TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

JANUARY 10, 2011
MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND

Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear which theorem you are using and why its use is justified. In problems with multiple parts, be sure to go on to the rest of the problem even if there is some part you cannot do. In working on any part, you may assume the answer to any previous part, even if you have not proved it.

Problem 1.

A continuous map between Hausdorff spaces is closed if the image of each closed set is closed. A continuous map is proper if the inverse image of each compact subset of the range is compact.

a) Let X, Y be metric spaces. Show every continuous proper map $f\colon X \to Y$ is closed.
b) Give an example of a map which is not closed and explain.

Problem 2.

In what follows, "surface" will mean a connected compact surface-with-boundary. χ denotes Euler characteristic. Prove or disprove the following statements (using standard results, which you can assume, of course).

(1) Two orientable surfaces A and B with $\partial(A) = \partial(B) = \emptyset$ are homeomorphic if and only if $\chi(A) = \chi(B)$.
(2) Let $E < 0$ be an integer. The number of homeomorphism classes of orientable surfaces A with $\chi(A) = E$ equals $1 - E$.
(3) If A is an orientable surface and B is a nonorientable surface then A cannot be homotopy-equivalent to B.
(4) If A is an orientable surface and B is a nonorientable surface then A cannot be homeomorphic to B.

Problem 3.

Let X be a locally path-connected space and

$$f_j : X \to S^1 = \{z \in \mathbb{C} \mid |z| = 1\}, \quad j = 0, 1,$$

be two continuous maps. Show f_0 and f_1 are homotopic if and only if there is a continuous function $\alpha : X \to \mathbb{R}$ such that $f_0(x) = f_1(x) \exp 2\pi i \alpha(x)$ for all $x \in X$.

Problem 4.

Fix $n \geq 1$, and let X_p denote the space obtained by attaching an $(n+1)$-cell to S^n by a map of degree p, where p is a prime.

a) Compute $H_*(X_p \times X_q, \mathbb{Z})$. Here p may or may not be equal to q.
b) If r is a prime (possibly equal to p) and $\mathbb{Z}_r$ is the cyclic group of order r, compute $H_*(X_p \times X_p, \mathbb{Z}_r)$.

Problem 5.

If X is a topological space, the *n-th symmetric power* $S^n X$ of X is defined to be the quotient of $X^n = X \times X \times \cdots \times X$ (n factors) by the action of the symmetric group Σ_n, acting by permutation of the factors. The quotient space is given the quotient topology.

(1) Show that if $X = \mathbb{CP}^1 \cong S^2$, then $S^n X$ is homeomorphic to $\mathbb{CP}^n$. (Hint: the elementary symmetric functions $\sigma_1(z_1, \cdots, z_n) = z_1 + \cdots + z_n, \cdots, \sigma_n(z_1, \cdots, z_n) = z_1 \cdots z_n$ give a homeomorphism from $S^n\mathbb{C}$ to $\mathbb{C}^n$ and you just need to extend it.)
(2) Compute (as explicitly as possible) the map of cohomology *rings* $H^*(\mathbb{CP}^n; \mathbb{Z}) \to H^*((S^2)^n; \mathbb{Z})$ induced by the quotient map $X^n \to S^n X, X = S^2$. In other words, give the structure of each cohomology ring (for this you can use standard results instead of computing from scratch), and then explain what each generator maps to.

Problem 6.

a) Let N^n be an orientable compact connected n-manifold without boundary. Show $H^n(N, \mathbb{Z}) \simeq \mathbb{Z}$.
b) Suppose M^n is another n-dimensional connected compact manifold without boundary and $f\colon M \to N$ is a continuous map such that

$$f_*\colon H_n(M, \mathbb{Z}) \to H_n(N, \mathbb{Z})$$

is onto. Show M is orientable and $f_*\colon H_r(M, \mathbb{Z}) \to H_r(N, \mathbb{Z})$ is onto for all $r \geq 0$.

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

AUGUST 4, 2010
MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND

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Problem 1.

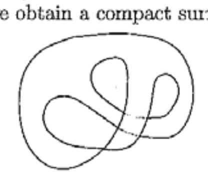
True-false: If true give a quick reason why. If false, give a counterexample.

a) If X is a metric space with metric d and $d(x,y) \leq 1$ for all $x, y \in X$, then X is compact.
b) Every connected topological space is path-connected
c) If $g : X \to Y$ is a continuous one-to-one surjection, then X and Y are homeomorphic.
d) If $f : \mathbb{R}^4 \to \mathbb{R}$ is given by $f(x,y,z,w) = x^2 + 2xy + z^2 + w^2$, then $f^{-1}(1)$ is a smooth submanifold of $\mathbb{R}^4$.
e) There exists a 2-manifold M (without boundary) with $M \times M$ homotopy equivalent to M.

Problem 2.

a) Let X be a closed, oriented surface and suppose there exists a covering space $p : Y \to X$ such that Y is homeomorphic to X but p is NOT a homeomorphism. Show X is homeomorphic to $S^1 \times S^1$.

b) The compact surface S pictured below has boundary homeomorphic to a circle (Note that there is a half-twist in the "bridge.") If M is the space obtained from the disjoint union of a 2 disk D and S by identifying the boundaries of each, we obtain a compact surface. What surface is it?



Problem 3.

Let S^3 be the unit sphere in $\mathbb{R}^4$ and let S^1 be embedded in S^3 as the intersection of S^3 with a two-dimensional plane through the origin Let M be the space obtained by from S^3 by identifying the embedded S^1 to a point Let G be a non-zero abelian group.

a) Compute the homology groups of M with coefficients in G.
b) What is the Euler characteristic of M?
c) Show M is not a manifold.
d) If $p : S^3 \to M$ is the quotient map, show $p_* : H_3(S^3, G) \to H_3(M, G)$ is not the zero map.

Problem 4.

Let X be the quotient space of $S^3 \times [0,1]$ with $(x, 0) \sim (\sigma(x), 1)$, where σ is the antipodal map on S^3.

a) Show that X can also be written as the quotient of $S^3 \times \mathbb{R}$ by the action of $\mathbb{Z}$ generated by $(x,t) \mapsto (\sigma(x), t+1)$.
b) Show that X is an orientable closed 4-manifold.
c) Compute the integral homology groups of X.
d) Compute the cohomology *ring* $H^*(X, \mathbb{Z})$ with $\mathbb{Z}$ coefficients.

Problem 5.

a) If n is even, show there does not exist a map $\mathbb{CP}^n \to \mathbb{CP}^n$ of degree -1.

b) Let $r : \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}^{n+1} - \{0\}$ be given by $r(z_0, z_1, \ldots, z_n) = (-z_0, z_1, \ldots, z_n)$ Then r induces a map $\bar{r} : \mathbb{CP}^n \to \mathbb{CP}^n$. What is the degree of $\bar{r}$?

Problem 6.

a) Suppose M is a compact, contractible, orientable n-manifold with boundary. Show ∂M is a homology $(n-1)$-sphere, that is, has the integral homology groups of a sphere.

b) Suppose $f : M \to N$ is a degree one map of closed, connected, orientable manifolds. Show $f_* : \pi_1(M) \to \pi_1(N)$ is onto. Hint: Argue by contradiction.

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

JANUARY 13, 2010
MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND

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Problem 1.

Let G be a topological group acting the topological space X. Let $G_x = \{g \in G \,|\, gx = x\}$ and $O(x) = \{gx \,|\, g \in G\}$.

(a) Show there exists a continuous bijection $\pi : G/G_x \to O(x)$ which is a homeomorphism if G is compact.

(b) Let X/G denote the orbit space, i.e., the set of equivalence classes of the equivalence relation $x \sim y$ if and only if there exists $g \in G$ with $x = gy$. Equip X/G with the quotient topology and let $p : X \to X/G$ be the natural map. Show p is an open map.

(c) Suppose G and X/G are connected. Show X is connected.

Problem 2.

Let C_a be the set of points $(z_1, z_2) \in \mathbb{C}^2$ with $z_1^2 = z_2^3 + a$, where $a \in \mathbb{C}$ is fixed.

(a) Show that for $a \neq 0$, C_a is a smooth submanifold of $\mathbb{C}^2$.

(b) Show that when $a = 0$, $w \mapsto (w^3, w^2)$ is a surjective map from $\mathbb{C}$ to C_0.

(c) When $a = 0$, is C_0 a *topological* manifold? Why or why not?

Problem 3.

a) Let X be a path-connected, locally path-connected space and suppose $H^1(X, \mathbb{Z}) = 0$. Show that every map $f : X \to S^1$ is null homotopic.

b) Let T_g denote the compact oriented surface of genus $g \geq 1$ (without boundary). For all $n \geq 1$ show there exists a unique T_h which is a regular n-fold covering space of T_g and determine h as a function of g and n.

Problem 4.

A complex line in the complex plane $\mathbb{C}^2$ is given by the equation $az+bw=c$ where $a,b,c \in \mathbb{C}$ and a and b are not both 0.

(a) Show that if two complex lines are disjoint they are parallel, i.e., given by $az+bw=c$ and $az+bw=d$ for some $a,b,c,d \in \mathbb{C}$ with a and b not both 0 and $c \neq d$.

(b) Show that if $l_1, \dots l_n$ are n disjoint lines in $\mathbb{C}^2$ then $\pi_1(\mathbb{C}^2 - \{\cup l_i\})$ is isomorphic to the free group on n generators.

Problem 5.

Let X be the two-dimensional CW complex obtained from S^1 by attaching two 2-cells by maps of degree a and b respectively where a and b are relatively prime.

(a) Show X is simply connected.

(b) Compute the integral homology groups of X.

(c) Show X is homotopy equivalent to S^2.

Problem 6.

Let M be a connected compact 4-manifold without boundary. (It need not be orientable.) Let $\mathbb{F}_2 = \mathbb{Z}/(2)$ be the field with two elements.

(a) Show that $H^4(M, \mathbb{F}_2)$ is one-dimensional.

(b) Show that there is a unique class $w \in H^2(M, \mathbb{F}_2)$ such that $x \cup w = x \cup x$. (Hint: show that $x \mapsto \langle x \cup x, [M] \rangle$ is a *linear* functional on $H^2(M, \mathbb{F}_2)$.)

(c) Assume now (in addition) that M is orientable. Show that if T is the torsion subgroup of $H^2(M, \mathbb{Z})$ and W is the image of $H^2(M, \mathbb{Z})$ under the "reduction mod 2 map" $r\colon H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{F}_2)$, then rT and W are each other's annihilators under the cup product pairing.

(d) When M is orientable, use the result of (c) to show that w is the reduction mod 2 of a class in $H^2(M, \mathbb{Z})$.

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

AUGUST 3, 2009
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Problem 1.

Consider two topological spaces A and B with the same underlying set $[0,1]^{\mathbb{N}}$, consisting of all sequences $x = \langle x_n \rangle$, where $n = 1, 2, 3, \ldots$ and x_n lies in the interval $[0,1]$. Give A the *weak topology,* that is the smallest topology for which the projections $x \longmapsto x_n$ are continuous Give B the *metric topology,* defined by the metric

$$d(x,y) := \sqrt{\sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|^2}.$$

Prove or disprove the following statements:

(a) The identity map $A \longrightarrow B$ is continuous.
(b) The identity map $B \longrightarrow A$ is continuous.
(c) A and B are homeomorphic.

Problem 2.

Let G denote the set of all motions of the plane $\mathbb{R}^2$ of the form

$$f_{m,n} : (x,y) \mapsto ((-1)^m x + n, y + m),$$

where m and n are integers It is easy to verify (don't bother with this—you can take it on faith) that G is a group under composition

(a) For which m and n does $f_{m,n}$ preserve orientation?
(b) Show that G acts properly discontinuously and freely on $\mathbb{R}^2$, that the quotient $\mathbb{R}^2/G$ is a closed surface S, and that the quotient map $q\colon \mathbb{R}^2 \to S$ is a covering space.
(c) For any integer k, let $g_k : [0,1] \to \mathbb{R}^2$ be the map defined by $g_k(t) = (t, kt)$. Let $h_k\colon [0,1] \to S$ be the composition $q \circ g_k$ Show that h_k defines a nontrivial element of the fundamental group $\pi_1(S, q(0))$.

Problem 3.

Let $\mathbb{RP}^n$ denote, as usual, the real projective n-space. Corresponding to the inclusion $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ is an inclusion $\mathbb{RP}^n \hookrightarrow \mathbb{RP}^{n+1}$. **Prove or disprove** the following statements.

(a) The complement $\mathbb{RP}^{n+1} \setminus \mathbb{RP}^n$ is disconnected.
(b) $\mathbb{RP}^n$ is two-sided in $\mathbb{RP}^{n+1}$. (That is, it has a tubular neighborhood N such that $N \setminus \mathbb{RP}^n$ is disconnected.)
(c) Suppose A and B are manifolds of respective dimensions n and $n+1$, and $A \hookrightarrow B$ is the embedding of a submanifold. Suppose that A is two-sided in B. Then $B \setminus A$ is disconnected.

Problem 4.

If Y is a connected topological space, a *regular infinite cyclic cover* of Y is a connected regular covering space $\widetilde{Y} \to Y$ whose group of covering transformations is isomorphic to the integers $\mathbb{Z}$.

(a) Let X be a connected CW-complex Show $H^1(X, \mathbb{Z}) \neq 0$ if and only if there exists a regular infinite cyclic cover of X.
(b) Suppose X is a finite connected CW-complex and $\widetilde{X} \xrightarrow{p} X$ be a regular infinite cyclic cover of X. Let $T : \widetilde{X} \to \widetilde{X}$ be a generator of the group of covering transformations and let $C(\widetilde{X})$ (resp., $C(X)$) be the cellular chain complex of $\widetilde{X}$ (resp., X). Show there exists an exact sequence of chain complexes
$$0 \to C(\widetilde{X}) \xrightarrow{T-1} C(\widetilde{X}) \xrightarrow{p_*} C(X) \to 0.$$
(c) Suppose X is as in (b) and $\sum_0^\infty \dim_{\mathbb{Q}} H_i(\widetilde{X}, \mathbb{Q}) < \infty$. Show the Euler characteristic, $\chi(X)$, is zero.
(d) Let M_g be an oriented surface of genus $g \geq 2$. Show that if $\widetilde{M_g} \to M_g$ is an infinite cyclic cover of M_g then $H_1(\widetilde{M_g}, \mathbb{Z})$ is not a finitely generated abelian group.

Problem 5.

Let $f : S^{2n-1} \to S^n$ be defined as follows: Give S^n the usual cell decomposition with one 0-cell $*$ and one n-cell, and give $S^n \times S^n$ the product cell decomposition with n-skeleton the one-point union $S^n \vee S^n$ and with one additional $2n$-cell. Let c: $\partial D^{2n} = S^{2n-1} \to S^n \vee S^n$ be the attaching map of the top cell of $S^n \times S^n$, and let f be the composition of c with the folding map $S^n \vee S^n \to S^n$. Let $X = S^n \bigcup_f D^{2n}$

(a) Show that X can also be identified as $S^n \times S^n/\sim$, where $(x, *) \sim (*, x)$.
(b) For $n \geq 1$, calculate the integral cohomology **ring** of X.
(c) Show that X is not homotopy equivalent to a closed manifold for any $n \geq 1$.

Problem 6.

Let S_g be the closed oriented surface of genus g. Let C be a simple closed curve in S_g. Prove that S_g retracts to C if and only if C does not separate S_g. (Hints: 1. One way to do this is to use duality to analyze $H_0(S_g \setminus C)$. 2. Let $\mathbb{T}$ denote the unit circle in the complex plane, with its usual group structure. You may use the standard fact that the set of homotopy classes of maps from a space X (say with the homotopy type of a CW complex) to $\mathbb{T}$, with the group structure coming from pointwise multiplication of $\mathbb{T}$-valued functions, is isomorphic to $H^1(X, \mathbb{Z})$.)

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

JANUARY 14, 2009
MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND

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Problem 1.

Let X, Y be locally compact Hausdorff spaces. Recall that a *closed map* is a continuous map that sends closed sets to closed sets. Prove or give a counterexample to each of the following statements:

a) The Cartesian projection $X \times Y \longrightarrow X$ is a closed map.
b) Assume X is compact. The Cartesian projection $X \times Y \longrightarrow X$ is a closed map.
c) Assume Y is compact. The Cartesian projection $X \times Y \longrightarrow X$ is a closed map.
d) Assume both X and Y are compact. The Cartesian projection $X \times Y \longrightarrow X$ is a closed map.

Problem 2.

a) Let X be a path connected, locally path connected, and locally simply connected space. Let $p : \tilde{X} \to X$ be a simply connected covering space of X and let $A \subseteq X$ be a path connected, locally path connected subspace of X with $\tilde{A} \subseteq \tilde{X}$ a path component of $p^{-1}(A)$. Let $x_0 \in A$. Show $p : \tilde{A} \to A$ is a covering space of A and corresponds to the kernel of the map $\pi_1(A, x_0) \to \pi_1(X, x_0)$.

b) Let g and h be positive integers with $h > 1$ and let M_k denote the closed oriented surface of genus $k \geq 0$. Show M_g is a covering space of M_h if and only if there exists $n > 0$ such that $g = n(h-1)+1$.

Problem 3.

Let $M_n(\mathbb{R})$ be the space of $n \times n$ real matrices topologized as $\mathbb{R}^{n^2}$. Let $S(n)$ be the subspace of symmetric matrices and $F : M_n(\mathbb{R}) \to S(n)$ be the (smooth) map $F(A) = AA^t$ where A^t is the transpose of A.

a) If $D_AF : M_n(\mathbb{R}) \to S(n)$ is the tangent map of F at $A \in M_n(\mathbb{R})$, show D_AF sends $B \mapsto AB^t + BA^t$.
b) Show $\mathrm{Id} \in S(n)$ is a regular value of F.
c) Show $O(n) = F^{-1}(\mathrm{Id}) = \{A \in M_n(\mathbb{R}) | AA^t = \mathrm{Id}\}$ is a submanifold of $M_n(\mathbb{R})$ and determine its dimension.
d) Determine the tangent space to $O(n)$ at the identity matrix as a subspace of $M_n(\mathbb{R})$.

