S_k - \frac{k}{2}\right| > \sqrt{n}\right\} > 0$

5. Consider the Markov chain X_n with two states 0 and 1, and transition matrix

$$\mathbf{P} = \begin{pmatrix} 1/4 & 3/4 \\ 3/4 & 1/4 \end{pmatrix}$$

Determine the limit

$$\lim_{n\to\infty} P(X_n = 0, X_{n+1} = 0)$$

6. Let $U_1, U_2, ...$ be independent Uniform(0,1) random variables. Define a random variable X by

$$X + 1 = \min\left\{n : \prod_{i=1}^{n} U_i < e^{-\lambda}\right\}, \quad \text{where} \quad \prod_{i=1}^{0} \mathrm{U_i} \equiv 1.$$

Show that the distribution of X is Poisson(λ).

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
August, 2003

Probability (Ph. D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

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c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

---

1. Let $X_1, X_2, \ldots$ be an infinite sequence of independent identically distributed random variables having double exponential distribution with density

$$f(x) = (1/2)e^{-|x|}, \ -\infty < x < +\infty.$$

Prove that with probability one,

$$\limsup_{n\to\infty} \frac{|X_n|}{\ln n} = 1.$$

2. Let $\{X_n, n = 1, 2, \ldots\}$ be a random process with constant mean $E(X_n) = \mu$ and covariance $R(n-m) = \text{cov}(X_n, X_m)$ depending only on $n - m$. Prove that if $R(n) \to 0$, $n \to \infty$, then $(1/n)\sum_1^n X_i \to \mu$ in probability as $n \to \infty$.

3. Let $X_1, X_2, \dots$ be a sequence of independent random variables, X_n having a uniform distribution on $[-n, n]$, $n = 1, 2, \dots$. Prove that the Central Limit Theorem (CLT) holds: for $S_n = X_1 + \dots + X_n$, $(S_n - E(S_n))/\sqrt{\text{var}(S_n)}$ converges in distribution to $Z \sim N(0,1)$ as $n \to \infty$.

4. Let ξ_1, ξ_2, ξ_3 be random variables with finite second moments. Prove that the relations

$$E(\xi_1|\xi_2) = \xi_2,\ E(\xi_2|\xi_3) = \xi_3,\ E(\xi_3|\xi_1) = \xi_1$$

imply that with probability one $\xi_1 = \xi_2 = \xi_3$.

5. Let X be a random variable with symmetric distribution. Prove the following properties of its characteristic function $f(t) = E(e^{itX})$:
(i) $1 + f(2t) \geq 2[f(t)]^2$,
(ii) if $E|X| < \infty$, then $f(t)$ satisfies the Lipschitz condition, i.e.,

$$|f(t+\Delta) - f(t)| \leq C|\Delta| \text{ for some } C > 0 \text{ and all } t, \Delta.$$

6. For a sequence of random variables $X_1, X_2, \dots$ set $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ (the σ-algebra generated by $X_1, \dots, X_n$).
(i) Prove that if $(X_n, \mathcal{F}_n)$, $n = 1, 2, \dots$ is a martingale with $E|X_n|$ finite for all n, the sequence of variances $\sigma_n^2 = \text{var}(X_n), n = 1, 2, \dots$ is nondecreasing. You must consider the cases of both finite and infinite variances.
(ii) What are necessary and sufficient conditions for $\sigma_N^2 = \sigma_{N+1}^2 = \sigma_{N+2}^2 = \dots$ for some N?

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
January, 2003

Probability (MA Version)

Instructions to the Student

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c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

---

1. Suppose that X_n, Y_n, Z_n are triples of random variables for each n, satisfying

$$E(X_n \mid Z_n) = Y_n \quad , \qquad 0 < \mathrm{Var}(X_n) < \infty$$

Show that

(a) $\mathrm{Var}(Y_n) \le \mathrm{Var}(X_n)$, and

(b) if $\mathrm{Var}(Y_n)/\mathrm{Var}(X_n) \to 1$ as $n \to \infty$, then

$$\frac{X_n - E(X_n)}{\mathrm{Var}(X_n)} - \frac{Y_n - E(Y_n)}{\mathrm{Var}(Y_n)} \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty$$

2. Let $X_1, .., X_n$ be iid N$(0,1)$ random variables. Also let $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$, $a_2 = \frac{1}{n}\sum_{i=1}^n X_i^2$, and $M_2 = \frac{1}{n}\sum_{i=1}^n (X_i - \overline{X})^2$. Evaluate in terms of M_2 and n the conditional expectation

$$E(a_2 \mid X_1 - \overline{X}, ..., X_n - \overline{X})$$

3. Suppose that V_i, $i = 1, 2, \ldots$, are iid Unif$[-1, 1]$ random variables. Show that $\prod_{i=1}^n \left(1 + \frac{2}{\sqrt{n}} V_i\right)$ converges in distribution as $n \to \infty$, and find the limiting distribution exactly.

4. You are engaged in an infinite sequence of independent trials conducted under identical conditions. On any given trial the events A, B are mutually exclusive with fixed probabilities $P(A), P(B)$, respectively.

a. What is the probability that A will occur before B?

b. In repeated independent tossings of a pair of fair dice, what is the probability that the sum of 3 will appear before the sum of 7?

Note: The "sum of 3" and "sum of 7" refer to the total number of dots showing up on the faces of the two dice tossed in each trial, and that the 6 faces of each die have 1 through 6 dots on them.

5. Let X, Y be independent random variables. Prove that if for all u we have $P(X > u) \geq P(Y > u)$, then $P(X > Y) \geq 0.5$.