Problem 4.

a) Let X be a CW-complex of dimension n. Show $H_n(X, \mathbb{Z})$ is free abelian.

b) Let X be a 2-dimensional CW-complex with 1 zero-cell x_0, 2 one-cells and 3 two-cells. Suppose further $\pi_1(X, x_0) \cong S_3$, the symmetric group on 3 letters, i.e., the unique non-abelian group of order 6. Let $\tilde{X}$ be the connected cover of X corresponding to the unique subgroup of S_3 of index 2. Determine $H_*(X)$ and $H_*(\tilde{X})$.

Problem 5.

Let $n > 1$ be an odd integer, S^n the n-sphere, $\mathbb{RP}^n$ the real projective n-space and $p : S^n \to \mathbb{RP}^n$ the natural projection. Suppose we are given continuous maps $g : S^n \to S^n$ and $f : \mathbb{RP}^n \to \mathbb{RP}^n$ such that $pg = fp$. I.e, the following diagram commutes:

$$\begin{array}{ccc} S^n & \xrightarrow{g} & S^n \\ \downarrow p & & \downarrow p \\ \mathbb{RP}^n & \xrightarrow{f} & \mathbb{RP}^n \end{array}$$

a) Show degree(g) = degree(f).
b) Prove there exists a map $f' : \mathbb{RP}^n \to S^n$ such that $pf' = f$ if and only if degree(f) is even. (You may find cohomology with mod 2 coefficients useful).
c) Prove that if $h : S^n \to S^n$ commutes with the antipodal map of S^n then degree(h) is odd.

Problem 6.

Let M^n be a compact, connected manifold of dimension n. We do not assume M closed or orientable unless this is explicitly stated.

a) If n is odd and $\partial M = \emptyset$, show the Euler characteristic $\chi(M) = 0$.
b) If n is odd and $\partial M \neq \emptyset$, show $\chi(\partial M)$ is divisible by 2.
c) If $n = 3$, M is not orientable and $\partial M = \emptyset$, show $H_1(M, \mathbb{Z})$ is infinite.
d) If M is orientable, contractible, and $\partial M \neq \emptyset$, show ∂M has the same integral homology groups as S^{n-1}.

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

AUGUST 8, 2008
MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND

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Problem 1.

Let (M, d) be a metric space.

(a) Show that the topology on M induced by the metric is Hausdorff.
(b) Show that $d\colon M \times M \longrightarrow \mathbb{R}$ is continuous with respect to the product topology on $M \times M$.
(c) Find an example for which M is a smooth manifold, but $d\colon M \times M \longrightarrow \mathbb{R}$ is *not* smooth.

Problem 2.

Let X and Y, be manifolds, and let U and Z be submanifolds of Y.

(a) Assume that $f\colon X \to Y$ is a smooth map transversal to Z in Y, so that $W = f^{-1}(Z)$ is a submanifold of X. Prove that $T_x(W)$ is the preimage of $T_{f(x)}(Z)$ under the linear map $df_x\colon T_x(X) \to T_{f(x)}(Y)$.
(b) Assume that U is transversal to Z. Show that for $y \in U \cap Z$, $T_y(U \cap Z) = T_y(U) \cap T_y(Z)$.

Problem 3.

(a) Compute the fundamental group of the space obtained from the disjoint union of two spaces, each homeomorphic to the torus $S^1 \times S^1$, by identifying a circle $S^1 \times 1$ in one torus with the corresponding circle $S^1 \times 1$ in the other torus.
(b) Let $X \subset \mathbb{R}^m$ be the union of convex open sets $X_1, \cdots, X_n$ such that $X_i \cap X_j \cap X_k \neq \emptyset$ for all $i, j, k = 1, \ldots, n$. Show that X is connected and simply-connected.

Problem 4.

Let $\mathsf{TopPair}$ be the category of pairs of topological spaces and continuous maps (as usual, we identify a single space X with the pair $(X, \emptyset)$) and let $\mathsf{ChCompl}$ be the category of chain complexes $C_\bullet$ of abelian groups (with $C_n = 0$ for $n < 0$) and chain maps. Let $F\colon \mathsf{TopPair} \rightsquigarrow \mathsf{ChCompl}$ be a functor and define a "homology theory" H^F by $H_n^F(X) = H_n(F(X))$, $H_n^F(X, A) = H_n(F(X, A))$. Assume that for each $(X, A) \in \mathsf{TopPair}$, one has a natural short exact sequence

$$0 \to F_\bullet(A) \to F_\bullet(X) \to F_\bullet(X, A) \to 0.$$

Also assume that if X is contractible, then

$$H_n^F(X) \cong \begin{cases} \mathbb{Z} \quad \text{(with a natural choice of generator)}, & n = 0, \\ 0, & n > 0. \end{cases}$$

(a) Suppose $x, y \in X$ lie the same path component of X. Show that the images of $H_0^F(x)$ and of $H_0^F(y)$ in $H_0^F(X)$ must be equal.
(b) Let $\mathsf{Sing}\colon \mathsf{TopPair} \rightsquigarrow \mathsf{ChCompl}$ be the singular chain functor. Show that there is a natural transformation $\Phi\colon \mathsf{Sing} \to F$ inducing an isomorphism $H_\bullet \to H_\bullet^F$ on contractible spaces. (Hint: *Naturality* is key; use the method of acyclic models.)
(c) Now assume in addition that the natural map $(D^n, S^{n-1}) \to (S^n, \mathrm{pt})$ (obtained by collapsing S^{n-1} to a point) induces an isomorphism on the relative H_n^F groups for all $n \geq 1$. (This is a weak form of the excision axiom.) Also assume that $F(X \amalg Y) = F(X) \oplus F(Y)$. (Here $\amalg$ denotes the disjoint union of spaces.) Deduce that Φ induces isomorphisms $H_\bullet(S^n) \to H_\bullet^F(S^n)$ for each n. (Hint: Start by proving this for $n = 0$, and proceed by induction on n.)

Problem 5.

Let $n \geq 3$ and suppose X is a CW complex with one 0-cell and all other cells of dimension $\geq n-1$. Suppose

$$H_n(X,\mathbb{Z}) \cong \mathbb{Z}^m \oplus F,$$

where F is a finite abelian group which is the direct sum of k finite cyclic groups.

(a) Show that you can attach $m+k$ $(n+1)$-cells to X, obtaining a new CW complex Y with $H_n(Y,\mathbb{Z}) = 0$ and $H_j(Y,\mathbb{Z}) \cong H_j(X,\mathbb{Z})$ for $j \neq n, n+1$. (Hint: The property you need for the attaching maps of the cells has something to do with the Hurewicz map.)
(b) What is $H_{n+1}(Y,\mathbb{Z})$?
(c) Show that if $X = S^n$, Y can be taken to be D^{n+1}.

Problem 6.

Suppose M^n is a compact orientable topological n-manifold with boundary ∂M a rational homology sphere, i.e., with $H_\bullet(\partial M, \mathbb{Q}) \cong H_\bullet(S^{n-1}, \mathbb{Q})$.

(a) Assuming n is odd, use Poincaré duality (with coefficients $\mathbb{Q}$) to show that M has Euler characteristic $\chi(M) = 1$.
(b) Assuming $n \equiv 2 \pmod 4$, show that the Euler characteristic $\chi(M)$ of M is odd. You will need the fact (which you can assume) that if a finite-dimensional vector space over $\mathbb{Q}$ admits a non-degenerate skew-symmetric bilinear form, then the vector space has even dimension.

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

JANUARY 9, 2008
MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND

Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear which theorem you are using and why its use is justified. In problems with multiple parts, be sure to go on to the rest of the problem even if there is some part you cannot do. In working on any part, you may assume the answer to any previous part, even if you have not proved it.

Problem 1.

Let S^1 denote the circle and $C = \mathbb{R} \times S^1$ be the cylinder.

(a) Find an example of a surjective continuous map $\mathbb{R} \longrightarrow S^1$ which is a local homeomorphism but not a covering space.

(b) Show that for any local homeomorphism $f : \mathbb{R} \longrightarrow S^1$, the induced map $\pi_1(f) : \pi_1(\mathbb{R}) \longrightarrow \pi_1(S^1)$ is injective.

(c) Find an example of a local homeomorphism $f : X \longrightarrow C$ for which the induced map $\pi_1(f) : \pi_1(X) \longrightarrow \pi_1(C)$ is not injective.

Problem 2.

Let Σ be a closed orientable surface of genus g, where $g \geq 3$. An orientation-preserving homeomorphism $\theta : \Sigma \to \Sigma$ of order n is said to act freely if for all $x \in \Sigma$, $\theta^k(x) = x$ if and only if $n \mid k$.

(a) Show that if there exists an orientation-preserving homeomorphism of Σ acting freely, then $n \mid (g-1)$.

(b) Show that there exists an orientation-preserving homeomorphism $\theta : \Sigma \to \Sigma$ of order $g-1$ which acts freely on Σ.

(c) Suppose $g = 2$. Does there exist an orientation-preserving homeomorphism $\theta : \Sigma \to \Sigma$ of order 2 acting freely?

Problem 3.

Let X be a topological space and let $Y \subset X$ be a subspace. Then Y is a *retract* of X if and only if there exists a continuous map $r : X \longrightarrow Y$ such that $r(y) = y$ for all $y \in Y$.

(a) Let Y be a retract of X. Show that if X is contractible, then Y is contractible.

(b) Let Y be a retract of X. Show that if X is connected, then Y is connected.
(c) A space Z is said to have the fixed point property if every continuous map $h : Z \to Z$ has a fixed point. Give an example of a pair $Y \subset X$ where Y is a retract of X, Y has the fixed point property and X does not.

Problem 4.

If X is any locally compact Hausdorff space, a singular n-cochain on X, $f : C_n(X) \to \mathbb{Z}$, is said to have compact support if there is a compact subset $Y \subseteq X$ such that f vanishes on any singular simplex $\sigma : \Delta^n \to X - Y$.

Let $\Delta^1 = \{(t_0, t_1) \mid t_0 + t_1 = 1, t_i \geq 0\}$ be the standard 1-simplex and let $e_0 = (0, 1)$, $e_1 = (1, 0)$. Define a singular 1-cochain $f \in \mathrm{C}^1(\mathbb{R}, \mathbb{Z})$ by

$$f(\sigma) = \left\{ \begin{array}{cl} 1 & \sigma(e_0) \leq 0 < \sigma(e_1) \\ -1 & \sigma(e_1) \leq 0 < \sigma(e_0) \\ 0 & \text{otherwise} \end{array} \right\}$$

for any singular 1-simplex $\sigma : \Delta^1 \to \mathbb{R}$.

(a) It is a fact that f is a 1-cocycle. Show that f has compact support.
(b) Write f as a coboundary, i.e. find a 0-cochain $g \in \mathrm{C}^1(\mathbb{R}, \mathbb{Z})$ such that $f = \delta g$.
(c) Show that there is no 0-cochain h with compact support such that $f = \delta h$.

Problem 5.

(a) Let M^n be a connected, compact, *non-orientable* n-manifold. Show that $H^n(M, \mathbb{Z}) \simeq \mathbb{Z}/2$.

(b) Show that if a closed (i.e., compact, connected with no boundary) orientable manifold of dimension $2k$ has $H_{k-1}(M, \mathbb{Z})$ torsion free, then $H_k(M, \mathbb{Z})$ is also torsion free.

Problem 6.

(a)(*3 points*) For each $n > 0$, show that the complex projective space $\mathbb{CP}^n$ can be obtained from $\mathbb{CP}^{n-1}$ by attaching a cell of dimension $2n$. (Hint: View $\mathbb{CP}^{n-1}$ as the subspace of $\mathbb{CP}^n$ consisting of points with homogeneous coordinates $[z_0, \ldots, z_{n-1}, 0]$.)

(b)(*7 points*) In the situation of a), show that the attaching map, $h : S^{2n-1} \to \mathbb{CP}^{n-1}$, of the $2n$-cell is not homotopic to a constant, provided $n > 1$, by using the structure of the cohomology ring $H^*(\mathbb{CP}^n, \mathbb{Z})$.

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

AUGUST 6, 2007
MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND

Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear which theorem you are using and why its use is justified. In problems with multiple parts, be sure to go on to the rest of the problem even if there is some part you cannot do. In working on any part, you may assume the answer to any previous part, even if you have not proved it.

Problem 1.

Let U be a connected open set in $\mathbb{R}^n$.

(a) Show that any two points in U can be connected by a piecewise straight line. Define $\mathrm{dist}(p, q)$ to be the infimum of the lengths of all such curves joining points p and q in U.

(b) Show that dist is a metric on U and that $\mathrm{dist} \geq D$, where D is the Euclidean distance.

(c) Show that dist defines the same topology as D.

(d) Show that $D = \mathrm{dist}$ if and only if the closure of U is convex.

Problem 2.

Represent the two-torus as the quotient Lie group

$$T^2 := \mathbb{R}^2/\mathbb{Z}^2.$$

(a) Prove that the map

$$\mathbb{R}^2 \xrightarrow{\tilde{f}} \mathbb{R}^2$$

$$(x, y) \longmapsto (-x, -y)$$

defines a diffeomorphism $T^2 \xrightarrow{f} T^2$ of period 2.

(b) Notation as in (a), let G be the cyclic group of diffeomorphisms of T^2 generated by f. Prove or disprove: The quotient map $X \xrightarrow{q} X/G$ is a covering space.

(c) Prove or disprove: X/G is a (topological) manifold.

(d) Prove or disprove: X/G is homeomorphic to a 2-sphere.

Problem 3.

In this problem, assume X is a connected finite CW-complex with one 0-cell, two 1-cells, three 2-cells, and some 3-cells.

(a) Suppose X has the homotopy type of a closed 3-dimensional manifold. How many 3-cells are there? (Hint: Euler characteristic)

(b) If X^1 is the 1-skeleton of X, show that $X^1 \cong S^1 \vee S^1$ and that $\pi_1(X^1)$ is a free group on two generators a and b represented by the two 1-cells in X.

(c) If the three 2-cells are attached by (based) maps $g_i : S^1 \to S^1 \vee S^1$ and $[g_i] = w_i(a,b) \in \pi_1(X^1)$, ($w_i(a,b)$ is a word in a and b), give necessary and sufficient conditions on $\{w_i\}$ so that $H_1(X, \mathbb{Z}) = 0$.

Problem 4.

(a) Suppose M^n is any connected, oriented, closed n-manifold. By considering a small disc about a point p in M, show there exists a map $f\colon M \to S^n$ which induces an isomorphism on H_n.

(b) Suppose M^n and f are as in (a), $n = 3$, and $H_1(M, \mathbb{Z}) = 0$. Show that $f\colon M^3 \to S^3$ induces an isomorphism on *all* homology groups. Does f have to be a homotopy equivalence?

(c) Show by example if M^n is a connected, oriented, closed n-manifold, there *need not be* a map $g : S^n \to M^n$ which induces an isomorphism on H_n.

Problem 5.

Suppose M is a compact 5-manifold such that $H_0(M) = \mathbb{Z}$, $H_1(M) = \mathbb{Z}/3$ and $H_2(M) = \mathbb{Z}$.

(a) Is M orientable?

(b) What are $H_3(M), H_4(M)$ and $H_5(M)$?

(c) Assume that M can be chosen to be of the form $S^2 \times N$ for some 3-manifold N. What would the homology groups of N be? Find such an N.

Problem 6.

Let V be a closed orientable $2n-2$ submanifold of $\mathbb{CP}^n$, not necessarily connected.

(a) Prove or disprove: The complement of V in $\mathbb{CP}^n$ is connected.

(b) Show by example that the complement of V in $\mathbb{CP}^n$ need not be simply connected. **Hint:** What happens in the case $n = 1$?

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

JANUARY 19, 2007

In any problem you may assume the answer to a previous part even if you have not proven it.

1. Let (X, d) be a compact metric space. For any $r > 0$ and any closed subset A of X let $\mathcal{N}(A, r) = \{x \in X \mid \exists a \in A \text{ with } d(a, x) < r\}$ denote the set of points in X at distance less than r to A. For any two non-empty closed subsets A, B of X define

$$D(A, B) = \inf\{r > 0 \mid \mathcal{N}(A, r) \text{ contains } B \text{ and } \mathcal{N}(B, r) \text{ contains } A\}.$$

Show D is a metric on the space of non-empty closed subsets of X. (Make sure you explain why it is well defined.)

2. The *Cantor ternary set* $T \subset [0, 1]$, consisting of all real numbers whose ternary (base-3) expansion consists entirely of 0's and 2's. It is the image of the map

$$\prod_{i=1}^{\infty}\{0, 2\} \xrightarrow{\tau} [0, 1], \qquad (d_1, \ldots) \longmapsto \sum_{i=1}^{\infty} d_i 3^{-i}.$$

where $\prod_{i=1}^{\infty}\{0, 2\} = \{(d_1, d_2, \ldots) \mid d_i = 0, 2\}$ denotes the countably infinite product of the set $\{0, 2\}$.

a) If $\{0, 2\}$ is given the discrete topology and $\prod_{i=1}^{\infty}\{0, 2\}$ the product topology, show $\tau : \prod_{i=1}^{\infty}\{0, 2\} \to [0, 1]$ is continuous.
b) Show τ is injective.
c) Prove or disprove: $\prod_{i=1}^{\infty}\{0, 2\}$ is homeomorphic to T.
d) Prove or disprove: $[0, 1]$ is homeomorphic to the Cantor set T.

3. Let X be the figure eight and $\bar{x}$ the wedge point.

a) Give a set of generators and relations for $\pi_1(X, \bar{x})$.
b) Show that if Y is a circle with 3 other circles appropriately attached, then Y can be made into a three fold regular covering space of X. (A well drawn picture of Y and a description of the covering projection $p : Y \to X$ should suffice.)
c) Choose a point $\bar{y} \in p^{-1}(\bar{x})$. Compute the fundamental group $\pi_1(Y, \bar{y})$ and describe the map $p_* : \pi_1(Y, \bar{y}) \to \pi_1(X, \bar{x})$.

d) Let S^1 be the unit circle in the complex plane. Show that for any continuous $f\colon (S^1, 1) \to (X, \bar{x})$ there is a continuous $g\colon (S^1, 1) \to (Y, \bar{y})$ so that $p \circ g(z) = f(z^3)$ for all $z \in S^1$.

4. Let $\mathbb{RP}^2$ denote the real projective plane.
   a) Prove $\mathbb{RP}^2$ is homeomorphic to a space obtained from the Möbius band by adding a cone to the boundary.
   b) Using a), show that $\mathbb{RP}^2$ embeds in $\mathbb{R}^4 \subseteq S^4$. (The embedding need not be smooth.)
   c) Determine the homology groups $\hat{H}_*(S^4 - \mathbb{RP}^2)$ with $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients for any embedding of $\mathbb{RP}^2$ in S^4.

5. Let A be an abelian group and $n \geq 2$ an integer. A CW-complex, $M(A, n)$ is called a Moore space of type (A, n) if $H_0(M(A, n), \mathbb{Z}) = \mathbb{Z}$, $H_n(M(A, n), \mathbb{Z}) \simeq A$ and $H_j(M(A, n), \mathbb{Z}) = 0$ for $j \neq 0, n$.
   a) Let $\mathbb{Z}_k$ be the cyclic group of order $k > 1$. Show there exists a simply connected $M(\mathbb{Z}_k, n)$ of dimension $n + 1$.
   b) Show the one-point union (i.e., wedge) of a Moore space of type (A, n) and a Moore space of type (B, n) is a Moore space of type $(A \oplus B, n)$. If A is any finitely generated abelian group show there exists a simply connected $M(A, n)$ of dimension $\leq n + 1$.
   c) Determine all finitely generated abelian groups $A_1, A_2, \ldots, A_s$ and $n_1, n_2, \ldots, n_s$ so that
   $$M(A_1, n_1) \vee M(A_2, n_2) \vee \cdots \vee M(A_s, n_s)$$
   has the homotopy type of a closed orientable 4-manifold.

6. Recall that points in $\mathbb{CP}^n$ can be represented by equivalence classes $[z_0, z_1, \ldots, z_n]$, where $z_j \in \mathbb{C}$ are not all zero, and $[z_0, z_1, \ldots, z_n] = [\lambda z_0, \lambda z_1, \ldots, \lambda z_n]$ for $\lambda \in \mathbb{C}^*$.
   a) Show that
   $$([z_0, z_1], [w_0, w_1]) \mapsto [z_0 w_0, z_0 w_1, z_1 w_0, z_1 w_1]$$
   defines a continuous $f : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^3$.
   b) Compute the induced map on $H_2(-, \mathbb{Z})$.
   c) Let u and v be the canonical generators of $H^2(\mathbb{CP}^1, \mathbb{Z})$ for each of the two copies of $\mathbb{CP}^1 \simeq S^2$, respectively and let t be the canonical generator of $H^2(\mathbb{CP}^3, \mathbb{Z})$. Use your answer in (b) to determine $f^*(t^2)$.

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

AUGUST 11, 2006
MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND

Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear which theorem you are using and why its use is justified.

Problem 1.

Let X and Y be CW-complexes. A map $f : X \longrightarrow Y$ is *a local homeomorphism* if $\forall x \in X$, $\exists U$ open neighborhood of $x \in X$ such that the restriction of f to U is a homeomorphism. A map $f : X \longrightarrow Y$ *evenly covers* Y if $\forall y \in Y$, $\exists V$ open neighborhood of $y \in Y$ such that for each connected component U of $f^{-1}(V)$, the restriction of f to U is a homeomorphism onto V.

Prove or disprove the following statements:

(1) A map is a local homeomorphism if it evenly covers its image.
(2) A map is a local homeomorphism only if it evenly covers its image.
(3) Suppose the surjective map $f : X \longrightarrow Y$ evenly covers Y. If Y is 1-connected and X is connected, then f is a homeomorphism.

Problem 2.

Let X be a compact metric space and $X \xrightarrow{f} X$ an isometry. Prove that f is onto. (*Hint:* Look at the iterates of f acting on a point.)

Problem 3.

Suppose $k \geq 0$ is an integer, $n = 4k + 2$. Let M^n be a compact n-dimensional manifold with $\partial M = \emptyset$. Let $b_{2k+1} = \dim H^{2k+1}(M; \mathbb{Q})$ be the $(2k + 1)$-st rational Betti number of M.

Prove or disprove:

(1) If M is orientable, then b_{2k+1} is even.
(2) If M is nonorientable, then b_{2k+1} is even.

Problem 4.

Let M, N be compact connected 2-dimensional manifolds, possibly with boundary. Such a manifold is said to be *closed* if its boundary is empty. Suppose that $f : M \longrightarrow N$ is a homotopy-equivalence.

Prove or disprove:

(1) M is closed if and only if N is closed.
(2) Suppose that M and N are both closed. Then M is orientable if and only if N is orientable.
(3) Suppose that neither M nor N is closed. Then M is orientable if and only if N is orientable.

Problem 5.

Consider a two-by-two integer matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

of determinant 1. Let T^2 be the quotient torus $\mathbb{R}^2/\mathbb{Z}^2 \approx S^1 \times S^1$. Let $T^2 \xrightarrow{f_A} T^2$ denote the map induced by A on the quotient $\mathbb{R}^2/\mathbb{Z}^2$. Consider two closed solid tori $M_1 = S^1 \times D^2$, $M_2 = D^2 \times S^1$ with respective boundaries $\partial M_i = S^1 \times S^1 \approx T^2$ for $i = 1, 2$. Let $M(A)$ denote the identification space of $M_1 \coprod M_2$ by the equivalence relation $x_1 \sim x_2 \iff x_2 = f_A(x_1)$ for $x_i \in \partial M_i$.

Prove or disprove:

(1) f_A is always a homeomorphism.
(2) $M(A)$ is always a manifold.
(3) The fundamental group $\pi_1(M(A))$ is always cyclic.
(4) The fundamental group $\pi_1(M(A))$ is always finite.
(5) The homology group $H_2(M(A); \mathbb{Z})$ is always trivial.
(6) $M(A)$ is never simply-connected.

Problem 6.

Let X be the space obtained from $S^1 \vee S^1$ by attaching two 2-cells by the words a^5b^{-3} and $b^3(ab)^{-2}$ in $\pi_1(S^1 \vee S^1)$ = the free group with generators a and b. Recall that a space is *acyclic* $\iff$ it has only trivial homology groups in positive dimensions. Prove or disprove:

a) X is acyclic.
b) The map $a \mapsto (1, 2, 4, 5, 3)$ and $b \mapsto (2, 3, 4)$ defines a homomorphism $\pi_1(X) \to A_5$, the alternating group on 5 letters.
c) X is contractible.

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND
JANUARY 18, 2006

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Problem 1.

This problem concerns five statements. Some of these statements are true, some of them are false, and some might even be unknown! Prove the correct statements and disprove the false ones. You are not expected to prove or disprove the ones which are unknown.

(1) A path-connected space is connected.
(2) A connected manifold is path-connected.
(3) A CW-complex with one 0-cell is connected.
(4) A CW-complex with one 1-cell is connected.
(5) A CW-complex with one 0-cell and no 1-cells is simply connected.

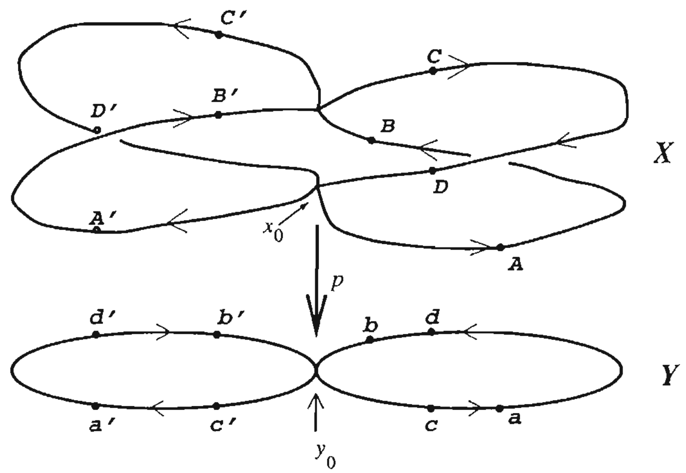
Problem 2.

Let G be a Lie group, i.e. G is a smooth manifold and a group, and the group operations $\mu : G \times G \to G$ and $i : G \to G$ given by $\mu(g, h) = gh$ and $i(g) = g^{-1}$ are smooth maps. Let e denote the identity of G.

(1) Show that $\mu_{*e}(X, Y) = X + Y$ where X, $Y \in T_eG$.
(2) Show that $i_{*e}(X) = -X$ for $X \in T_eG$.
(3) Show that $\mu : G \times G \to G$ is a submersion.

Problem 3.

Let $p : X \to Y$ be the double cover of the wedge of two circles pictured below.



Here $p(A) = a$, $p(B) = b$, etc.

(1) Determine X up to homotopy type.
(2) In terms of appropriate generators of $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$, compute $p_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

Problem 4.

Recall that for K^n and L^n, smooth connected n-manifolds without boundary, we can form a new n-manifold, denoted $K \# L$, called the **connected sum** of L and N, by taking smooth embeddings $f \colon \mathbb{R}^n \to K$ and $g \colon \mathbb{R}^n \to L$ and gluing $K - \{f(0)\}$ to $L - \{g(0)\}$ by identifying $f(tu)$ with $g(t^{-1}u)$ for u in the unit sphere S^{n-1} and $t \in (0, \infty)$. Let M be the connected sum of $\mathbb{RP}^4$ and $S^1 \times S^3$.

(1) Determine $\pi_1(M)$ and the integral homology groups of M. (Recall that $H_q(\mathbb{RP}^n) = \mathbb{Z}$ for $q = 0$ and $q = n$ if n is odd, $\mathbb{Z}_2$ if q is odd and $1 \leq q \leq n-1$ and zero otherwise.)
(2) Is M orientable?
(3) Find the cohomology ring $H^*(M, \mathbb{Z}_2)$.

Problem 5.

Let Γ be the cyclic group of order 2.

(1) Use cup products to show that any action of Γ on $\mathbb{CP}^2$ preserves orientation, and then use the Lefschetz Fixed-Point Theorem to show that there is no free action of Γ on $\mathbb{CP}^2$.
(2) More generally, suppose M is a compact, oriented, topological 4-manifold without boundary and with finite fundamental group. Show that if Γ acts freely on M, preserving the orientation, then $b_2(M) = \dim H_2(M, \mathbb{Q}) \geq 2$. Show that this bound is sharp, by exhibiting a free orientation-preserving action of Γ

on a manifold with $b_2 = 2$. (Hint: M does not have to be very complicated.)

Problem 6.

(1) Show that for any $n \geq 1$ and any m, there exists a continuous map $f : S^n \to S^n$ of degree m. 3

(2) Let k and l be integers and attach two $(n+1)$-cells to S^n by maps of degree k and l respectively. Let K be the resulting CW-complex. Compute the integral homology groups $H_*(K)$. 4

(3) Find all k, l and n for which K has the homotopy type of a closed manifold of dimension $n + 1$. 3

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

UNIVERSITY OF MARYLAND

Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem. If you do, it is your responsibility to clarify exactly which theorem you are using and to justify its use. In any part of a problem with multiple parts, you may assume the answer to any previous part, even if you have not proved it.

(1) Consider the following two topological spaces. Let A be the union of an infinite number of circles in the plane, all tangent at the origin,

$$A = \bigcup_{n=1}^{\infty} \{(x,y) \mid x^2 + (y - 1/n)^2 = 1/n^2\}$$

let $K = \{1, 1/2, 1/4, \ldots\} = \{2^{-n} \mid 0 \leq n \in \mathbb{Z}\}$ and let B be the quotient space of the half open interval $(0,1]$ with K crushed to a point

$$B = (0,1]/K$$

Each of these spaces is a countably infinite number of circles with a point in common.

(a) Which of A or B is connected?
(b) Which of A or B is compact?
(c) Are A and B homeomorphic?

(2) Let X_0 and X_1 be arcwise connected, locally arcwise connected, locally relatively simply connected spaces and let $p_i \colon \tilde{X}_i \to X_i$ be their universal coverings. Suppose $f \colon X_0 \to X_1$ is continuous.

(a) Show that there is a continuous map $\tilde{f} \colon \tilde{X}_0 \to \tilde{X}_1$ so that $p_1 \circ \tilde{f} = f \circ p_0$.
(b) If $\pi_1(X_1, x_1) \approx \mathbb{Z}_6$, how many different such $\tilde{f}$ are there?

Date: 12 August 2005.

(3) Let $f(x, y, z) = x^3 + 6xz + y^2 - 3z^2$.
  (a) Explain why $M = f^{-1}(10)$ is a smooth submanifold of $\mathbb{R}^3$.
  (b) Find the tangent plane to M at the point $(1, 3, 2)$.
  (c) Is the vertical line $x = 1$, $y = 3$ transverse to M at $(1, 3, 2)$?

(4) Let M be a compact, connected, orientable, three dimensional manifold with nonempty connected boundary ∂M. Suppose that $\pi_1(M, x_0) = \mathbb{Z} * \mathbb{Z}_6$ and $H_2(M; \mathbb{Z}) = \mathbb{Z}$.
  (a) Duality for manifolds with boundary implies that

$$\cap[M] \colon H_i(M, \partial M; \mathbb{Z}) \to H^{3-i}(M; \mathbb{Z})$$

  is an isomorphism for all i. Write down the long exact homology sequence for the pair $(M, \partial M)$ (with $\mathbb{Z}$ coefficients) and evaluate all of its terms except for $H_1(\partial M; \mathbb{Z})$.
  (b) Show that $H_1(\partial M; \mathbb{Z})$ is all torsion.

(5) Let A be the the unit circle in the xy plane in $\mathbb{R}^3$ and let A_+ and A_- be two of its semicircles,

$$\begin{aligned} A &= \{(x, y, z) \mid z = 0, x^2 + y^2 = 1\} \\ A_+ &= \{(x, y, z) \mid z = 0, x^2 + y^2 = 1, x \geq 0\} \\ A_- &= \{(x, y, z) \mid z = 0, x^2 + y^2 = 1, x \leq 0\} \end{aligned}$$

  (a) Find $H_*(\mathbb{R}^3 - A_+; \mathbb{Z})$.
  (b) Find $H_*(\mathbb{R}^3 - A_-; \mathbb{Z})$.
  (c) Find $H_*(\mathbb{R}^3 - A; \mathbb{Z})$.

(6) Recall $\mathbb{CP}^n$ is complex projective n space and S^2 is the two dimensional sphere.
  (a) Compute the cohomology ring $H^*(S^2 \times \mathbb{CP}^2; \mathbb{Z})$.
  (b) Suppose $f \colon S^2 \times \mathbb{CP}^2 \to \mathbb{CP}^3$ and $a \in H_6(S^2 \times \mathbb{CP}^2; \mathbb{Z})$. Show $f_*(a)$ is divisible by 3. (Hint: Use the cohomology ring structure.)

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

JANUARY 14, 2005

MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND

Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear exactly which theorem you are using and why its use is justified. In problems with multiple parts, you may assume the answer to any previous part, even if you have not proved it.

Problem 1.

Let X and Y be locally compact Hausdorff spaces and let $f\colon X \to Y$ be a continuous map. Recall that we say f is proper if $f^{-1}(K)$ is compact for all compact subsets $K \subset Y$.

a) Let $X^\infty = X \cup \{\infty\}$ and $Y^\infty = Y \cup \{\infty\}$ be the one point compactifications of X and Y. Let $f^\infty\colon X^\infty \to Y^\infty$ be defined by $f^\infty|_X = f$ and $f^\infty(\infty) = \infty$. Show that f is proper if and only if f^∞ is continuous. (Note: It is not sufficient to say "This is a theorem in Bredon". You must prove this directly from the definition of the one point compactification.)

b) Show that f is a homeomorphism if and only if f is proper, one to one, and onto.

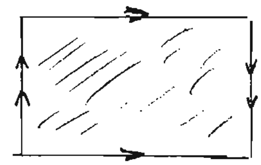
c) Give an example of locally compact Hausdorff X and Y and a (nonproper) one to one onto map $f\colon X \to Y$ so that X and Y are not homeomorphic.

Problem 2.

a) Let X and Y be non-empty, Hausdorf, path connected, and locally path connected spaces. Suppose X is compact and $f : X \to Y$ a local homeomorphism. Show f is onto and $f : X \to Y$ is a covering map.

b) Now suppose X and Y are non-empty smooth n-dimensional manifolds without boundary with X compact, connected and Y simply connected. Let $f : X \to Y$ be a smooth map whose Jacobian $f_* : T_pX \to T_{fp}Y$ is non-singular for all $p \in X$. Show f is a diffeomorphism.

Problem 3.



a) Show the Klein bottle K (the rectangle with the sides identified as shown) is the union of two Möbius bands with the boundary circles identified.

b) Use a) and the van Kampen theorem to determine a presentation of $\pi_1(K, x_0)$.

c) Determine $H_*(K, \mathbb{Z})$ and $H^*(K, \mathbb{Z}_2)$.

Problem 4.