6. Let X be a continuous random variable and assume $\mathrm{E}|\mathrm{X}|^{\mathrm{k}} < \infty$ for some $k > 0$.

a. Show that as $n \to \infty$

$$n^k P\{|X| > n\} \to 0$$

b. Show that

$$\mathrm{E}|\mathrm{X}|^{\mathrm{k}} = \mathrm{k}\int_0^\infty \mathrm{x}^{\mathrm{k}-1}\mathrm{P}\{|\mathrm{X}| > \mathrm{x}\}\mathrm{dx}$$

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
January, 2003

Probability (Ph. D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

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c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

---

1. Suppose that X_n, Y_n, Z_n are triples of random variables for each n, satisfying

$$E(X_n \,|\, Z_n) = Y_n \quad , \qquad 0 < \mathrm{Var}(X_n) < \infty$$

Show that

(a) $\mathrm{Var}(Y_n) \le \mathrm{Var}(X_n)$, and

(b) if $\mathrm{Var}(Y_n)/\mathrm{Var}(X_n) \to 1$ as $n \to \infty$, then

$$\frac{X_n - E(X_n)}{\mathrm{Var}(X_n)} - \frac{Y_n - E(Y_n)}{\mathrm{Var}(Y_n)} \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty$$

2. X_n and Y_n are sequences of random variables. Suppose $X_n \to X$ in distribution as $n \to \infty$, and assume that for any finite c,

$$\lim_{n\to\infty} P(Y_n > c) = 1$$

Show that we also have,

$$\lim_{n\to\infty} P(X_n + Y_n > c) = 1$$

3. Suppose that V_i, $i = 1, 2, \ldots,$ are iid Unif$[-1, 1]$ random variables.

(a) Show that $\prod_{i=1}^{n} \left(1 + \frac{2}{\sqrt{n}} V_i\right)$ converges in distribution as $n \to \infty$, and find the limiting distribution exactly.

(b) Show that $\prod_{k=1}^{n} \left(1 + k^{-2/3} V_k\right)$ converges almost surely, as $n \to \infty$.

4. You are engaged in an infinite sequence of independent trials conducted under identical conditions. On any given trial the events A, B are mutually exclusive with fixed probabilities $P(A), P(B)$, respectively.

a. What is the probability that A will occur before B?

b. In repeated independent tossings of a pair of fair dice, what is the probability that the sum of 3 will appear before the sum of 7?

Note: The "sum of 3" and "sum of 7" refer to the total number of dots showing up on the faces of the two dice tossed in each trial, and that the 6 faces of each die have 1 through 6 dots on them.

5. Let $X_1, X_2, \ldots$ be iid Bernoulli random variables and let $Y_n = \frac{1}{n} \sum_{i=1}^{n} X_i$.

(a) Show that for $p \neq 1/2$ we have the convergence in distribution

$$\sqrt{n}[Y_n(1 - Y_n) - p(1 - p)] \to \mathrm{N}[0, (1 - 2\mathrm{p})^2 \mathrm{p}(1 - \mathrm{p})]$$

(b) Obtain the asymptotic distribution of $Y_n(1 - Y_n) - 1/4$ when $p = 1/2$.

6. Let $\{\epsilon_t\}$, $t = 0, \pm 1, \pm 2, \cdots$, be a sequence of uncorrelated real-valued random variables with mean zero and variance σ_ϵ^2 (i.e. white noise) and define a real-valued weakly stationary stochastic process $\{Z_t\}$ by the stochastic difference equation

$$Z_t = \phi_1 Z_{t-1} + \epsilon_t, \qquad t = 0, \pm 1, \pm 2, \cdots$$

where $|\phi_1| < 1$.

a. Prove that the partial sums

$$\sum_{j=0}^{n} \phi_1^j \epsilon_{t-j}$$

converge to Z_t in mean square as $n \to \infty$.

b. Obtain $E[Z_t]$, $E[Z_t^2]$, and $\mathrm{Cov}[Z_t, Z_{t-k}]$, $k = 0, \pm 1, \pm 2, \cdots$.

c. Describe the behavior of $\{Z_t\}$ in the three cases when ϕ_1 is 0, positive, and negative.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
AUGUST, 2002

Probability (MA Version)

Instructions to the Student

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d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

---

1. A r.v. Y with pdf $g(x)$ is called *stochastically bigger* than a r.v. X with pdf $f(x)$ if $P(Y > u) \geq P(X > u)$ for all $u \in \mathbb{R}$. Prove that if the likelihood ratio $g(x)/f(x)$ is monotone increasing, then Y is stochastically bigger than X.

2. Let X be a uniform (0,1) r.v., and suppose the r.v. Y is independent of X, with arbitrary distribution. Prove that the r.v. Z, which is the fractional part of $X + Y$, is a uniform (0,1) r.v.

3. The lifetimes of n computer systems are assumed to be independent and exponentially distributed with mean 1.
(a) Calculate the pdf of L, the life time of the system that survives the longest.
(b) Show that the times between failures are independent.

4. Let X_n be independent Poisson r.v. with mean a_n, and $S_n = X_1 + \ldots + X_n$. Prove that if $\sum a_n = \infty$, then $S_n/E(S_n) \to 1$ in probability.

5. Pick $(2n+1)$ numbers at random in $(0, 1)$, i.e., assume the numbers are independent and uniformly distributed. Let V_{n+1} be the $(n+1)$th largest number, and $Y_n = \sqrt{2n}(2V_{n+1} - 1)$.
(a) Calculate the pdf of V_{n+1}.
(b) Prove that Y_n converges in distribution, and find the limit distribution.