Let S^n be the n-sphere and $f : S^n \to S^n$ a continuous map.

a) Show that if f is not surjective, then f is homotopic to a constant map.

b) Construct an example of a surjective map $f : S^n \to S^n$ which is homotopic to a constant.

c) Is every map $f : S^n \to S^n$ homotopic to a constant? (Either give a proof or a counterexample.)

Problem 5.

Let $S^3 = \{(z, w) \in \mathbb{C}^2 \,|\, |z|^2 + |w|^2 = 1\}$ and $S^2 = \mathbb{C} \cup \{\infty\}$. Define maps $f, g, h : S^3 \to S^2$ by the following formulas i) $f(z, w) = 0$, ii) $g(z, w) = z$ and iii) $h(z, w) = z/w$. Denote by X_k the space $S^2 \cup_k e^4$ where k is one of f, g or h.

a) Write down the cellular chain complexes for the spaces X_k (including boundary maps) and compute the integral homology groups of the spaces X_k.

b) Which of the spaces X_k are homotopy equivalent and which are not? (The space X_h can be shown to be homeomorphic to $\mathbb{CP}^2$, complex projective 2-space.)

Problem 6.

a) Show any map $f : S^{k+l} \to S^k \times S^l$ with k, $l > 0$ induces the zero map $f_* : H_{k+l}(S^{k+l}, \mathbb{Z}) \to H_{k+l}(S^k \times S^l, \mathbb{Z})$.

b) Let M and N be k-dimensional compact, connected oriented manifolds without boundary and let $f : M \to N$ be a continuous map. Suppose M is simply connected and $H_{k-1}(N, \mathbb{Z}) \neq 0$. Show $f_* : H_k(M, \mathbb{Z}) \to H_k(N, \mathbb{Z})$ is the zero map. (This is really a covering space problem.)

Written Qualifying Examination
Geometry/Topology
Friday, August 20, 2004

Instructions. Answer each question on a separate numbered answer sheet. In problems with multiple parts, whether the parts are related or not, the parts are graded independently of one another. Be sure to go on to subsequent parts even if there is some part you cannot do.

You are allowed to appeal to "standard theorems" proved in class or in the textbook, but if you do so, it's your responsibility to state clearly exactly what you're using and why it applies.

1. Let U be a connected subset of $\mathbb{R}^n$. Define $d_l\colon U \times U \to \mathbb{R}$ by $d_l(x, y) =$ inf of the lengths of all broken straight line segments joining x and y in U. (If there are no such paths, let $d_l(x, y) = +\infty$.)

(a) Prove that $d_l' = \min(d_l, 1)$ is a metric.

(b) Let d be the ordinary Euclidean metric on U. Show that the identity on U, mapping from the topology induced by d_l' to the topology induced by d, is continuous.

(c) If U is open, show that the map in (b) is a homeomorphism.

(d) Give a counterexample to part (c) if U is not assumed to be open.

2. Let $h\colon M \to N$ be a submersion from a smooth manifold M onto a smooth manifold N.

(a) Show that $h^{-1}(x)$ is a smooth manifold without boundary, for each $x \in N$.

(b) Suppose that h is proper, that is, $h^{-1}(K)$ is compact for any compact set K in N. By (a), $h^{-1}(x)$ is a smooth compact manifold without boundary, for each $x \in N$. In this case, one can show (you do not need to do this) that for each $x \in N$, there is an open neighborhood U such that $h^{-1}(U)$ is diffeomorphic to $U \times P$, where $P = h^{-1}(x)$, in such a way that the restriction of h to $h^{-1}(U)$ can be identified with the projection $U \times P \to U$. In other words, h is *locally* the projection in a product. Give an example where h is proper and is not *globally* the projection in a product, i.e., where M does not split as $N \times P$ for any P.

(c) If $N = \mathbb{R}$ and h is proper, show (using (b)) that M is homeomorphic to $P \times \mathbb{R}$ for some compact smooth manifold P.

(d) Give an example of a submersion $h\colon M \to \mathbb{R}$ which is not proper and with M not a product with $\mathbb{R}$.

3. Let M be a connected manifold with $H_1(M, \mathbb{Z}) = 0$. Show that any continuous $f\colon M \to T^2$ is null homotopic, where T^2 is the torus $S^1 \times S^1$.

4. A compact connected 7-manifold M (without boundary) has the following homology groups:

$$\begin{cases} H_1(M,\mathbb{Z}) & \cong \mathbb{Z}/3, \\ H_2(M,\mathbb{Z}) & \cong \mathbb{Z}, \\ H_3(M,\mathbb{Z}) & \cong \mathbb{Z} \oplus \mathbb{Z}/3. \end{cases}$$

(a) Compute all the remaining homology groups of M.
(b) Compute all the cohomology groups of M.
(c) Give a concrete example of a manifold with these homology groups. You can take M to be of the form $N^4 \times L^3$, with N simply connected.

5.
(a) Show that the 2-torus T^2 and $S^1 \vee S^1 \vee S^2$ both have CW decompositions with four cells: one 0-cell, two 1-cells, and a 2-cell. Recall that $\vee$ denotes the "one-point union" of two spaces, obtained from the disjoint union $\coprod$ by identifying basepoints. Then show that T^2 and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups.
(b) Show that T^2 and $S^1 \vee S^1 \vee S^2$ have different fundamental groups, hence are *not* homotopy equivalent.
(c) Show that the suspensions

$$ST^2, \quad S\left(S^1 \vee S^1 \vee S^2\right) = S^2 \vee S^2 \vee S^3$$

are homotopy equivalent. (The *reduced suspension* of a based space (X, x) is the smash product with S^1, i.e., $SX = (S^1 \times X)/(S^1 \times \{x\} \cup \{*\} \times X)$.) **Hint**: ST^2 has a CW decomposition with one 0-cell, two 2-cells, and a 3-cell. The attaching map of the 3-cell is the suspension of the attaching map of the 2-cell in T^2. From knowledge of the attaching map of the 2-cell in T^2, show that this attaching map is null-homotopic. You may assume that $\pi_2(S^2 \vee S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$.

6. Let p be the quotient map from $\mathbb{CP}^n$ to $\mathbb{CP}^n/\mathbb{CP}^k$, $k < n$.
(a) Show that p^* is a monomorphism on integral cohomology.
(b) Describe the ring structure on $H^*(\mathbb{CP}^n/\mathbb{CP}^k, \mathbb{Z})$. Give necessary and sufficient conditions on n and k for all cup products to be trivial.
(c) Show that there is no retraction from $\mathbb{CP}^m/\mathbb{CP}^k$ to $\mathbb{CP}^n/\mathbb{CP}^k$, assuming that $n < 2k + 2 \leq m$.
(d) Show that $\mathbb{CP}^n/\mathbb{CP}^{n-1}$ is homeomorphic to S^{2n}.

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

JANUARY 22, 2004

In any problem you may assume the answer to a previous part, even if you have not proven it. $H_*(Z)$ of a space Z alway means homology with integer coefficients.

1. Let Z be a space, $\pi_0(Z)$ the set of path components of Z and $\pi_0'(Z)$ the set of components of Z Suppose $f : X \to Y$ is a continuous map.

a) Show f induces a map $f_\# : \pi_0(X) \to \pi_0(Y)$ and that if $f \simeq g$ (homotopic) then $f_\# = g_\#$.
b) Show f induces a map $f_\# : \pi_0'(X) \to \pi_0'(Y)$ and that if $f \simeq g$ (homotopic) then $f_\# = g_\#$.
c) Suppose Z is a space for which the path components and the components coincide and let W be homotopy equivalent to Z. Show the path components and components of W agree.

2. Let $X = S^1 \times S^1$ with base point $(1,1)$ and $A = (S^1 \times \{1\}) \cup (\{1\} \times S^1)$, the union of the longitude and meridian circles in X.

a) What are the fundamental groups of X and A?
b) Let $\tilde{X}$ be the universal cover of X, $p : \tilde{X} \to X$ the projection and $\tilde{A} = p^{-1}(A) \subseteq \tilde{X}$. Show $p : \tilde{A} \to A$ is a connected covering of A and draw a picture of $\tilde{A} \subseteq \tilde{X}$.
c) Under the correspondence of coverings of A and subgroups of $\pi_1(A)$ show that $p : \tilde{A} \to A$ corresponds to the commutator subgroup of $\pi_1(A)$.
d) Let Z be a path-connected, locally path-connected space with $\pi_1(Z)$ finite. Show that every $f : Z \to X$ is null-homotopic.

3. Some of the following statements are true and some are false. Separate the true from the false and give reasons for your conclusions.

a) $S^3 - \{\text{three points}\}$ is simply connected.
b) $S^3 - \{\text{three points}\}$ is homotopy equivalent to $S^2 - \{\text{two points}\}$.
c) There exists a smooth map $f : S^3 \to S^1 \times S^2$ such that the differential $df_x : T_xS^3 \to T_{f(x)}(S^1 \times S^2)$ is an isomorphism for all x.

4. Let $X = S^1 \times S^1$ and identify $H_1(X)$ with $\mathbb{Z} \times \mathbb{Z}$ by choosing the standard generators of the torus. Then any $h : H_1(X) \to H_1(X)$ is identified with a 2×2 matrix with integer coefficients.

a) Given any 2×2 matrix with integer coefficients A, show there exists a map $f : X \to X$ with $f_* : H_1(X) \to H_1(X)$ equal to A.
b) If $f : X \to X$ and $f_* : H_1(X) \to H_1(X)$ corresponds to the matrix A (with respect to the standard generators) how does one describe $f^* : H^1(X) \to H^1(X)$ with respect to the dual generators?
c) Let $f : X \to X$ be a map of degree d, i.e., $f_*[X] = d[X]$ where $[X]$ is the fundamental class of X. Show $f_* : H_1(X) \to H_1(X)$ has determinant equal to d.

5. a) Let $n \geq 3$ and $r : S^{n-1} \to S^{n-1}$ be the reflection in the hyperplane $x_n = 0$. I.e., $r(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, -x_n)$. The mapping torus of r is the identification space $Y = S^{n-1} \times [0,1]/\sim$ where one identifies $(u, 0)$ with $(r(u), 1)$. Y is an n-dimensional manifold. Use the Meyer-Vietoris sequence to show Y is non-orientable with $H_{n-1}(Y) \approx \mathbb{Z}_2$.

b) Let $f : S^5 \to S^5$ be a map of degree d and let X be RP^5 with a 6-cell attached by the map $\pi \circ f$ where $\pi : S^5 \to RP^5$ is the natural projection. Determine the homology and cohomology groups of X.

6. Let M be a closed, connected 3-manifold. For all k, $H_k(M)$ is a finitely generated abelian group, and in particular $H_1(M) \approx \mathbb{Z}^r \oplus F$, where F is a finite group.

a) Show $H_2(M) \approx \mathbb{Z}^r$ if M is orientable.
b) Show $H_2(M) \approx \mathbb{Z}^{r-1} \oplus \mathbb{Z}_2$ if M is non-orientable. (Hint: For any abelian group G, $H_3(M, G) \approx \{g \in G | 2g = 0\}$ if M is non-orientable.)
c) Show that if M is non-orientable then $\pi_1(M)$ is infinite.
d) Let U and V be connected n-manifolds with $n \geq 3$ and let $U \# V$ be the connected sum. Show $H^1(U \# V) \approx H^1(U) \oplus H^1(V)$ and conclude that if $r \geq 0$ then there exists an orientable 3-manifold M with $H_2(M) \approx \mathbb{Z}^r$. (Recall the connected sum of two n-manifolds is obtained by removing the interior of a closed n-disc from each and identifying the bounding $(n-1)$-spheres. The identification is done in such a way that the connected sum of two orientable manifolds is orientable.)

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

AUGUST 18, 2003

MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND

Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear exactly which theorem you are using and why its use is justified. In problems with multiple parts, you may assume the answer to any previous part, even if you have not proved it.

Problem 1. Let L be the set of all lines in the plane $\mathbb{R}^2$ (not just the ones passing through the origin). Let X be the set of ordered pairs of distinct points in $\mathbb{R}^2$, i.e., $X = \{(u,v) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid u \neq v\}$, with the subspace topology. Let $\pi : L \to L$ be the map that takes a line to the line parallel to it through the origin. Let $\psi : X \to L$ be the map which takes a pair of points to the line through the points. We may define a topology on L by saying a set U is open if $\psi^{-1}(U)$ is open in X.

a) Show that this defines a topology on L.
b) Show that π is continuous.
c) Show that L is homeomorphic to a well known (non-compact) two-dimensional manifold. Which one?

Problem 2. Recall that a genus n surface is the closed orientable surface with n handles, i.e., obtained from the sphere by connected sum with n tori.

a) Show by drawing a picture of the identifications made by the non-trivial element of the deck group that a genus 3 surface is a two-fold cover of a genus 2 surface.
b) Show again by drawing a picture, that a genus 3 surface can cover a nonorientable surface.
c) Deduce that there are two different closed surfaces with the same Euler characteristic.

Problem 3. View the vector space of real 2 by 2 matrices as $\mathbb{R}^4$ (with the usual topology).

a) Show that the space $GL(2)$ of invertible matrices is an open subset.
b) Show that the space $SL(2)$ of matrices with determinant 1 is a smooth submanifold of $\mathbb{R}^4$.
c) Show that matrix multiplication in $SL(2)$ is a smooth map.
d) Find the critical points for the distance function from the zero matrix to points in $SL(2)$.

Problem 4. Let X be the space obtained by attaching a closed Möbius band to the real projective plane $\mathbb{RP}^2$ by a homeomorphism of the boundary of the closed Möbius band to any non-contractible embedded $S^1 \subseteq \mathbb{RP}^2$.

a) Use the Meyer-Vietoris sequence to compute the integral homology groups of X.
b) Compute $H_*(X;\mathbb{Z}_2)$ and $H^*(X;\mathbb{Z})$.

Problem 5. Recall that $\mathbb{CP}^{n+1}$ is obtained from $\mathbb{CP}^n$ by attaching a $(2n+2)$-cell by the canonical map $S^{2n+1} \to \mathbb{CP}^n$. Let $\mathbb{CP}^\infty = \cup_n \mathbb{CP}^n$ be the CW-complex whose $2n$-skeleton is $\mathbb{CP}^n$.

a) Show $H^*(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}[u]$ as rings where $\mathbb{Z}[u]$ is a graded polynomial ring with u of degree 2. (If you know it, you may assume without proof the ring structure of $H^*(\mathbb{CP}^n;\mathbb{Z})$.)
b) Compute the cohomology ring $H^*(Y;\mathbb{Z})$ where Y is the quotient space $Y = \frac{S^1 \times \mathbb{CP}^\infty}{S^1 \times *}$ and show it is isomorphic to the cohomology ring of $S^3 \times \mathbb{CP}^\infty$.

Problem 6. Let M and N be closed, connected, oriented n-dimensional manifolds.

a) Let $f : M \to N$ be a map of degree one, i.e. $f_*[M] = [N]$. Show there exists a map $\gamma : H_*(N;\mathbb{Z}) \to H_*(M;\mathbb{Z})$ such that $f_* \circ \gamma$ is the identity on $H_*(N;\mathbb{Z})$. (This map does NOT have to be induced by a map of spaces.) (Hint: Use the cap product and Poincaré duality).
b) Suppose $\pi_1(M)$ is a finite group and $H^1(N,\mathbb{Z}) \neq 0$. Show that if $f : M \to N$ is any continuous map, then $f_* : H_n(M;\mathbb{Z}) \to H_n(N;\mathbb{Z})$ is zero. (Hint: Show N has a covering space with group of covering transformations isomorphic to $\mathbb{Z}$.)

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

UNIVERSITY OF MARYLAND

Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem. If you do, it is your responsibility to clarify exactly which theorem you are using and to justify its use. In any part of a problem with multiple parts, you may assume the answer to any previous part, even if you have not proved it.
NOTE: On this exam not all the problems are equally weighted. Problem 5 is worth 20 points and problems 1-4 are each worth 10.

(1) Let (X, d_X) and (Y, d_Y) be metric spaces. A map $\pi : X \longrightarrow Y$ is called a *submetry* if for every $x \in X$, and any $r > 0$,
$$\pi\big(D(x,r)\big) = D(\pi(x), r)$$
where $D(x,r)$ denotes the closed r-ball about x.
  (a) Show that π is surjective if X is nonempty.
  (b) Show that π is continuous.
  (c) Show that π is open. [A map $f : A \longrightarrow B$ is *open* if and only if for every open subset $U \subset A$, the image $f(U)$ is open in B.]
  (d) Suppose that $y_1, y_2 \in Y$. Suppose that $x_1 \in X$ satisfies $\pi(x_1) = y_1$. Show that there exists $x_2 \in X$ such that $\pi(x_2) = y_2$ and $d_X(x_1, x_2) = d_Y(y_1, y_2)$.

(2) Let $F : \mathbb{R}^4 \longrightarrow \mathbb{R}$ be the quadratic function
$$F(x,y,z,t) = 4x^2 + 3y^2 + 3z^2 + t^2.$$
Let $f : S^3 \to \mathbb{R}$ be the restriction of F to the unit sphere $S^3 \subset \mathbb{R}^4$.
  (a) Let $\mathbb{RP}^3$ be real projective space and let $\pi : S^3 \longrightarrow \mathbb{RP}^3$ be the 2-fold covering map. Give $\mathbb{RP}^3$ the unique differentiable structure for which π is a local diffeomorphism.
  Prove that f descends to a smooth function $\bar{f}$ on $\mathbb{RP}^3$; that is, there exists a smooth function $\bar{f}$ on $\mathbb{RP}^3$ such that $\bar{f} \circ \pi = f$.
  (b) Find the critical points of $\bar{f}$.

Date: 22 January 2003.