6. Consider a population in which each member acts independently, and gives birth at exponential rate 1. That is, each member gives birth to one new member after an exponential waiting time with mean 1, independently of all other members. This birth mechanism applies to each new member, and all exponential waiting times are independent. Suppose that no one ever dies. Let X_t represent the population size at time t, and $X_0 = 1$. Assume that the age of the member at time 0 is 0. Prove that the expected sum of the ages at time t is $e^t - 1$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
AUGUST, 2002

Probability (Ph. D. Version)

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---

1. A r.v. Y with pdf $g(x)$ is called *stochastically bigger* than a r.v. X with pdf $f(x)$ if $P(Y > u) \geq P(X > u)$ for all $u \in \mathbb{R}$. Prove that if the likelihood ratio $g(x)/f(x)$ is monotone increasing, then Y is stochastically bigger than X.

2. The lifetimes of n computer systems are assumed to be independent and exponentially distributed with mean 1.
(a) Calculate the pdf of L, the lifetime of the system that survives the longest.
(b) Show that the times between failures are independent.

3. Let $X_1, X_2, \ldots$ be positive random variables. Prove that if X_n converges to X in probability, and $E(X_n)$ converges to $E(X)$, then X_n converges to X in L^1, as n tends to ∞.

4. Pick $(2n+1)$ numbers at random in $(0, 1)$, i.e., assume the numbers are independent and uniformly distributed. Let V_{n+1} be the $(n+1)$th largest number, and $Y_n = \sqrt{2n}(2V_{n+1} - 1)$.
(a) Calculate the pdf of V_{n+1}.
(b) Prove that Y_n converges in distribution, and find the limit distribution.

5. Consider a population in which each member acts independently, and gives birth at exponential rate 1. That is, each member gives birth to one new member after an exponential waiting time with mean 1, independently of all other members. This birth mechanism applies to each new member, and all exponential waiting times are independent. Suppose that no one ever dies. Let X_t represent the population size at time t, and $X_0 = 1$. Assume that the age of the member at time 0 is 0. Prove that the expected sum of the ages at time t is $e^t - 1$.

6. Consider successive flips of a fair coin. Use optional stopping theorems of martingales to compute the expected number of flips until the following sequences appear.
(a) H
(b) HT

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
JANUARY, 2002

Probability (MA Version)

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---

1. Let Z be a standard normal r.v., and let $S = 1, -1$ be the sign of Z.
(a) Calculate the conditional density $f_{Z|S}(z|s)$ of Z given S.
(b) Calculate the conditional variance of Z given $S = 1$.

2. Suppose that an ordinary deck of 52 playing cards (containing 4 Aces) is thoroughly shuffled, and that cards are dealt one by one (without replacement) from the top.
(a) Calculate, to a numerical value. the expected number of cards seen up to and including the first Ace dealt. Hint: consider $n \geq 4$ instead of $n = 52$, and relate larger value of n to a smaller one.
(b) Calculate, to a numerical value, the expected number of cards seen up to and including the fourth Ace dealt.

3. Suppose that there is an infinite backlog of tasks in a queue at time 0, to be serviced in order, that the first task in line is begun just at time 0. and that the service times for successive tasks are independent and identically distributed discrete random variables each equal to either 1, or 2 with equal probability.

(a) Calculate the probability that a task is completed (thus, a new task is begun) right at time n.

(b) Let K_n be the total number of tasks completed up to and including time n. Find positive numbers a, b such that $\frac{K_n - an}{b\sqrt{n}}$ converges in distribution to a standard normal distribution. Prove this by first writing down the strong law of large numbers and central limit theorem for S_m, the total time needed to complete the first m tasks, and then exploiting relations between K_n and S_m. For example, $S_{K_n} \leq n$ and $K_{S_m} = m$.

4. Assume that r.v's X_1, X_2, X_3 are independent and the moment generating functions $\psi_i(t) = E[e^{tX_i}]$ are finite for all $t \in R$, and $i = 1, 2, 3$. Prove that if $X_1 + X_3$ and $X_2 + X_3$ have identical distribution, then X_1 and X_2 also have identical distribution.

5. Suppose that the characteristic function of r.v. X is

$$\varphi_X(t) = E(e^{itX}) = \frac{3\sin t}{t^3} - \frac{3\cos t}{t^2},$$

which is defined to be 1 at $t = 0$.

(a) Show that X and $-X$ have identical distribution.

(b) Show that $E[X^2] = 1/5$.

6. Let (X_n) be i.i.d. Bernoulli with $p = 1/2$ and

$$S_n = \Sigma_{i=1}^{n}(2X_i - 1), \quad S_0 = 0.$$

Let $m, k,\ m < k$ be positive integers.
(a) Prove that the probability is m/k that the random walk $(m + S_n)$ visits k before 0.
(b) The random walk $(1 + S_n)$ never visits 0 if and only if it visits 2 before 0, then visits 3 before 0, then visits 4 before 0, and so on, indefinitely. Use this fact and the result of (a) to show

$$P\{(1 + S_n)\ visits\ 0\} = 1,$$

$$P\{(1 + S_n)\ visits\ 0\ infinitely\ often\} = 1.$$

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
JANUARY, 2002

Probability (Ph. D. Version)

Instructions to the Student

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d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

---

1. What is the minimum number of points a sample space must contain in order that there exist 10 independent, identically distributed r.v's $X_1, X_2, \ldots, X_{10}$ with $P\{X_1 = j\} > 0$ for all integers j, $1 \le j \le 6$? Why?