(3) The picture on the following page illustrates the map $p : X \longrightarrow Y$ of adjunction spaces X, Y which we describe precisely as follows. For $n = 1, 2, 3, 4, 5$ let C_n denote the circle $\{(e^{i\theta}, n) \mid \theta \in \mathbb{R}\}$. Choose basepoints

$$\begin{aligned} a_1 &= (1,2) \in C_2 \\ b_1 &= (-1,2) \in C_2 \\ c_1 &= (1,1) \in C_1 \end{aligned}$$

and

$$\begin{aligned} a_2 &= (1,3) \in C_3 \\ b_2 &= (e^{2\pi i/3}, 3) \in C_3 \\ c_2 &= (e^{-2\pi i/3}, 3) \in C_3. \end{aligned}$$

Let X denote the identification space of $C_1 \coprod C_2 \coprod C_3$ under the equivalence relation defined by:

$$\begin{aligned} a_1 &\sim a_2, \\ b_1 &\sim b_2, \\ c_1 &\sim c_2. \end{aligned}$$

Let $a, b, c \in X$ be the corresponding images in X. Let Y denote the identification space of $C_4 \coprod C_5$ under the equivalence relation defined by $(1,4) \sim (1,5)$ and let $y \in Y$ be the common image of these points in Y.

There is a continuous map $p : X \longrightarrow Y$ defined as follows:

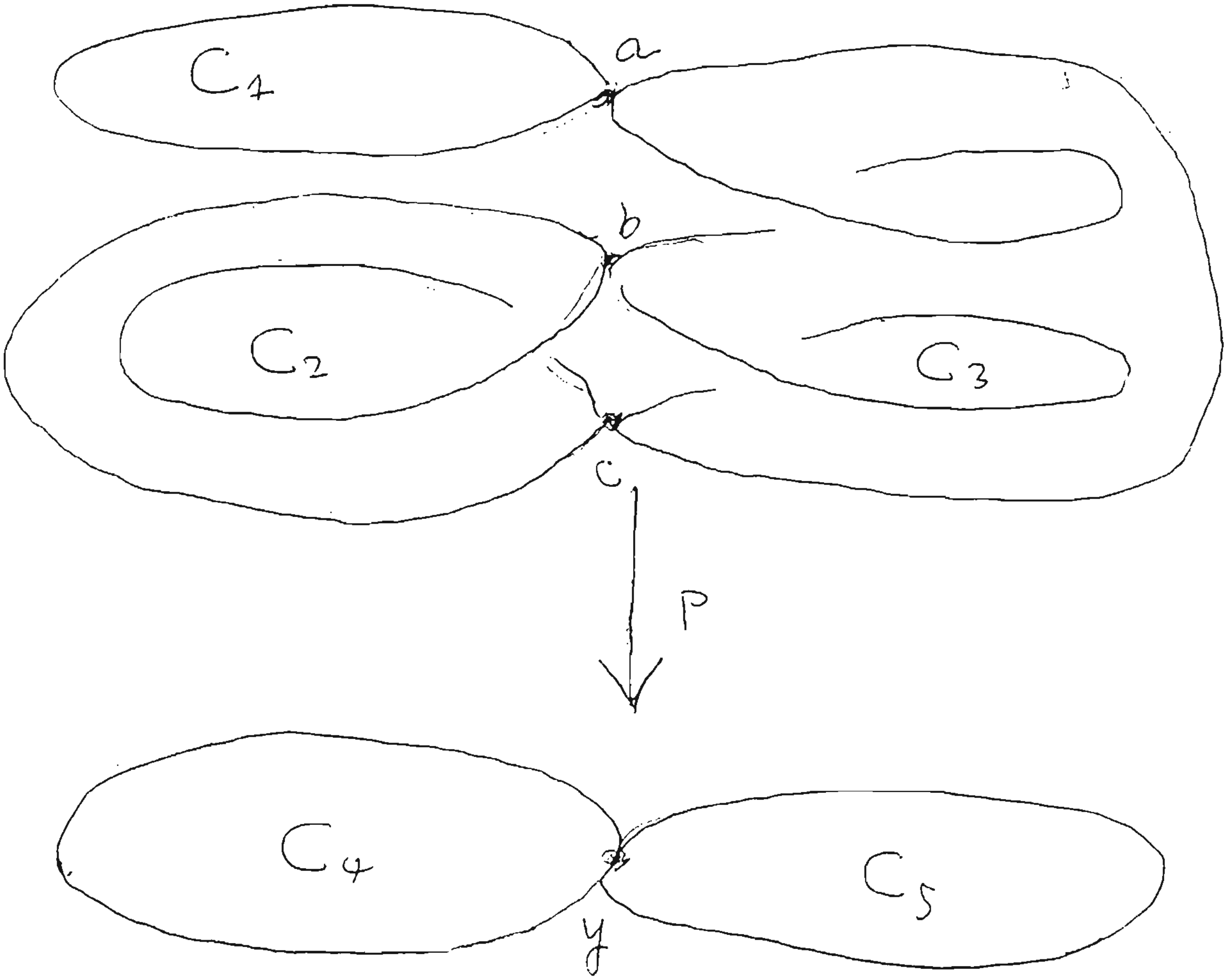
$$(\zeta, n) \longmapsto \begin{cases} (\zeta, 4) & \text{if } n = 1 \\ (\zeta^2, 4) & \text{if } n = 2 \\ (\zeta^3, 5) & \text{if } n = 3. \end{cases}$$

Informally, p maps the circle C_1 once around C_4 and C_2 twice around C_4. The circle C_4 is attached to C_5 at the point y, and p wraps C_3 three times around C_5. The points a, b, c comprise the inverse image $p^{-1}\{y\}$.

(a) Show that p is a covering space.

(b) Determine k such that X is homotopy equivalent to a wedge of k copies of S^1.

(c) Prove or disprove: *p is a regular covering space.*



(4) Let p, q be relatively prime integers. Consider the following CW-complex: X has one 0-cell x_0, two 1-cells labelled a and b, and two 2-cells labelled c, d. The boundary ∂c is attached to the 1-skeleton

$$X^1 = x_0 \cup a \cup b$$

by the map $a^p b^q$. That is, the attaching map for ∂c wraps p times around the a-circle and then q-times around the b-circle. The boundary ∂d is attached to X^1 by the map $aba^{-1}b^{-1}$, that is the map which wraps ∂d first around a, then around b, then around a in the opposite direction, and finally around b in the opposite direction.

(a) Compute the fundamental group and the integral homology groups of X.

(b) Show X is homotopy equivalent to S^2 with two points identified. [Hint: Think about $(p, q) = (1, 0)$.]

(5) In the following 20-point problem, any part may be used (even if you didn't prove it) in any later part. (χ denotes Euler characteristic. By definition a manifold is *closed* if it is compact and has empty boundary.)

(a) Suppose that M is a closed, connected, orientable odd-dimensional manifold. Show that $\chi(M) = 0$.

(b) Suppose X is a compact, connected, oriented n-manifold with or without boundary. Use Poincaré-Lefschetz duality to show $H_{n-1}(X, \mathbb{Z})$ is free abelian. (You may assume all homology groups are finitely generated abelian groups.)

(c) Let $n \geq 1$ be an integer. Show that there exists a connected, closed, orientable n-dimensional manifold M with $\chi(M) = 0$.

(d) If $M\#N$ denotes the orientable, connected sum of the closed, orientable n-manifolds M and N, show

$$\chi(M\#N) = \chi(M) + \chi(N) - (1 + (-1)^n).$$

(The *connected sum* of $M\#N$ is obtained by gluing together complements $M \backslash D^n_M$ and $N \backslash D^n_N$, where D^n_M and D^n_N are discs in M and N respectively, by an orientation-reversing homeomorphism $\partial D^n_M \approx \partial D^n_N$ of their boundaries.)

(e) Suppose there exists a closed, orientable n-dimensional manifold with $\chi(M)$ an odd integer greater than 1. Show that for any integer l there exists a connected, closed, orientable n-dimensional manifold W with $\chi(W) = l$.
[Hint: Try to find closed orientable manifolds of arbitrarily large even or odd Euler characteristic.]

(f) Suppose n is a positive integer divisible by 4 and m is an integer. Show there exists an closed, orientable n-dimensional manifold of Euler characteristic m.

(g) Suppose M is a closed orientable $2k$-dimensional manifold where k is an integer ≥ 1. Let F be a field of characteristic $\neq 2$. Use the fact that if A is a $m \times m$ skew-symmetric matrix with entries in F having nonzero determinant then m is even to show the following: *Any closed orientable $4n+2$-dimensional manifold has even Euler characteristic.*

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION
AUGUST 12, 2002

MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND

Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear exactly which theorem you are using and why its use is justified. In problems with multiple parts, you may assume the answer to any previous part, even if you have not proved it.

(1) Let $\mathbb{C}^*$ be the set of nonzero complex numbers. Let $f : \mathbb{C}^* \to \mathbb{C}^*$ be given by $f(z) = z^2$. Show that f is a 2-fold covering map.

(2) Recall the Theorem of "invariance of domain": *If A and B are homeomorphic subsets of $\mathbb{R}^n$ and A is open, then so is B.*
  (a) Use this to show that the sphere S^2 is not homeomorphic to a subset of the plane $\mathbb{R}^2$.
  (b) Show by example that invariance of domain need not hold if $\mathbb{R}^n$ is replaced by a closed interval.

(3) Let X be a ~~path-connected~~ topological space. Let $A, B \subset X$ be open subsets such that $X = A \cup B$. Suppose that A and B are each path-connected and simply connected.
  (a) Prove that X is path-connected. if $A \cap B \neq \emptyset$.
  (b) Assume $A \cap B$ is path-connected. Prove that X is simply connected.
  (c) Find an example where X is not simply connected.

(4) Let $T = S^1 \times S^1$ be the torus, let

$$M = \Big([0,1] \times [0,1]\Big) / \sim$$

be the Möbius band, where the equivalence relation is defined by:

$$(t, 0) \sim (1 - t, 1)$$

for $t \in [0, 1]$. Let $\mathbb{RP}^2$ be the projective plane.

(a) Let X be the space obtained by attaching the boundary of M to T via a homeomorphism with $S^1 \times \{x_0\}$ where $x_0 \in S^1$ is a point. Compute the homology groups $H_*(X; \mathbb{Z})$.

(b) Compute the homology groups $H_*(X \times \mathbb{RP}^2; \mathbb{Z})$.

(5) Consider a closed oriented 3-dimensional manifold M covered by S^3 where the group G of deck transformations is a group of order 120 which equals its commutator subgroup $[G, G]$ (the normal subgroup generated by $\{ghg^{-1}h^{-1} \mid g, h \in G\}$).

(a) Compute the integral homology groups of M. (Remark: This part is independent of the next two parts.)

(b) If N is any oriented closed 3 manifold and $d \equiv 0 \pmod{120}$ show that there is a map $f : N \to M$ of degree d.

(c) If in addition N is simply connected, show these are the only possible degrees d of maps $f : N \to M$.

(6) Let $f : S^2 \to S^2$ be a map of degree $k > 1$ and $h : S^3 \longrightarrow S^2$ be the Hopf map. Let X be the cell complex $e^0 \cup e^2 \cup e^4$ where the 2-cell e^2 is attached to the 0-cell e^0 by the constant map and the 4-cell e^4 is attached to the 2-skeleton $e^0 \cup e^2 \approx S^2$ by the Hopf map h. Let Y be the cell complex $e^0 \cup e^2 \cup e^3 \cup e^4$ where the 2-cell e^2 is attached by the constant map as for X, the 3-cell e^3 is attached to the 2-skeleton $e^0 \cup e^2 \approx S^2$ by the map f of degree k, and the 4-cell e^4 is attached by the constant map $\partial e^4 \longrightarrow e^0$.

(a) Compute the homology and cohomology of X and Y with integer coefficients and show that $H^*(X; \mathbb{Z}) \cong H^*(Y; \mathbb{Z})$ as rings.

(b) Show that $H^*(X; \mathbb{Z}/k)$ and $H^*(Y; \mathbb{Z}/k)$ are isomorphic as **groups** but not isomorphic as **rings**.

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND

Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear exactly which theorem you are using and why its use is justified. In problems with multiple parts, you may assume the answer to any previous part, even if you have not proved it.

As in Bredon, the symbol $\mathbb{Z}_k$ refers to the cyclic group of order k. For real numbers a, b the symbol (a, b) refers to the open interval between a and b.

(1) Let M be a compact, connected orientable smooth 6-dimensional manifold without boundary. Suppose its universal cover $p : M' \to M$ is a 7-fold cover. Suppose also the Euler characteristic of M is 5 and $H_3(M; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2$.
   (a) Compute $H_4(M; \mathbb{Z})$.
   (b) Compute $H^4(M; \mathbb{Z})$.

(2) Let the CW complex Y be obtained from the 2-sphere S^2 by attaching two 3-disks, one via a map of degree 6, and one via a map of degree 9.
   (a) Compute $H_*(Y; \mathbb{Z})$.
   (b) Compute $H_*(Y; \mathbb{Z}_3)$.
   (c) Compute $H^*(Y; \mathbb{Z}_2)$.

(3) Let $W = S^2 \vee S^4$ be the one point union of a 2-sphere and a 4-sphere. Let $f : S^4 \to W$ be inclusion.
   (a) Show that $f^* : H^4(W; \mathbb{Z}) \to H^4(S^4; \mathbb{Z})$ is an isomorphism.
   (b) If $\alpha, \beta \in H^2(W; \mathbb{Z})$ show that $\alpha \cup \beta = 0$.
   (c) Show that W and complex projective space $\mathbb{CP}^2$ are not homotopy equivalent even though they have isomorphic homology and cohomology groups. (You may use standard facts about $\mathbb{CP}^2$.)

Date: 18 January 2002.

(4) Let $\mathbb{L}$ be the set of all lines in the plane (not necessarily passing through the origin). Let $l_0 \in \mathbb{L}$ be a line. Define $U(l_0)$ to be the subset of $\mathbb{L}$ consisting of lines l which intersect l_0 in exactly one point. For $l \in U(l_0)$, let $\psi_{l_0}(l) = (p, \theta)$ where $p \in l_0$ is the point of intersection of l with l_0 and $\theta \in (0, \pi)$ is the angle at which l intersects l_0.

(a) Show that the collection of all (U_{l_0}, ψ_{l_0}) gives $\mathbb{L}$ the structure of a (topological) manifold.

(b) Show that this manifold is homeomorphic to the open Möbius band (the compact Möbius band with its boundary removed).

(5) Prove or disprove: The fundamental group of a metric space is commutative.

(6) Let X be a topological space and $X_1 \subset X_2 \subset \cdots \subset X_n \subset \dots$ a sequence of subsets, each with the subspace topology. Suppose that $X = \cup X_i$, and has the *weak topology* with respect to this union: $\forall O \subset X$, O is open $\Leftrightarrow O \cap X_i$ is open in X_i, $\forall i$.

Recall that a topological space S is $T_1 \Leftrightarrow \{p\}$ is a closed subset of S, for each $p \in S$. Suppose that each X_i is T_1.

(a) Let $S \subset X$. Suppose that for each i, the intersection $S \cap X_i$ is finite. Prove that S is closed.

(b) Suppose that each X_i is T_1. Suppose that K is *sequentially compact* (that is, every infinite sequence has a convergent subsequence). Prove that $K \subset X_n$ for some n.

TOPOLOGY QUALIFYING EXAM

AUGUST, 2001

(1) (**Math 730**)Compute the fundamental group of the open subset Ω of $\mathbb{R}^3$ obtained by removing the three coordinate axes.

(2) (**Math 730**)We suppose that all topological groups are semi-locally path-connected so that the theory of covering spaces applies.
   (a) Show that a discrete normal subgroup N of a connected topological group G lies in the center of G.
   (b) Let G, H be connected n-dimensional topological groups and $f : G \longrightarrow H$ a homomorphism which is a covering space. Show that the kernel of f is abelian.
   (c) Show that the fundamental group of a connected topological group must be abelian.

(3) (**Math 730**)
   (a) Let X, Y be topological spaces. Suppose that X is compact and Y is Hausdorff. Let $f : X \longrightarrow Y$ be continuous and bijective. Prove that f is a homeomorphism.
   (b) Find a counterexample when X is only assumed to be locally compact.

(4) (Math 734)

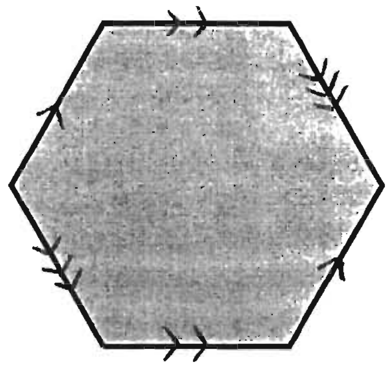
Let $f : X \to \mathbb{CP}^n$ be a continuous map from a CW complex X to complex projective n-space. Suppose that the map $f_* : H_{2n}(X;\mathbb{Z}) \to H_{2n}(\mathbb{CP}^n;\mathbb{Z})$ is nonzero.

(a) If β is the generator of $H^{2n}(\mathbb{CP}^n;\mathbb{Z})$, show that $f^*(\beta) \neq 0$.

(b) Show that $H^2(X;\mathbb{Z})$ and $H_2(X;\mathbb{Z})$ are both nontrivial groups. (Hint: for the last part tensor with $\mathbb{Q}$ to obtain that $f^*\beta$ is not torsion.)

(5) (Math 734)

Let X be the topological space obtained by taking a solid regular hexagon and identifying the opposite edges by parallel translation.



Calculate the integral homology of X.

(6) (Math 734)

Recall that the connected sum $M\#N$ of two oriented n-manifolds is obtained by removing an open n-disk from each and identifying the boundaries of the disks by an orientation-preserving homeomorphism.

(a) Express the Euler characteristic $\chi(M\#N)$ in terms of $\chi(M)$ and $\chi(N)$.

(b) Suppose that $n > 2$. Express the fundamental group $\pi_1(M\#N)$ in terms of $\pi_1(M)$ and $\pi_1(N)$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAM

JANUARY 2001

GEOMETRY/TOPOLOGY (Ph.D. Version)

Instructions

a. Answer all six questions.

b. Each question will be assigned a grade from 0 to 10. If some problems have multiple parts, be sure to go on to subsequent parts even if there is a part you cannot do.

c. Use a different set of sheets for each question. Write the problem number and your **code number** (not your name) on the outside sheets.

d. Keep scratch work on separate pages or on separate set of sheets.

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION
JANUARY, 2001

UNIVERSITY OF MARYLAND MATHEMATICS DEPARTMENT

Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear exactly which theorem you are using and why its use is justified. In problems with multiple parts, you may assume the answer to any previous part, even if you have not proved it.

1. (a) Let (M, d) be a metric space. Define the topology on M induced by the metric. Prove that this topology is Hausdorff.
   (b) If (M, d) is a metric space, show that the distance function $d : M \times M \longrightarrow \mathbb{R}$ is continuous with respect to the product topology on $M \times M$.
   (c) Suppose that M is a smooth manifold with a metric (M, d) compatible with the manifold topology. Find an example for which $d : M \times M \longrightarrow \mathbb{R}$ is *not* smooth.

2. (a) Let X be a topological space whose fundamental group $\pi_1(X)$ is abelian, and let $\pi : Y \longrightarrow X$ be a covering space. Show that π is regular.
   (b) Give an example of a covering which is not regular but for which $\pi_1(X)$ is finite.

3. Let X be a compact space and $f : X \rightarrow Y$ be a continuous map to a Hausdorff space Y. Show that the image $f(X)$ is homeomorphic to a quotient space of X.

4. Let $X = S^1 \vee S^1$, so that $\pi_1(X)$ is free group on two generators, say a and b. Let $f : S^1 \to X$ represent the element $a^{-1}aa^{-1}b^2a$. (We can write this out as a map on [0,1] subdivided into 6 subintervals.) Let $Z = X \bigcup_f D^2$.
   (a) Show that $\chi(Z)$, the Euler characteristic, is 0.
   (b) Compute the fundamental group of Z.
   (c) Determine $H_*(Z)$ as an abelian group.
   (d) Determine $H^*(Z; \mathbb{Z}/2)$ as a ring.

5. (a) Let $g : M \to N$ be a degree one map of n-manifolds. Show that $g_\# : \pi_1(M) \to \pi_1(N)$ is onto.
   (b) Show that the homology groups of a closed connected orientable 3-manifold M are determined by $\pi_1(M)$.

6. Use cup products to compute the degree of the map $\mathbb{CP}^n \to \mathbb{CP}^n$ which raises each coordinate to the d-th power. (Try this first for $n = 1$).

Department of Mathematics
University of Maryland
Written Graduate Qualifying Exam
Topology
August, 2000

Instructions

1. Answer all six questions. Each one will be assigned a grade from 0 to 10. In problems with multiple parts, the parts are graded independently of one another. Be sure to go on to subsequent parts even if there is some part you cannot do. You may assume the answer to any part in subsequent parts of the same problem.

2. Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear exactly which theorem you are using and why its use is justified.

1. Consider the following topological spaces:

a) the subset X of $\mathbb{R}^2$ consisting of all rays $\{(x, x/n) : x > 0\}$, as n runs over the positive integers, with the subspace topology from $\mathbb{R}^2$.

b) the subset Y of $\mathbb{R}^2$ defined by

$$\{(0,0)\} \cup \{(x,y) : -1 < x < 1, y > 0\},$$

with the subspace topology from $\mathbb{R}^2$.

c) the quotient space $Z = W/\sim$ obtained from the subspace

$$W = \{(n,y) : n \in \mathbb{Z}, y \in \mathbb{R}\}$$

of $\mathbb{R}^2$ (with the subspace topology from $\mathbb{R}^2$), where $(n,0) \sim (0,0)$ for all $n \in \mathbb{Z}$ (and there are no other identifications). Note that Z is to be given the quotient topology.

d) the quotient space $Q = \mathbb{R}/\sim$, where $x \sim y$ if $x-y$ is of the form $n + m\sqrt{2}$, $n, m \in \mathbb{Z}$. Note that Q is to be given the quotient topology.

Which of these spaces are locally compact? Which are Hausdorff? Which are metrizable? Give explicit reasons for your answers.

2. Let M^m be a smooth connected m-manifold (without boundary), whose fundamental group is finite of odd order.

a) If $m < n$, show that any continuous map $f\colon M \to \mathbb{RP}^n$ is null-homotopic (homotopic to a constant map).

b) If M is compact, $m = n$, and n is odd, show that there exists a continuous map $f\colon M \to \mathbb{RP}^n$ which is *not* null-homotopic. (This is also true if n is even, though you don't have to deal with this case.)

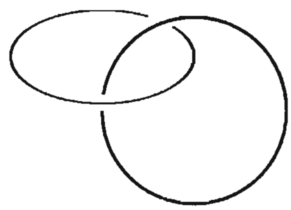
3. Let M^m and N^n be *disjoint* oriented compact connected smooth submanifolds of $\mathbb{R}^{k+1}$, with $\dim M + \dim N = m + n = k$. Define $\lambda: M \times N \to S^k$ by

$$\lambda(x,y) = \frac{x-y}{|x-y|}, \quad x \in M,\ y \in N.$$

Let $\operatorname{Lk}(M,N) = \deg \lambda$ (called the *linking number* of M and N in $\mathbb{R}^{k+1}$.

i) If $M = \partial W$, where W is a compact oriented manifold with boundary in $\mathbb{R}^{k+1}$, and $W \cap N = \emptyset$, show that $\operatorname{Lk}(M,N) = 0$.

ii) Compute (up to sign) $\operatorname{Lk}(S^1, S^1)$ for the following link in $\mathbb{R}^3$:



4. Recall that for K^n and L^n, smooth connected n-manifolds without boundary, we can form a new n-manifold, denoted $K\#L$, called the **connected sum** of L and N, by taking smooth embeddings $f: \mathbb{R}^n \to K$ and $g: \mathbb{R}^n \to L$ and gluing $K \setminus f(0)$ to $L \setminus g(0)$ by identifying $f(tu)$ with $g(t^{-1}u)$ for u in the unit sphere S^{n-1} and $t \in (0,\infty)$. Let $M = \mathbb{RP}^4 \# \mathbb{CP}^2$.

(a) Compute the fundamental group, $\pi_1(M)$ and the homology groups $H_*(M)$ of M. You may use the fact that

$$H_q(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}, & q = 0, \text{ and if } n \text{ is odd also } q = n, \\ \mathbb{Z}_2, & q \text{ odd},\ 1 \le q \le n-1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Is M orientable?

(c) Compute the cohomology **groups** (not the ring structure) of M.

5. Let $f : S^{2n-1} \to S^n$ be defined as follows: Consider S^{2n-1} to be the boundary of $D^{2n} = D^n \times D^n$ where D^n is the n-disk. If $c : D^n \to S^n$ is the map that collapses the boundary to the base-point $*$, then $c \times c : D^{2n} \to S^n \times S^n$ carries S^{2n-1} to the one-point union $S^n \vee S^n$. Then f is the composition of this with the folding map $S^n \vee S^n \to S^n$. Let $X = S^n \cup_f D^{2n}$.

(a) Show that X can also be identified as $S^n \times S^n/\sim$, where $(x,*) \sim (*,x)$.

(b) For $n \ge 2$, calculate the integral cohomology **ring** of X. (Hint: use the map $S^n \times S^n \to X$ from (a).)

(c) Show that X is not homotopy equivalent to a closed manifold for any $n \ge 2$.

6. Recall that $H^*(\mathbb{CP}^n) = \mathbb{Z}[a]/(a^{n+1})$, for $a \in H^2(\mathbb{CP}^n)$.

(a) Determine the cohomology **ring** of $\mathbb{CP}^2 \times \mathbb{CP}^2$.

(b) Show that any homotopy equivalence $f : \mathbb{CP}^2 \to \mathbb{CP}^2$ is orientation preserving.

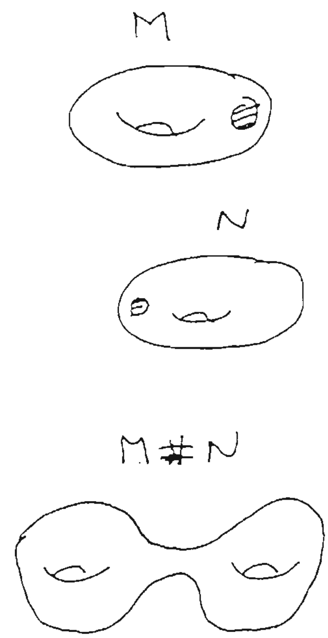
TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION
JANUARY, 2000

Answer all six questions. Each one will be assigned a grade from 0 to 10. In problems with multiple parts, the parts are graded independently of one another. Be sure to go on to subsequent parts even if there is some part you cannot do. You may assume the answer to any part in subsequent parts of the same problem. Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear exactly which theorem you are using and why its use is justified.

1. (730) Let A and B be closed subsets of a topological space X, so that $X = A \cup B$.
   (a) If A and B are compact, is X necessarily compact? If so, prove it. If not, give a counterexample and prove it under the simplest additional hypothesis you can think of which guarantees that X is compact.
   (b) If A and B are connected, is X necessarily connected? If so, prove it. If not, give a counterexample and prove it under the simplest additional hypothesis you can think of which guarantees that X is connected.
   (c) If A and B are Hausdorff, is X necessarily Hausdorff? If so, prove it. If not, give a counterexample and prove it under the simplest additional hypothesis you can think of which guarantees that X is Hausdorff.

2. (730) Let M be a connected smooth manifold (without boundary) and let $f\colon M \to \mathbb{R}$ be a continuous function such that $f^{-1}(-1)$ and $f^{-1}(1)$ are both nonempty.
   (a) Show that $f^{-1}(0)$ separates M, i.e., that $M - f^{-1}(0)$ is not connected.
   (b) For this part assume in adddition that M is compact. (This is not really necessary but avoids one technical problem.) For any neighborhood U of $f^{-1}(0)$, show that there is a smooth closed submanifold N of M, of codimension 1, such that $N \subset U$ and N separates M.

3. (730) Let M^n and N^n be smooth connected n-manifolds without boundary. Form a new n-manifold, denoted $M\#N$, called the connected sum of M and N, by taking smooth embeddings $f\colon \mathbb{R}^n \to M$ and $g\colon \mathbb{R}^n \to N$ and gluing $M - f(0)$ to $N - g(0)$ by identifying $f(tu)$ with $g(t^{-1}u)$ for u in the unit sphere S^{n-1} and $t \in (0, \infty)$. (See picture.) It turns out that the diffeomorphism type of $M\#N$ may depend on the choice of embeddings if M and N are both orientable.
   (a) Show that if $n \geq 3$, then $\pi_1(M\#N)$ (relative to a basepoint located on $f(S^{n-1})$) can be naturally identified with the free product of $\pi_1(M)$ and $\pi_1(N)$ (taken relative to the corresponding basepoints of M and N).
   (b) Show that the result of (a) is false if $n = 1$. Where does your proof of (a) break down when $n = 1$?
   (c) Show that the result of (a) is false if $n = 2$. Where does your proof of (a) break down when $n = 2$?



4. (734) In parts (a) and (b), let m and n be positive integers.
   (a) Show that there exists a map of degree m from the n-sphere to itself.
   (b) Show that there exists a finite CW-complex X such that

$$\widetilde{H}_q(X) = \begin{cases} \mathbb{Z}_m, & q = n \\ 0 & q \neq n \end{cases}.$$

   (c) Let $\{A_n\}_{n\geq 1}$ be a sequence of finitely generated abelian groups. Show that there is a space X such that $H_n(X) = A_n$ for all $n \geq 1$.

5. (734) Let X be a finite CW-complex, and F_p the field of p elements for some prime p.
   (a) Show that $\chi(X) = \sum_{n\geq 0}(-1)^n \dim_{F_p} H_n(X; F_p)$.
   (b) Let X be an odd dimensional closed manifold (compact without boundary), not necessarily orientable. Show $\chi(X) = 0$.
   (c) Show that any non-orientable 3-manifold has an infinite fundamental group.

6. (734) Let p be the projection from $\mathbb{RP}^n$ to $\mathbb{RP}^n/\mathbb{RP}^k$, $k < n$.
   (a) Show $p^* : H^*(\mathbb{RP}^n/\mathbb{RP}^k; \mathbb{Z}_2) \to H^*(\mathbb{RP}^n; \mathbb{Z}_2)$ is a monomorphism.
   (b) Describe the ring structure of $H^*(\mathbb{RP}^n/\mathbb{RP}^k; \mathbb{Z}_2)$.
   (c) Assuming that $m \geq 2k + 2 > n$, show that $\mathbb{RP}^n/\mathbb{RP}^k$ is not a retract of $\mathbb{RP}^m/\mathbb{RP}^k$.

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION
AUGUST 1999

MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND, COLLEGE PARK

Instructions: You may use any standard theorem, providing you first quote it precisely, and then explain how it applies in the particular situation. In any part of multipart problem, you may assume the result of a previous part, even if you have not proved it.

1. (730) Suppose that X is a normal Hausdorff space, and a finite group G acts freely on X (i.e., if $x \in X$ and $g \in G$, $g \neq 1$, then $gx \neq x$).
   (a) Show that the quotient space X/G is Hausdorff and that the quotient map $q : X \to X/G$ is a covering map.
   (b) Give an example to show that both conclusions of (a) may fail is G is infinite, but the hypotheses are otherwise the same.
2. (730) For $t \geq 0$ let M_t be the space of all real 2×2 matrices of determinant 1 and trace t. Show that M_t is a manifold if $t < 2$. Can you argue analogously when $t = 2$? Explain your answer.
3. (730) Let X be a connected locally contractible Hausdorff space (for example a smooth manifold) and denote by $[X, S^1]$ the set of homotopy classes of continuous maps from X to the circle. You may assume the following standard fact: $[X, S^1]$ has a natural abelian group structure coming from pointwise multiplication of maps, when we identify S^1 with the set of complex numbers of modulus 1 (which is a topological group under multiplication).
   (a) Fix a basepoint x_0 in X. Show that there is a natural injective homomorphism Φ from $[X, S^1]$ to $\operatorname{Hom}(\pi_1(X, x_0), \mathbb{Z})$ that sends the homotopy class of a map $\phi : X \to S^1$ to
   $$\phi_* : \pi_1(X, x_0) \to \pi_1(S^1, \phi(x_0)) \cong \mathbb{Z}.$$
   N.B. You need to first show that Φ is well defined and a homomorphism, and then show that it is injective.
   (b) Suppose that X is an identification space
   $$X = \overbrace{\left(S^1 \vee S^1 \vee \cdots \vee S^1\right)}^{k} \cup_f D^2,$$

in other words the disjoint union of a one-point union of k circles (a "k-petal flower") and D^2, with $\partial D^2 \cong S^1$ identified to a subset of $S^1 \vee S^1 \vee \cdots \vee S^1$ via a map $f: S^1 \to S^1 \vee S^1 \vee \cdots \vee S^1$. Prove that Φ is an isomorphism. (The result is actually true much more generally, but this is the crucial case.)

4. (734) Let Y be a 3-sphere with a 4-cell attached by a map of degree 6, and let $X = \mathbb{R}P^3 \times Y$. Calculate the homology and cohomology groups of X.
5. (734)Let X be a CW-complex with one 0-cell, three 1-cells, two 2-cells, and no other cells. Assume that X is homotopy equivalent to a compact orientable 3-manifold M.
   (a) Show that $\partial M \neq \emptyset$.
   (b) Show that if ∂M is connected, it is homeomorphic to a torus.
   (c) Show that in general either every component of ∂M is homeomorphic to a torus, or at least one component is homeomorphic to a sphere.
6. (734) Recall that the connected sum $M_1 \# M_2$ of two closed n-manifolds is gotten by removing open n-disks D_1, D_2 from M_1, M_2 respectively, taking the union and identifying ∂D_1 with ∂D_2.
   (a) Let M be a connected orientable n-manifold, and D an open n-disk (whose closure is a closed n-disk). Prove that $H^t(M - D) \cong H^t(M)$ for $t \neq n$, and $H^n(M - D) = 0$.
   (b) Let M_1, M_2 be connected orientable closed n-manifolds. Prove that the cohomology ring of $M_1 \# M_2$ is exactly the quotient of $H^*(M_1) \oplus H^*(M_2)$ by the identification of the two multiplicative identities in H^0, and the identification of the two n-dimensional generators up to a sign (depending on the choice of identification of ∂D_1 with ∂D_2).
   (c) Calculate the cohomology ring of $\mathbb{C}P^2 \# \mathbb{C}P^2$.
   (d) Show that $\mathbb{C}P^2 \# \mathbb{C}P^2$ and $S^2 \times S^2$ cannot be homotopy equivalent.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION

JANUARY 1999

TOPOLOGY (Ph.D. Version)

Instructions

a. Answer all six questions.

b. Each question will be assigned a grade from 0 to 10. If some problems have multiple parts, be sure to go on to subsequent parts even if there is a part you cannot do.

C. Use a different set of sheets for each question. Write the problem number and your **code number (not** your name) on the outside sheets.

D. Keep scratch work on separate pages or on separate set of sheets.

Geometry/Topology Ph.D Written Examination: January 1999

1. [730] Let A be an $n \times n$ matrix all of whose entries are real and positive. Show that A has a positive real eigenvalue λ, and that A has an eigenvector $v \in \mathbb{R}^n$ (for this eigenvalue λ) all of whose coordinates v_j are positive. (*Hint:* Use Brouwer's fixed point theorem applied to a self-map induced by A of a suitable compact subset of $\mathbb{R}^n$.)