2. Suppose that an ordinary deck of 52 playing cards (containing 4 Aces) is thoroughly shuffled, and that cards are dealt one by one (without replacement) from the top.
(a) Calculate, to a numerical value, the expected number of cards seen up to and including the first Ace dealt. Hint: vary the number 52.
(b) Calculate, to a numerical value, the expected number of cards seen up to and including the fourth Ace dealt.

3. Suppose that there is an infinite backlog of tasks in a queue at time 0, to be serviced in order, that the first task in line is begun just at time 0, and that the service times for successive tasks are independent and identically distributed discrete random variables each equal to either 1, or 2 with equal probability.
(a) Calculate the probability that a task is completed (thus, a new task is begun) right at time n.
(b) Let K_n be the total number of tasks completed up to and including time n. Find positive numbers a, b such that $\frac{K_n - an}{b\sqrt{n}}$ converges in distribution to a standard normal distribution. Prove this by first writing down the strong law of large numbers and central limit theorem for S_m, the total time needed to complete the first m tasks, and then exploiting relations between K_n and S_m.

4. Assume that r.v's X_1, X_2, X_3 are independent and

$$P\{|X_i| > a\} < e^{-a}, i = 1, 2, 3, \ a > 0.$$

Prove that if $X_1 + X_3$ and $X_2 + X_3$ have identical distribution, then X_1 and X_2 also have identical distribution.

5. Let Y_1, Y_2, ... be i.i.d. r.v's and $S_n = \Sigma_{i=1}^n Y_i$, and let T be a stopping time of the process Y. Assume both Y_1 and T have finite mean. Prove Wald's identity

$$E[S_T] = E[Y_1]\, E[T]$$

by verifying and using the identity $S_T = \Sigma_{k=1}^{\infty} Y_k 1(T \geq k)$, where $1(T \geq k)$ is the indicator function of $\{T \geq k\}$.

6. Let (X_n) be i.i.d. Bernoulli with $p = 1/2$ and

$$S_n = \Sigma_{i=1}^{n}(2X_i - 1), \quad S_0 = 0.$$

Let m, k, $m < k$ be positive integers.

(a) Prove that the probability is m/k that the random walk $(m + S_n)$ visits k before 0.

(b) The random walk $(1 + S_n)$ never visits 0 if and only if it visits 2 before 0, then visits 3 before 0, then visits 4 before 0, and so on, indefinitely. Use this fact and the result of (a) to show

$$P\{(1 + S_n) \ visits\ 0\} = 1,$$

$$P\{(1 + S_n) \ visits\ 0\ infinitely\ often\} = 1.$$

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
AUGUST, 2001

Probability (Ph. D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (**NOT YOUR NAME**) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

---

1. Let $X_1, X_2, \ldots$ be i.i.d. r.v.'s with the common probability density function $f(x) = \frac{1}{2}e^{-|x|}, -\infty < x < \infty$. Let the r.v. B_n be given by

$$B_n = \frac{\Sigma_{m=1}^n X_m}{\sqrt{\Sigma_{m=1}^n |X_m|}}.$$

Prove that B_n converges in distribution, and find the limiting distribution as n tends to ∞.

2. Let $X_1, X_2, \ldots$ be i.i.d. uniform on $(0,1)$, $S_n = X_1 + X_2 + \ldots + X_n$, and $T = \min\{n : S_n > 1\}$.
(a) Calculate $P(T > n)$, and $E(T)$.
(b) Calculate $E(S_T)$.

3. Let F be the interarrival distribution which has a density function (with respect to Lebesgue measure) and a finite mean. Let $H(t)$ be the

probability that there are an even number of renewals in $(0, t]$, given an arrival at time 0, where interarrival times are assumed independent.
(a) Write a renewal equation for $H(t)$, in terms of F.
(b) Use a renewal theorem to find $\lim_{t\to\infty} H(t)$.

4. A spider hunting a fly moves between locations 1 and 2 according to a Markov chain with transition matrix P,

$$P_{11} = P_{22} = 0.7, \qquad P_{12} = P_{21} = 0.3,$$

starting in location 1. The fly, unaware of the spider, starts in location 2 and moves according to a Markov chain with transition matrix Q,

$$Q_{11} = Q_{22} = 0.4, \qquad Q_{12} = Q_{21} = 0.6.$$

The spider catches the fly and the hunt ends at the first time when they occupy the same location. The progress of the hunt, except for knowing the location where it ends, can be described by a three-state Markov chain with a single absorbing state representing the end of the hunt, and the other two states exactly representing the spider and fly at distinct locations.
(a) Define the three states precisely, and obtain the transition matrix for this three-state Markov chain.
(b) Find the average duration of the hunt.
(c) Find the probability that at time n the spider and fly are both at their initial locations with the hunt still in progress.

5. Let X be a positive r.v. with finite mean. Let $Y_{i,j},\ i,\ j \geq 1$ be r.v.'s and

$$X_{m,n} \;=\; E[\,X \mid Y_{i,j},\ i \leq m,\ j \leq n].$$

(a) Prove that $(X_{m,n},\ m, n \geq 1)$ is a uniformly integrable family of r.v.'s.
(b) Explain why the r.v.'s $X_{m,n}$ converge for each fixed n as $m \to \infty$ and describe exactly the limiting r.v.

6. Let $(X_n)_{n\geq 1}$ be a positive martingale. Prove from first principles (*not* by directly citing a theorem) that for each $n \geq 1$ and $a > 0$,

$$P(\max_{k\leq n} X_k \geq a) \leq \frac{E(X_n)}{a}.$$

Probability Exam, MA Level, January 2001

Instructions

a. Answer all six questions. In problems with multiple parts, the parts are graded indpendently of one another. You may assume the answer to any part in later parts of the same problem.

b. Use a different booklet for each question. Write the problem number and your **code number** (not your name) on the outside cover of each booklet.

c. Keep scratch work on separate pages of the same booklet.

d. You may appeal to any 'well known theorems' in your solution to a problem, but if you do, you must say exactly which theorem you are using and why its use is justified.

e. The following standard abbreviations are used without comment in the problem statements: *iid* = independent and identically distributed, *r.v.* = random variable, *d.f.* = distribution function.

---

1. Suppose that U is a Uniform random variable on the interval $(-1/2, 1/2)$.

(a) What is the conditional expectation of $Y \equiv U^2$, given $X \equiv U^3$?

(b) What is the conditional expectation of X given Y?

(c) What is the conditional expectation of $Y = U^2$ given the first 2 digits (A_1, A_2) of the binary expansion of $Z = \frac{1}{2} + U$?

(Recall that the binary expansion of an irrational real number $x \in (0,1)$ is the sequence $\{b_k : k = 1, 2, \ldots\} \in \{0, 1\}^\infty$ for which $x = \sum_{k=1}^{\infty} 2^{-k} b_k$.)

2. (a) I roll one 6-sided die, and you roll four dice independently. Assume all dice are balanced (i.e., fair). What are the probabilities that the number of dots showing on my die is respectively greater to, equal to, or less than L, the *largest* of the numbers of dots showing on your four dice ?

(b) Let T be the total number of times I must roll a die (independently) until the number of dots is **less than or equal** to your number L in (a). Find $E(T)$.

In problem 2, your answers need not be reduced arithmetically: each answer may involve an arithmetic summation-expression with 6 or fewer summands.

3. Suppose that W, X, and Y are mean 0 jointly normally distributed r.v's. Assume that X, Y are independent standard normal, and that

$E[W^2] = 9$, $E[WX] = 2$, $E[WY] = 1$. Find the conditional **distribution** of W given (X, Y), and use it to find the (unconditional) expectation of $WX\,|Y|$.

4. Suppose that ξ_n is a Markov chain with 2 states, 1 and -1. The transition probability matrix is

$$P = \begin{pmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{pmatrix}$$

Establish the existence of and calculate the in-probability limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (\xi_{k+1} - \xi_k)^2$$

5. If $X_1, X_2, \ldots, X_n, \ldots$ are independent identically distributed random variables with $E(X_k) = 0$ and $\mathrm{Var}(X_k) < \infty$, then calculate the limit as $n \to \infty$ of

$$P\left(\sum_{i=1}^{n} X_i \Big/ \sqrt{\sum_{i=1}^{n} X_i^2} < 2 \right)$$

6. Suppose that $X_t = N_t - M_t$ is a (right-continuous) continuous-time stochastic process defined from independent Poisson(1) processes M_t, N_t.

(a) Show that X_t is a continuous-time Markov process.

(b) Show that the counting process $Z_t = \#\{\, s \in [0,t] \,:\, X_t \neq \lim_{s\nearrow t} X_s \,\}$ is a renewal counting-process, where $\#$ denotes the number of elements in a set.

(c) Show that with probability 1, the process X_t re-visits the state 0 infinitely often.

Probability Exam, Ph.D. Level, January 2001

Instructions

a. Answer all six questions. In problems with multiple parts, the parts are graded indpendently of one another. You may assume the answer to any part in later parts of the same problem.

b. Use a different booklet for each question. Write the problem number and your **code number** (not your name) on the outside cover of each booklet.

c. Keep scratch work on separate pages of the same booklet.

d. You may appeal to any 'well known theorems' in your solution to a problem, but if you do, you must say exactly which theorem you are using and why its use is justified.

e. The following standard abbreviations are used without comment in the problem statements: *iid* = independent and identically distributed, *r.v.* = random variable, *d.f.* = distribution function, and $I_A(x) \equiv 1$ if $x \in A$, and $\equiv 0$ if $x \notin A$.

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1. Let $\epsilon_1, \epsilon_2, \ldots$ be an *iid* sequence of discrete r.v.'s with $P(\epsilon_k = 1) = P(\epsilon_k = -1) = 1/2$.

(a) For which positive values of the parameter α is the characteristic function of $S_n = \sum_{k=1}^{n} k^{-\alpha}\, \epsilon_k$ pointwise convergent to a characteristic function as $n \to \infty$?

(b) For which positive values of the parameter α is the limiting characteristic function in (a) twice-differentiable ?

2. Suppose that $W, X,$ and Y are mean 0 jointly normally distributed r.v's. Assume that X, Y are independent standard normal, and that $E[W^2] = 9$, $E[WX] = 2$, $E[WY] = 1$.

(a) Find the **conditional density** of W^3 given (X, Y).

(b) Find the unconditional expectation $E\Big(W^2\, X\, Y\Big)$.

3. Suppose that ξ_n is a Markov chain with 2 states, 1 and -1. The transition probability matrix is

$$P \quad = \quad \begin{pmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{pmatrix}$$

Establish the existence of and calculate the probability-1 limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (\xi_{k+1} - \xi_k)^2$$

[If you cannot establish convergence with probability 1, then do it in probability.]

4. If $X_1, X_2, \ldots, X_n, \ldots$ are independent identically distributed random variables with $E(X_k) = 0$ and $\mathrm{Var}(X_k) < \infty$, then calculate the limit as $n \to \infty$ of

$$P\left(\sum_{i=1}^{n} X_i \Big/ \sqrt{\sum_{i=1}^{n} X_i^2} \ < \ 2 \right)$$

5. If X_i for $i = 1, 2, \ldots$ are independent identically distributed random variables with a continuous distribution function F, and if

$$S_n \;=\; \sum_{i=1}^{n} I_{[F(X_i) \geq 1 - 2/n]}$$

then find $E(S_n)$ and the limit as $n \to \infty$ of $P(S_n \geq 5)$.