2. [730]Let U be an open subets of $\mathbb{R}^n$, and let $f : U \to \mathbb{R}$ be a smooth map. Prove that for every $\varepsilon > 0$, there exists $c \in \mathbb{R}^n$ with $||c|| < \varepsilon$, such that $x \to f(x) + \langle x, c \rangle$ has *isolated* critical points. (*Hint:* Use Sard's Theorem. Here $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product and $|| \cdot ||$ the Euclidean norm on $\mathbb{R}^n$.)

3. [730] Compute the fundamental group $\pi_1(U)$ of the open set U of $\mathbb{R}^3$ obtained by removing the three coordinate axes.

4. [734] Let M^{2k} be a compact oriented manifold of dimension $2k$. For $a, b \in H^k(M, \mathbb{R})$ define $< a, b > = < a \cup b, [M] > \in \mathbb{R}$ where $[M]$ is the fundamental class of M.

   (a) Show that $< \, . \, >$ satisfies

   (1) $< a, b > = (-1)^k < b, a >$.

   (2) If $a \neq 0$ then there exists b such that $< a, b > \neq 0$.

   (b) Let M^{4k} be the connected sum $\mathbb{CP}^{2k} \# (S^{2k} \times S^{2k})$. Compute $< e_i, e_j >$ for an appropriate basis of $H^{2k}(M)$. You may use the fact that the cohomology ring of $\mathbb{CP}^n$ is an integral polynomial ring on one generator α of degree 2 with the relation $\alpha^{n+1} = 0$.

5. [734] For any map $f : S^k \to X$, let $C_f = X \bigcup_f D^{k+1}$ be the space obtained by attaching a $k+1$-cell to X via the map f.

   Let $f : S^{2n-1} \to S^n$ be any map.

   (a) Calculate the cohomology of C_f and show that $H^n(C_f)$ and $H^{2n}(C_f)$ are infinite cyclic.

   Call their generators a_1 and a_2, respectively, and define $H(f) \in \mathbb{Z}$ by the equation $a_1 \cup a_1 = H(f) a_2$.

   (b) Show that if $f \sim g$ are homotopic maps, then $H(f) = \pm H(g)$.

   (c) Show that if n is odd, $H(f) = 0$ for all f.

   (d) Show that if $n = 2$, there exists a map $f : S^3 \to S^2$ with $H(f) = \pm 1$.

6. [734] Let X be a finite CW-complex such that $H^i(X, \mathbb{Z})$ is $\mathbb{Z}$ for $i = 0, 2n$, is the direct sum $\mathbb{Z} + \mathbb{Z}_6$ for $i = n$, and is 0 otherwise. Calculate

   (a) $H_*(X; \mathbb{Z})$;

   (b) $H^*(X; \mathbb{Z}_2)$;

   (c) the Euler characteristic of X.

GEOMETRY/TOPOLOGY PH.D.
QUALIFYING EXAM - AUGUST, 1998

(1) (i) Let X and Y be manifolds, and let $f\colon X \to Y$ be a map transversal to a submanifold Z in Y. Then $W = f^{-1}(Z)$ is a submanifold of X. Prove that $T_x(W)$ is the preimage of $T_{f(x)}(Z)$ under the linear map $df_x\colon T_x(X) \to T_{f(x)}(Y)$.
(ii) If X and Z are transversal submanifolds of Y and $y \in X \cap Z$, prove that $T_y(X \cap Z) = T_y(X) \cap T_y(Z)$.

(2) Let X be a compact space and $f\colon X \to Y$ a continuous map to a Hausdorff space Y. Show that the image $f(X)$ is homeomorphic to a quotient space of X.

(3) Show that there does not exist a smooth map $f\colon S^3 \to S^1 \times S^2$ such that $df_x\colon TS^3_x \to T(S^1 \times S^2)_{f(x)}$ is an isomorphism for every $x \in S^3$.

(4) (a) Show there exists no map of $\mathbb{C}P^n \to \mathbb{C}P^n$ of degree -1 if n is even.
(b) Let $r : \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}^{n+1} - \{0\}$ be the map $r(z_0, z_1, \ldots, z_n) = (-z_0, z_1, \ldots, z_n)$. Then r induces a map $r_1 : \mathbb{C}P^n \to \mathbb{C}P^n$. What is the degree of r_1?

(5) Let X be a CW-complex with one 0-cell, one 1-cell, and two 2-cells, one attached by a map S^1 to S^1 of degree 4 and one by a map S^1 to S^1 of degree 2, and one 4-cell.
(a) What is the Euler characteristic of X?
(b) Determine $H_*(X, \mathbb{Z})$ and $H^*(X, \mathbb{Z}/2)$
(c) Can X have the homotopy type of a 4-manifold?

(6) Consider $\mathbb{R}P^1 \subseteq \mathbb{R}P^6$. Calculate the cohomology ring $H^*(\mathbb{R}P^6/\mathbb{R}P^1; \mathbb{Z}/2)$.

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION
JANUARY 1998

MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND, COLLEGE PARK

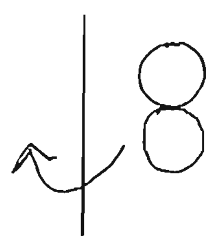
Begin each problem on a separate answer sheet. Write the problem number and your code number (NOT your name) on the answer sheets.

Justify your answers with clear, grammatical prose. Provide careful statements of any theorems you invoke. No credit will be given for arguments that do not directly lead to a solution.

1. Let M and N be smooth manifolds, let $f\colon M \to N$ be a smooth map, and let Γ be the graph of f.
   (a) Show that Γ is a smooth submanifold.
   (b) Using the natural identification of $T(M \times N)_{(x,y)}$ with $TM_x \times TN_y$, show that
   $$T\Gamma_{(x,f(x))} \subset TM_x \times TN_{f(x)}$$
   is the graph of df_x.

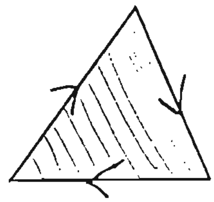
2. Let A be a closed subset of a normal topological space X, and let $f\colon A \to S^n \subset \mathbb{R}^{n+1}$ be a continuous map.
   (a) Show that there exists an open neighborhood $U \supset A$ and a continuous extension of f to a map $U \to S^n$.
   (b) Illustrate with an example that, without further restrictions, f cannot always be extended to a continuous map $X \to S^n$.
   (c) Show that if f is not onto, there exists a continuous extension to a map $X \to S^n$.

3. Let X be the subset of $\mathbb{R}^3$ obtained by rotating the union of two tangent circles in a plane around a disjoint axis in the same plane parallel to the line joining the centers of the two circles:



   Compute $\pi_1(X)$.

4. Consider the topological space X which is the quotient of a 2-simplex T (i.e., a filled triangle) by identifying the edges according to the pattern



Equivalently, X is the 2-complex obtained by attaching a 2-cell to the circle by a map of degree 3.
   (a) Compute the homology groups with $\mathbb{Z}$ coefficients and the cohomology groups with $\mathbb{Z}_3$ coefficients.
   (b) Compute the cup product $H^1(X;\mathbb{Z}_3) \times H^1(X;\mathbb{Z}_3) \to H^2(X;\mathbb{Z}_3)$.

5. Suppose that M is a compact 5-manifold with $H_0M = \mathbb{Z}$, $H_1M = \mathbb{Z}_3$, and $H_2M = \mathbb{Z}$.
   (a) Show that M is orientable.
   (b) Calculate the rest of the homology and cohomology with $\mathbb{Z}$ coefficients.

6. (a) Show that any compact 3-manifold (orientable or not) has Euler characteristic equal to zero.
   (b) Deduce that a connected, compact, non-orientable 3-manifold has an infinite fundamental group.

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION
AUGUST 1997

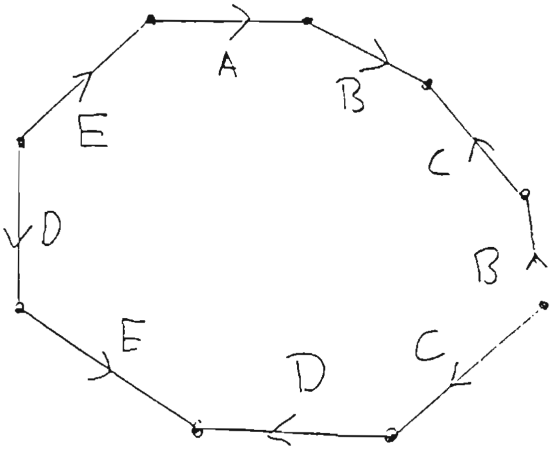
MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND, COLLEGE PARK

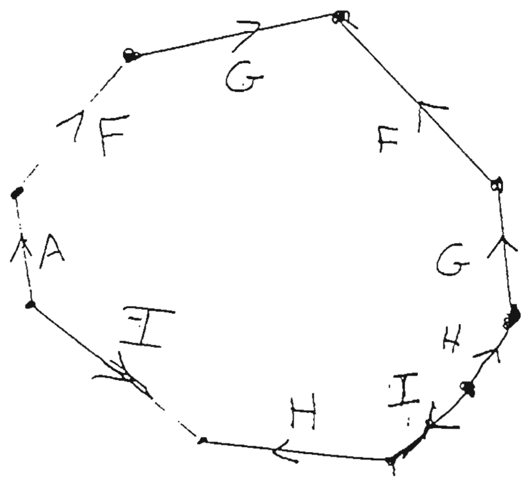
Your solutions will be evaluated as mathematical prose. Use clear and correct mathematical English. Graders only see what you write on the page. They cannot be expected to guess what you thought but could not communicate.

1. Let G be a topological group and let $\Omega(G)$ denote the space (given the compact-open topology) of all continuous maps $(S^1, s_0) \longrightarrow (G, e)$ where $s_0 \in S^1$ is a basepoint and $e \in G$ is the identity element. If $\alpha, \beta \in \Omega(G)$, define their *product* $\alpha \bar{\star} \beta \in \Omega(G)$ by
$$\alpha \bar{\star} \beta(s) = \alpha(s)\beta(s)$$
   (a) Show that $\bar{\star}$ induces ~~defines~~ a map
$$\star : \pi_1(G, e) \times \pi_1(G, e) \longrightarrow \pi_1(G, e)$$
   (b) Show that if $a, b \in \pi_1(G, e)$, then $a \star b$ equals the usual product ab in $\pi_1(G, e)$.
   (c) Show that $a \star b = b \star a$ and $\pi_1(G, e)$ is commutative.
2. Let $p : X \longrightarrow Y$ be a covering space and let Z be an topological arcwise connected space. Define
$$q : X \times Z \longrightarrow Y \times Z$$
$$(x, z) \longmapsto (p(x), z)$$
   (a) Show that q is a covering space.
   (b) If p is a regular covering space and G its group of covering transformations, show that q is a regular covering space. Compute the group of covering transformations of q.
3. (a) Construct a smooth map $S^1 \longrightarrow S^1$ with exactly one critical point.
   (b) Let $f : S^1 \longrightarrow \mathbb{R}$ be a smooth map with finitely many critical points, all of which are local maxima or minima. Show that f has exactly as many local maxima as local minima.

Date: July 15, 1997.

4. Let A be the cyclic group of order 45 and B the cyclic group of order 21, considered as $\mathbb{Z}$-modules. Write each of the following $\mathbb{Z}$-modules as a direct sum of cyclic modules:
   - $A \otimes B$,
   - $B \otimes A$,
   - $\mathrm{Hom(A, B)}$,
   - $\mathrm{Hom(B, A)}$,
   - $\mathrm{Ext(A, B)}$,
   - $\mathrm{Ext(B, A)}$,
   - $\mathrm{Tor(A, B)}$,
   - $\mathrm{Tor(B, A)}$.
5. Let $f : S^1 \longrightarrow S^2$ be the usual embedding and let $X = S^2/f(S^1)$ be the quotient space of S^2 with $f(S^1)$ collapsed to a point. Compute the homology of X with coefficients in $\mathbb{Z}/2$. Is X homotopy-equivalent to a compact manifold?
6. Consider the identification space of the two nonagons pictured below:
   (a) Prove that X is a topological manifold.
   (b) Prove that X is orientable.
   (c) Compute the genus of X.





a manifold and the quotient map $X \longrightarrow X/G$ is a covering space.)

5. Let K be a connected 1-complex. Suppose that M is a compact manifold (with empty boundary) homotopy-equivalent to K. Show that M is homeomorphic to S^1.
6. Let M and N be n-dimensional compact connected orientable manifolds with fundamental classes $\mu_M \in H_n(M, \mathbb{Z})$ and $\mu_N \in H_n(N, \mathbb{Z})$ respectively. A map $f : M \longrightarrow N$ has *degree* k if and only if $f_* \mu_M = k \cdot \mu_N$.
   (a) Suppose f has degree one. Show that the induced homomorphism $f^* : H^*(N) \longrightarrow H^*(M)$ is a *split monomorphism*, that is, there exists a homomorphism $\rho : H^*(M) \longrightarrow H^*(N)$ such that the composition $\rho \circ f^*$
   $$H^*(N) \xrightarrow{f^*} H^*(M) \xrightarrow{\rho} H^*(N)$$
   is the identity $H^*(N) \longrightarrow H^*(N)$.
   (b) Suppose that $\pi_1(M)$ is finite and $H^1(N; \mathbb{Z}) \neq 0$. Show that any map $f : M \longrightarrow N$ has degree zero.

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION
AUGUST 1996

MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND, COLLEGE PARK

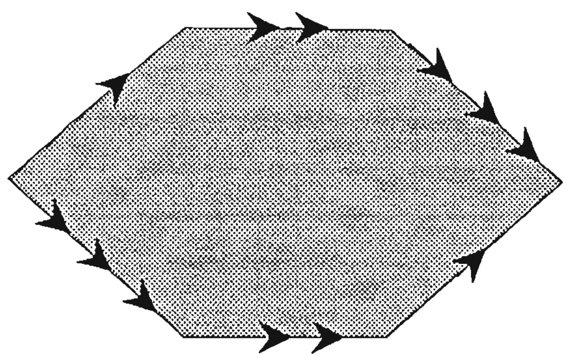
Do ANY six problems. Only the first six problems which you do will be graded, so there is no point in turning in more than six problems. This test is designed so that anyone who has covered the syllabi for two of the basic courses (730,734,740,742) should be able to do six problems. The indicated course numbers are included only as a guide and are not intended to restrict your choice of problems (you may safely ignore them if you wish).

Your answers will be graded as mathematical prose. We expect grammatically correct sentences and clear exposition.

1. **(730)** Let (X, d) be a complete metric space. Suppose $A, B \subset X$ are compact subsets.
   (a) Prove that, for every $a \in A$, the minimum $\min_{b \in B} d(a, b)$ exists (i.e., there is a point $b_a \in B$ such that $d(a, b_a) = \min_{b \in B} d(a, b)$). Show that the maximum
   $$h(A, B) = \max_{a \in A} \min_{b \in B} d(a, b)$$
   also exists.
   (b) Show that $h(A, B) = 0$ if and only if $A \subseteq B$.
   (c) Let $\mathcal{K}(X)$ denote the set of all nonempty compact subsets of X. If $A, B \in \mathcal{K}(X)$, define
   $$D(A, B) = \max(h(A, B), h(B, A))$$
   One can show (but we don't ask you to do this) that $(\mathcal{K}(X), D)$ is a complete metric space.
   Let $f : X \longrightarrow \mathcal{K}(X)$ be the map
   $$f(x) = \{x\}$$
   Show that f is an isometric embedding of metric spaces, i.e., $D(f(x), f(y)) = d(x, y)$.
   (d) Let X_n be the subset of $[0, 1]$ consisting of rational numbers whose denominator divides 2^n. Show that X_n is a Cauchy sequence in $\mathcal{K}([0, 1])$, and compute its limit.
2. **(730)**
   (a) Give an example of a surjective continuous map $\mathbb{R} \longrightarrow S^1$ which is a local homeomorphism but not a covering map.
   (b) Let $C = \mathbb{R} \times S^1$ be the cylinder. Find an example of a surjective local homeomorphism $f : X \longrightarrow C$, where X is a surface, for which the induced map
   $$\pi_1(f) : \pi_1(X) \longrightarrow \pi_1(C)$$
   is not injective.
   (c) Show that a contractible topological space is path-connected.

3. (730)
   (a) Suppose X is a compact Hausdorff space and $x \in X$. Show that the one-point compactification of the complement $X - \{x\}$ is homeomorphic to X.
   (b) Use the previous part to show the one-point compactification of the Euclidean space $\mathbb{R}^n$ is homeomorphic to the n-sphere
   $$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$
   (c) Suppose X is homeomorphic to a disjoint union of k circles, and that the cone on X is homeomorphic to a manifold with boundary. Determine k.
4. (734) Let (X, x) and (Y, y) be pointed connected topological spaces (i.e., each is a connected space together with a base point). We define the wedge $X \vee Y$ to be the topological space obtained from the disjoint union $X \amalg Y$ by identifying x and y to a single point z. We choose the resulting point z as the base point for $X \vee Y$.

   Assume now that X and Y are manifolds.
   (a) Is $X \vee Y$ a manifold?
   (b) Show that there is an isomorphism of abelian groups
   $$\phi\colon \tilde{H}^*(X \vee Y, z) \to \tilde{H}^*(X, x) \oplus \tilde{H}^*(Y, y)$$
   (c) Compute the multiplication law induced on the direct sum by the cup product on $\tilde{H}^*(X \vee Y, z)$, i.e., find a formula for
   $$\phi\left(\phi^{-1}(a_1, b_1) \cup \phi^{-1}(a_2, b_2)\right)$$
   for $a_1, a_2 \in \tilde{H}^*(X, x)$ and $b_1, b_2 \in \tilde{H}^*(Y, y)$.
   (d) Deduce that $S^2 \times S^2$ is not homotopy equivalent to $S^2 \vee S^2 \vee S^4$.
5. (734) Let Z be a hexagon with opposite sides identified as shown. Calculate $H^*(Z)$.