6. Suppose that $X_t = N_t - M_t$ is a (right-continuous) continuous-time stochastic process defined from independent Poisson(1) processes M_t, N_t .

(a) Show that X_t is a continuous-time Markov process.

(b) Show that the counting process $Z_t = \#\{\, s \in [0,t] \; : \; X_t \neq \lim_{s \nearrow t} X_s \,\}$ is a renewal counting-process, where $\#$ denotes the number of elements in a set.

(c) Show that with probability 1, the process X_t re-visits the state 0 infinitely often.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
AUGUST 2000

Probability (M.A. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

b. Use a different booklet for each question. Write the problem number and your code number (NOT YOUR NAME) on the outside cover.

c. Keep scratch work on separate pages in the same booklet.

d. If you use a "well known" theorem in your solution to any problem, it is your responsibility to make clear which theorem you are using and to justify its use.

e. The following standard abbreviations will be used throughout:

d.f. = distribution function
r.v. = random variable
ch.f. = characteristic function
pr. = probability
iid = independent and identically distributed
i.d. = infinitely divisible
i.o. = infinitely often
a.e. = almost everywhere

---

1. Let Z_1 and Z_2 be independent $N(0,1)$ random variables.
   (a) Find the conditional density of $X = Z_1 + 3Z_2$ given that $Y = Z_1 - Z_2 = y$.
   (b) Find numbers α, β such that $Z_1 + 2Z_2$ and $\alpha Z_1 + \beta Z_2$ are independent.

2. Let Y be a discrete random variable with $P[Y = 1] = P[Y = -1] = 1/2$ and let Z be a standard normal random variable independent of Y. Prove that $X = YZ$ has a standard normal distribution and that X and Y are uncorrelated. Prove also that X and Z are not independent.

3. Consider a Markov chain with state space $\{1, 2\}$, initial state 1, and transition matrix

$$\mathrm{P} = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix},$$

for $\alpha, \beta \in (0, 1)$. Let τ_n be the time of the nth return to state 1. Prove that τ_n has a limiting normal distribution as $n \to \infty$ and calculate the parameters.

4. Suppose that $\{X_n\}$ is a sequence of independent exponential r.v.'s with common parameter 1. Let $S_n = \sum_{k=1}^{n} X_k$ and let $\tau = \inf\{n | S_n \geq 1\}$. Verify that $E[\tau] < \infty$ and $E[\tau] = E[S_\tau]$.

5. Let X_1 and X_2 be independent exponential r.v.'s with parameters λ_1 and λ_2, respectively. Define $Y = \min\{X_1, X_2\}$ and $\Delta = I\{X_1 < X_2\}$. Prove that Y is exponential and find its density. In addition, prove that Y and Δ are independent.

6. Let X be a random variable with a gamma density:

$$f(x; \alpha, \beta) = \frac{x^{\alpha-1}}{\beta^\alpha \Gamma(\alpha)} \exp(-\frac{x}{\beta}),$$

where $x > 0$. Find constants c_1 and c_2, depending on α and β, such that $c_1(X - c_2)$ has a limiting $N(0, 1)$ distribution as $\alpha \to \infty$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
AUGUST 2000

Probability (Ph. D. Version)

Instructions to the Student

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---

1. Let Z_1 and Z_2 be independent $N(0,1)$ random variables.

(a) Find the conditional density of $X = Z_1 + 3Z_2$ given that $Y = Z_1 - Z_2 = y$.

(b) Show that the formula derived in (a) yields a formula for conditional probabilities for X given $\sigma\{Y\}$, the σ-field generated by Y, which satisfies the measure-theoretic definition of conditional probability.

2. Let $X_1, \ldots, X_n, \ldots$ be a sequence of i.i.d. random variables with $E[X_k] = \mu > 0$ and $\operatorname{Var} X_k = \sigma^2 < \infty$. Also, let $Y_1, \ldots, Y_n, \ldots$ be a sequence of i.i.d. random variables with $P[Y_k = 1] = 1/2 = P[Y_k = -1]$. In addition, assume the sequences $\{X_k\}$ and $\{Y_k\}$ are independent. Calculate the limiting distribution of

$$\frac{\sqrt{n}\sum_{k=1}^n X_k Y_k}{\sum_{k=1}^n X_k}$$

as $n \to \infty$.

3. Consider a Markov chain with state space $\{1, 2\}$ and transition matrix

$$\mathrm{P} = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix},$$

for $\alpha, \beta \in (0, 1)$. Suppose that two particles move independently according to this chain, starting from state 1 at time 0. Let X_n be the position of the first particle and let Y_n be the position of the second particle at time n. Define $Z_k = I\{X_k = Y_k\}$, where $I\{\cdot\}$ denotes the indicator function, and define $W_n = \sum_{k=0}^n Z_k$. Prove that $\lim_{n\to\infty} W_n/n$ exists and calculate its value.

4. Suppose that $\{X_n\}$ is a sequence of independent exponential r.v.'s with common parameter 1. Let $S_n = \sum_{k=1}^n X_k$ and let $\tau = \inf\{n \mid S_n \geq 1\}$. Verify that $E[\tau] < \infty$ and $E[\tau] = E[S_\tau]$.

5. Let X_1 and X_2 be independent exponential r.v.'s with parameters λ_1 and λ_2, respectively. Define $Y = \min\{X_1, X_2\}$ and $\Delta = I\{X_1 < X_2\}$. Prove that Y is exponential and find its density. In addition, prove that Y and Δ are independent.

6. Let X_1, X_2, ... be independent random variables with a common uniform distribution on $[0, 1]$. The index n is interpreted as a discrete time parameter By definition, a *record* occurs at time n if $X_n = \max\{X_i \mid 1 \leq i \leq n\}$.

(a) Let R_{kn} be the *rank* of X_k among the first n X's. That is, R_{kn} is the number of X_j, $j = 1, \ldots, n$ which are less than or equal to X_k:

$$R_{kn} = \sum_{i=1}^{n} I\{X_i \leq X_k\}.$$

Use the invariance of the joint distribution of $(X_1, \ldots, X_{n+1})$ under permutations of the coordinates to show that

$$P\{\text{ record at } n+1 \mid R_{in},\ 1 \leq i \leq n\}$$

is a constant depending only upon n, and find that constant.

(b) Show that with probability one, there are infinitely many indices m such that no records occur between times $m+1$ and $2m$ (inclusive).

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
JANUARY, 2000

Probability (M. A. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

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i.o. = infinitely often
a.e. = almost everywhere

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1. A random variable X is distributed exponentially with $E[X] = 1$. Find the conditional probability $P[X = x|\sin X = y]$.

2. Suppose that $\{X_{i,j} : i = 1, \ldots, 10^4, j = 1, \ldots, 100\}$ is a doubly-indexed array of independent identically distributed random variables having the uniform distribution on $[-1, 1]$. Find and justify good approximate numerical values (expressed in terms of standard tabulated quantities and quantities obtainable from a scientific calculator) for

(a) $P(X_{1,1} + X_{2,1} + \cdots + X_{10000,1} \geq 100)$

(b) $\sum_{j=1}^{100} I\{X_{1,j} + X_{2,j} + \cdots + X_{10000,j} \geq 100\}$.

3. Demonstrate that each of the following functions of t is the moment generating function of a random variable by explaining how that random variable can be defined in terms of others with standard distributions:

$$\begin{aligned} M_1(t) &= \left(\frac{1}{1-4t}\right)\left(\frac{1}{1+2t}\right)\exp(-t+2t^2) \\ M_2(t) &= \frac{1}{3}\sum_{j=1}^{3}\exp(5jt + \frac{1}{2}j^2t^2) \end{aligned}$$

4. Suppose that r balls are placed at random into n boxes. That is, all n^r possible assignments of balls to boxes have equal probabilities. Let A_i be the event that the ith box is empty and let N_n denote the number of empty boxes.

(a) Calculate $P[A_i]$ and $E[N_n]$.

(b) Calculate the variance of N_n.

(c) Suppose that $n \to \infty$ and that $\lim_{n\to\infty} r/n \to c$, a constant. Prove that N_n/n converges in probability to a constant and evaluate the limit.

5. Travelers arrive at a railroad station according to a Poisson process with rate λ. If the train departs at time t, compute the expected waiting time of travelers arriving at the station in the interval $(0, t)$.

6. A round table has 10 seats, numbered from 1 to 10. A person sitting at the table randomly moves one seat to the right or one seat to the left based on tossing a fair coin.

(a) What is the expected number of moves before the person returns to his original seat?

(b) Let π_n be the probability mass function of the person's position after n moves. Does π_n converge? If so, what is the limit of π_n?

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
AUGUST, 1999

Probability (Ph. D. Version)

Instructions to the Student

a. Answer all six questions. Each will be graded from 0 to 10.

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1. Let X, Y, Z be random variables on a probability space $\Omega, \mathcal{F}, P)$. Assume $E(X^2) < \infty$.

(a) Prove that

$$\text{Var}\,[E(X|Y,Z)] \geq \text{Var}\,[E(X|Y)].$$

(b) Suppose that $\mathcal{G}$ and $\mathcal{H}$ are sub-σ-algebras such that $\mathcal{F} \supset \mathcal{G} \supset \mathcal{H}$. State an analog of the inequality of part (a) in terms of conditional moments of X given $\mathcal{G}$ and/or $\mathcal{H}$.

2. Let X_k, $k = 1, 2, \ldots$, be a sequence of independent random variables such that

$$P[X_k = 1] = \frac{1}{2}\left(1 - \frac{1}{k^2}\right) = P[X_k = -1],$$

$$P[X_k = k] = \frac{1}{2k^2} = P[X_k = -k]$$

Define $S_n = X_1 + \cdots + X_n$.

(a) Show that the asymptotic behavior of $S_n/\sqrt{n}$ is the same as it would have been if $P[X_k = 1] = 1/2 = P[X_k = -1]$.

(b) Calculate $\lim_{n\to\infty} \mathrm{Var}\,[S_n/\sqrt{n}]$ and discuss the relationship of this result with the result of (a).

3. Let $X_1, X_2, \ldots$ be iid with $P[X_j = 1] = 1/2 = P[X_j = -1]$. Let $\mathcal{F}_n$ denote the σ-field generated by $X_1, X_2, \ldots, X_n$ and define

$$S_0 = 0, \qquad S_n = X_1 + X_2 + \cdots + X_n,$$

$$\tau = \min\{m \geq 1 | S_m \geq 1\}, \qquad \tau_+ = \tau + 1, \qquad \tau_- = \tau - 1.$$

(a) Which, if any, of τ, τ_+, τ_- are stopping times?

(b) What is $E[S_{\tau\wedge 100}]$?

(c) What is $E[S_\tau]$?

(d) Define

$$\sigma = \begin{cases} 1 & \text{if } S_2 \leq 0 \\ 2 & \text{if } S_2 > 0. \end{cases}$$

Compute $E[S_\sigma]$.

4. Let the random vector $\mathrm{U}_n = (U_{n1}, \ldots, U_{n1})$ be uniformly distributed over the sphere in R^n, centered at the origin with radius 1. Find a sequence of numbers $\{c_n\}$ such that $c_n U_{n1}$ converges to a nondegenerate distribution as $n \to \infty$ and identify the limiting distribution.

5. Let $X_1, X_2, \ldots$, be i.i.d. with finite second moment. Show that

$$n^{-1/2} \max_{1 \le k \le n} |X_k| \to 0 \quad \text{in pr.}$$

[*Hint:* Consider $\lim_{n\to\infty} nP[|X_1| \ge \varepsilon\sqrt{n}]$.

6. Consider the Markov chain $\{X_n, n = 0, 1, 2, \ldots\}$ with state space $\{1, 2\}$ and transition probability matrix

$$\mathrm{P} = \begin{bmatrix} 1/4 & 3/4 \\ 3/4 & 1/4 \end{bmatrix}.$$

Calculate $\lim_{n\to\infty} \sum_{k=1}^{n} X_{k-1}X_k$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
JANUARY, 1999

Probability (Ph. D. Version)

Instructions to the Student

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1. (i) Prove that if $P(A_n) \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} P(A_n^c \cap A_{n+1}) < \infty$ then $P(A_n \ i.o.) = 0$.

   (ii) Find an example of a sequence to which the result in (i) can be applied but the Borel–Cantelli lemma cannot.

2. Let X be a r.v. with d.f. G supported on the whole real line and with mean μ and variance σ^2. Define random vectors (Z_i, Y_i), $i = 1, 2$, defined as follows:

$$\begin{aligned} Z_1 &= X, \qquad Y_1 = -X, \\ Z_2 &= X^+, \qquad Y_2 = X^-. \end{aligned}$$

For both (Z_i, Y_i), $i = 1, 2$, find:

(i) the joint d.f. of (Z_i, Y_i),

(ii) the marginal d.f.'s of Z_i and Y_i,

(iii) $\mathrm{Cov}(Z_i, Y_i)$,

(iv) the regions of the z-y plane supporting the joint d.f.'s of (Z_i, Y_i), $i = 1, 2$.

3. Let $X_1, X_2, \ldots$ be independent r.v.'s with

$$P(X_n = 1) = P(X_n = -1) = (2n)^{-1}, \qquad P(X_n = 0) = 1 - n^{-1}.$$

Define $Y_1 = X_1$ and, for $n \geq 2$,

$$\begin{aligned} Y_n &= X_n && \text{if } Y_{n-1} = 0 \\ &= nY_{n-1}|X_n| && \text{if } Y_{n-1} \neq 0. \end{aligned}$$

(i) Show that $(Y_n, \mathcal{F}_n)$ is a martingale with respect to the sequence of σ-algebras

$$\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$$

(ii) Show that Y_n does not converge a.e. but does converge to zero in pr.

(iii) Explain why the martingale convergence theorem does not apply.

4. Suppose that $\{X_n\}$ is a sequence of independent r.v.'s, all of which have zero means and finite variances σ_n^2 such that X_n/σ_n are identically distributed and

$$\max_{k\leq n} \frac{\sigma_k^2}{\sum_{j=1}^n \sigma_k^2} \to 0 \quad \text{as } n \to 0.$$

Carefully state the Lindeberg Central Limit Theorem and apply it to prove that as $n \to \infty$,

$$\frac{\sum_{k=1}^n X_k}{\sqrt{\sum_{j=1}^n \sigma_k^2}} \to N(0,1)$$

in distribution.

5. Let X be a r.v. with a ch.f. of the form:

$$\phi(t) = \exp(-|t|^\alpha) \quad \text{where } 0 < \alpha \leq 2.$$

(i) For what values of α do the mean and variance of X exist? In particular, check whether the mean and variance exist if $\alpha = 1$ or if $\alpha = 2$.

(ii) What happens if $\alpha > 2$?.

(iii) Find, for each n, the ch.f. of the r.v.

$$n^{-1/\alpha}(X_1 + \cdots + X_n),$$

where $X_1, X_2, \ldots$ are iid r.v.'s with the same ch.f. as X.

6. Consider a random walk $\{X_t, 0 \le t < \infty\}$ on the nonnegative integers in continuous time, described by the intensities

$$\begin{aligned} q_{ij} &= \lambda \quad \text{for } i \ge 0,\ j = i+1 \\ &= \mu \quad \text{for } i \ge 1,\ j = i-1 \\ &= 0 \quad \text{for } |i-j| > 1, \end{aligned}$$

where

$$\begin{aligned} q_0 &= \lambda \\ q_i &= \lambda + \mu \quad \text{for } i \ge 1. \end{aligned}$$

Let T be the first entrance time to state 0 from an initial state $i \ge 1$, and write $F_i(t)$ for its d.f., conditional on $X_0 = i$.

(i) Prove that the quantity $F_i = \lim_{t \to \infty} F_i(t)$ satisfies the equation

$$\mu F_{i-1} - (\lambda + \mu) F_i + \lambda F_{i+1} = 0.$$

with $F_0 = 1$.

(ii) Verify that the solution of the above equation is

$$\begin{aligned} F_i &= (\mu/\lambda)^i \quad \text{if } \lambda > \mu \\ &= 1 \qquad\quad\ \ \text{if } \lambda \le \mu. \end{aligned}$$