6. (734) Let M be a compact 7 dimensional manifold without boundary. Suppose $a \in H_2(M)$ is nonzero, but $8a = 0$.
   (a) Suppose $a \in H_2(M)$ is nonzero, but $8a = 0$. If M is orientable, show that there is a nonzero $b \in H_4(M)$ so that $8b = 0$.
   (b) If M is connected and not orientable, compute $H^7(M)$. Show that there is a class in $H^6(M; \mathbb{Z}_2)$ that does not lie in the image of the natural map $H^6(M) \to H^6(M; \mathbb{Z}_2)$ induced by the onto homomorphism $\mathbb{Z} \to \mathbb{Z}_2$.

7. (740)
   (a) Let M be an n-dimensional Riemannian manifold with metric $\langle\ ,\ \rangle$, and let σ be a differentiable real-valued function. Define a new metric $(\ ,\ )$ on M by
$$(\ ,\ ) = e^{2\sigma}\langle\ ,\ \rangle.$$
   Denote by ∇ and $\widetilde{\nabla}$ the Riemannian connections of $\langle\ ,\ \rangle$ and $(\ ,\ )$. Show that ∇ and $\widetilde{\nabla}$ are related by the formula
$$(\widetilde{\nabla}_X Y, Z) = e^{2\sigma}\Big(\langle\nabla_X Y, Z\rangle + X(\sigma)\langle Y, Z\rangle + Y(\sigma)\langle X, Z\rangle - Z(\sigma)\langle X, Y\rangle\Big),$$
   where X, Y and Z are vector fields on M.
   (b) The Riemann curvature tensors R and $\widetilde{R}$ of ∇ and $\widetilde{\nabla}$ are defined by
$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y] \qquad \text{and} \qquad \widetilde{R}_{XY} = \widetilde{\nabla}_{[X,Y]} - [\widetilde{\nabla}_X, \widetilde{\nabla}_Y].$$
   In the case that σ is a constant function, show that
$$\widetilde{R}_{XY} = R_{XY}.$$
   (c) The Ricci curvature tensors ρ and $\widetilde{\rho}$ of R and $\widetilde{R}$ are defined by
$$\rho(X,Y) = \sum_{i=1}^{n}\langle R_{XE_i}Y, E_i\rangle \qquad \text{and} \qquad \widetilde{\rho}(X,Y) = \sum_{i=1}^{n}\langle \widetilde{R}_{X\widetilde{E}_i}Y, \widetilde{E}_i\rangle,$$
   where $\{E_1, \ldots, E_n\}$ is a local orthonormal frame for $\langle\ ,\ \rangle$ and $\{\widetilde{E}_1, \ldots, \widetilde{E}_n\}$ is a local orthonormal frame for $(\ ,\ )$. In the case that σ is a constant function, show that ρ and $\widetilde{\rho}$ are related by
$$\widetilde{\rho}(X,Y) = \rho(X,Y).$$

8. (740)
   Let M be a Riemannian manifold. Two k-tuples $(p_1, \ldots, p_k)$ and $(q_1, \ldots, q_k)$ of points in M are said to be **isometric** if $d(p_i, p_j) = d(q_i, q_j)$ for any $i, j \in \{1, \ldots, k\}$, where d denotes the distance function of M. The manifold M is said to be k-point homogeneous if for any two isometric k-tuples $(p_1, \ldots, p_k)$ and $(q_1, \ldots, q_k)$ of M there exists an isometry $A : M \to M$ with $A(p_i) = A(q_i)$ for $i = 1, \ldots, k$.
   (a) Show that any 1-point homogeneous manifold is (geodesically) complete.
   (b) Show that any 2-point homogeneous manifold has the property that the covariant derivative of the curvature tensor of M vanishes.

9. (740) Show that the following two statements are equivalent for a smooth n-manifold M:
   (a) M admits an atlas where all coordinate changes are (restrictions of) affine maps of $\mathbb{R}^n$
   (b) M admits a torsion free connection with zero curvature tensor.

10. (742)
   (a) Let V be a real finite-dimensional vector space, Δ the diagonal of $V \times V$. Suppose $A : V \to V$ is an invertible linear map. Show that $W = \{(v, Av) \mid v \in V\}$ is transverse to Δ if and only if 1 is not an eigenvalue of A.
   (b) Let M^n be a compact smooth manifold without boundary, and $f : M \to M$ a smooth map. Suppose that for each fixed point $x \in M$, the linear map $df_x : T_xM \to T_xM$ has no eigenvalues equal to 1. Show that there are only finitely many fixed points for f.

11. (**742**) Define an $(n-1)$-form on $\mathbb{R}^n - \{0\}$ by

$$\omega = \sum_{i=1}^{n} (-1)^i \frac{x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n}{r^n}, \quad \text{where } r = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2},$$

and the "hat" over dx_i indicates that dx_i is omitted.

(a) Show that ω is closed.

(b) Let S be a cube in $\mathbb{R}^n$ containing the origin, and B a ball in $\mathbb{R}^n$ containing the origin. Relate the integrals $\int_{\partial S} \omega$ and $\int_{\partial B} \omega$, where ∂S and ∂B have the orientations induced from the standard orientation on $\mathbb{R}^n$.

(c) Show that ω is not exact.

12. (**742**) Prove or disprove the following statements.

(a) For any closed, connected, oriented manifold M of dimension n, there is a degree one map $f : M \to S^n$.

(b) For any closed, connected, oriented manifold M of dimension n, there is a degree one map $f : S^n \to M$.

(c) If M^n and N^n are closed and connected, and $f : M \to N$ has nonzero degree, then f is surjective.

TOPOLOGY/GEOMETRY QUALIFYING EXAMINATION
JANUARY 1996

MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND, COLLEGE PARK

January 10, 1996

Do ANY six problems. Only the first six problems which you do will be graded, so there is no point in turning in more than six problems. This test is designed so that anyone who has covered the syllabi for two of the basic courses (730,734,740,742) should be able to do six problems. The indicated course numbers are included only as a guide and are not intended to restrict your choice of problems (you may safely ignore them if you wish).

(1) **(730)** Recall that a *connected sum* of two n-manifolds is obtained by removing an embedded n-ball from each one and identifying the resulting boundary components, each of which is homeomorphic to S^{n-1}. Compute the fundamental group of a connected sum M of real projective 4-space $\mathbb{R}P^4$ with the product $S^2 \times S^2$. Describe the universal covering space of M as connected sum.

(2) **(730)** Let Z be a two-dimensional linear subspace of $\mathbb{R}^3$. Construct an identification of the set S of all one-dimensional subspaces transverse to Z with the set of linear maps $\mathbb{R}^3/Z \longrightarrow Z$. Prove or disprove: the homomorphism $\pi_1(S) \longrightarrow \pi_1(\mathbb{R}P^2)$ is injective.

(3) **(730, 740)** Consider the following two metric spaces S_c and S_r. Each metric space has underlying set the unit circle S^1 in the complex line $\mathbb{C}$. For S_c, the distance is the chordal distance d_c defined by

$$d_c(e^{i\phi}, e^{i\psi}) = |e^{i\phi} - e^{i\psi}|$$

and for S_r the distance is the (Riemannian) distance defined by the covering space

$$\begin{aligned} p : \mathbb{R} &\longrightarrow S^1 \\ t &\longmapsto e^{it} \end{aligned}$$

as follows. If $a, b \in S^1$, then their distance $d_r(a, b)$ is the infimum of distances $|\tilde{a} - \tilde{b}|$ measured in $\mathbb{R}$, where $\tilde{a} \in p^{-1}(a)$ and $\tilde{b} \in p^{-1}(b)$. Prove or disprove:

(a) With the metric space topologies, S_r and S_c are homeomorphic.
(b) S_r and S_c are isometric.
(c) p is continuous with respect to the metric space topologies.
(d) p is an isometry.

(4) **(734)** Let M^n be an n-dimensional manifold and suppose $S^p \times D^{n-p} \hookrightarrow M^n$ is an embedding with $n > p \geq 0$. Remove the embedded $S^p \times D^{n-p}$ from M^n leaving a manifold $\bar{M}$ with boundary diffeomorphic to $S^p \times S^{n-p-1}$ which is also the boundary of $D^{p+1} \times S^{n-p-1}$. Then define

$$M' = \bar{M} \cup (D^{p+1} \times S^{n-p-1})$$

with the common boundary identified. (This is called a $(p, n-p)$ surgery on M.)
(a) **(734)** If $\chi(X)$ denotes the Euler characteristic of X show $\chi(M) = \chi(M')$ if n is odd and $\chi(M) = \chi(M') \pm 2$ if n is even.
(b) Give an example of a $(p, n-p)$ surgery with M' not connected and M connected.

(5) **(734)** Let X be a compact, orientable manifold of dimension n with or without boundary.
(a) Show $H_{n-1}(X; \mathbb{Z})$ is torsion free.
(b) Prove that, if n is odd and $\partial X = \emptyset$ then the Euler characteristic of X is zero.
(c) Show any 3-dimensional non-orientable compact manifold without boundary has infinite fundamental group.

(6) **(734)**
(a) Show $H_*(\mathbb{C}P^3, G) \simeq H_*(S^2 \times S^4, G)$. for every coefficient group G but that $\mathbb{C}P^3$ is not homotopy equivalent to $S^2 \times S^4$.
(b) Show any continuous map $f : \mathbb{C}P^m \to \mathbb{C}P^n$ where $m > n$ induces the zero map $f^* : H^p(\mathbb{C}P^n) \to H^p(\mathbb{C}P^m)$ for all $p > 0$.

(7) **(730, 742)** Show that the Möbius band defined as the quotient of $\mathbb{R}^2$ by the cyclic group generated by the homeomorphism

$$(x, y) \longmapsto (x + 1, -y)$$

is a *nontrivial* real line bundle over the circle $\mathbb{R}/\mathbb{Z}$.

(8) **(740, 742)** Let G be an n-dimensional Lie group. Construct an isomorphism of its tangent bundle with the product bundle $G \times \mathbb{R}^n \longrightarrow G$. Find an example of a connected Lie group G and a closed subgroup $H \subset G$ such that the tangent bundle of the homogeneous space G/H is nontrivial.

(9) **(740)** Let X be the product of two spheres. Show that through any point $x \in X$ there exists an embedded totally geodesic flat torus.

(10) **(740)** Let ∇ be the Levi-Civita connection on $\mathbb{R}^3$ and let $\bar{\nabla}$ be the Levi-Civita connection on the cylinder $C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$. Let ξ be the

vector field

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} - \frac{\partial}{\partial z}$$

(a) Show that ξ is tangent to C.
(b) Compute the covariant derivative $\nabla_\xi \xi$ on $\mathbb{R}^3$.
(c) Compute the covariant derivative $\tilde{\nabla}_\xi \xi$ on C.

(11) (742) Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ be the map

$$f(x, y, z, w) = (x^3 - 3xy^2 + z^2 - w^2, 3x^2y - y^3 + 2zw)$$

(the map $C^2 \longrightarrow \mathbb{C}$ defined by $(u, v) \mapsto u^3 + v^2$). Let W be the line $\{1\}$ $\mathbb{R} \subset \mathbb{R}^2$. Show that $f^{-1}(W)$ is a smooth submanifold of $\mathbb{R}^4$ and compute dimension.

(12) (742,740) Let ω be the 1-form $xdy + dz$ in $\mathbb{R}^3$ and ξ the vector field

$$x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

(a) Find the flow $\{\phi_t\}_{t\in\mathbb{R}}$ tangent to ξ.
(b) Using the preceding result, compute the pullback $\phi_t^*\omega$.
(c) Using the preceding result, compute the Lie derivative $\mathfrak{L}_\xi\omega$.
(d) Using the preceding result, verify the Cartan formula

$$\mathfrak{L}_\xi\omega = d\iota_\xi\omega + \iota_\xi d\omega$$

where ι_ξ denotes interior product with respect to ξ.

Geometry/Topology Ph.D Written Examination: August 1995

Instructions: Answer any six questions in separate answer booklets. Make sure the front page of each booklet bears the question number and your examination code number. Under no circumstances will more than six answer booklets be accepted from any student or more than one question graded from any answer booklet. Although your choice of six questions is unrestricted, each question has been labeled for your guidance with the number of the course on which it is based. Recent MATH 730 students may wish to note that question 10 bears two labels.

1. [730] Let

$$X = (0,1) \cup 2 \cup (3,4) \cup 5 \cup \ldots \cup (3n, 3n+1) \cup 3n+2 \cup \ldots, \text{ and}$$
$$Y = (0,1] \cup (3,4) \cup 5 \cup \ldots \cup (3n, 3n+1) \cup 3n+2 \cup \ldots,$$

where both X and Y are topologized as subspaces of $\mathbb{R}$.

(a) Find *continuous* bijections $f : X \to Y$ and $g : Y \to X$.
(b) Show that X and Y are not homeomorphic.

2. [730]

(a) Let $p : \mathbb{R} \to X$ be a covering map. Show that X must be either $\mathbb{R}$ itself of the circle S^1.
(b) Let $p : S^n \to X$ be a covering map. The possible values of the Euler characteristic $\chi(X)$ depends on the value of n. Give examples of all possibilities.

3. [730] Let X be a disk with boundary C_0. Let D_1 and D_2 be disjoint closed disks contained in the interior of X, and let Y be X with the interiors of D_i and D_2 removed. The boundary of Y consists of three circles: C_0, the outer circle, and C_1 and C_2, the boundaries of D_1 and D_2. Identify points on as Y follows: on C_0 identify antipodal points, on C_1 identify points 120° apart, and on C_2 identify points 90° apart; points not on the boundary of Y are identified only with themselves. Let W be the resulting quotient space. Calculate the fundamental group of W.

4. [734] Let $f: X \to \mathbb{CP}^n$ be a continuous function from a CW complex X to complex projective space $\mathbb{CP}^n$ (which has real dimension $2n$). Suppose that the map $f_* : H_{2n}(X, \mathbb{Z}) \to H_{2n}(\mathbb{CP}^n, \mathbb{Z})$ is nonzero.

(a) If β is the generator of $H^{2n}(\mathbb{CP}^n, \mathbb{Z})$, show that $f^*(\beta) \neq 0$.
(b) Show that $H^2(X, \mathbb{Z})$ and $H_2(X, \mathbb{Z})$ are both nontrivial groups. (Hint: The cohomology algebra of $\mathbb{CP}^n$ is generated by a single element in $H^2(\mathbb{CP}^n, \mathbb{Z})$.)

5. [734] Let Y be a CW complex so that

$$H_i(Y, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, 2 \text{ or } 4, \\ \mathbb{Z}/12\mathbb{Z}, & \text{if } i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Calculate the cohomology groups $H^*(Y, \mathbb{Z})$ and $H^*(Y, \mathbb{Z}/2\mathbb{Z})$.
(b) Could Y have the homotopy type of a compact orientable manifold without boundary?
(c) Could Y have the homotopy type of a compact nonorientable manifold without boundary?

6. [734] For any positive integer n, let Z_n be the quotient space of the unit disk $\{z \in \mathbb{C}$ with $|z| \leq 1\}$ with the identifications $z \sim e^{2\pi i/n} z$ for all z with $|z| = 1$ and also $1 \sim e^{\pi i/n}$. Alternatively, Z_n is the space obtained by gluing the disk to a figure eight via a map which wraps the boundary of the disk n times around the figure eight. Calculate $H_*(Z_n, \mathbb{Z})$ and $H_*(Z_n, \mathbb{Z}/n\mathbb{Z})$ and $\pi_1(Z_n, 1)$.

7. [740] Let (x, y, z) be the standard coordinates on $\mathbb{R}^3$, and let e_1, e_2, and e_3 be the standard coordinate vector fields. Let ρ be the symmetric $(0,2)$ tensor on $\mathbb{R}^3$ defined by:

$$\rho(e_i, e_j) = \begin{cases} 0 & \text{if } i \neq j, \\ -1 & \text{if } i = j = 3, \\ 1 & \text{otherwise.} \end{cases}$$

Consider the submanifold $\mathbb{H} = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 - z^2 = -1, z > 0\}$. The restriction of ρ to $\mathbb{H}$ is a Riemannian metric on $\mathbb{H}$.

(a) Show that reflection in the x-z plane is an isometry of $\mathbb{H}$.
(b) Write down a parametrization of the unit speed geodesic in $\mathbb{H}$ with initial velocity $(1, 0, 0)$ at the point $(0, 0, 1)$.
(c) Show that there are no closed geodesics in $\mathbb{H}$ through the point $(0, 0, 1)$.

8. [740] Let M be a complete Riemannian manifold, and N_1 and N_2 disjoint compact submanifolds.

(a) Show that there is a length minimizing curve from N_1 to N_2.
(b) Let γ be a length minimizing curve from N_1 to N_2. Show that γ is a geodesic and is orthogonal to both N_1 and N_2.
(c) Suppose that $\dim(N_1) + \dim(N_2) \geq \dim(M)$. Show that there is a nonzero parallel vector field along γ that is tangent to N_1 and N_2 at the endpoints of γ.

9. [740]

(a) Let M be a three-dimensional Riemannian manifold. Let X, Y, Z be an orthonormal basis of $T_x(M)$. Give an expression forthe sectional curvature ofthe plane spanned by X and Y in terms of the Ricci curvatures of X, Y and Z.
(b) A Riemannian manifold is said to be ***Einstein*** if $Ric(X, Y) = c\langle X, Y\rangle$, where Ric is the Ricci tensor, c is a constant, and $\langle , \rangle$ is the metric tensor. Show that a three-dimensional Einstein manifold has constant sectional curvature.
(c) Show that a four-dimensional Einstein manifold does not necessarily have constant sectional curvature by considering the manifold $S^2 \times S^2$.

10. [730, 742] Consider the function $f : \mathbb{R}^4 \to \mathbb{R}$ defined by

$$f(x_1, x_2, x_3, x_4) = x_1 x_2 + x_3 x_4.$$

(a) Find the set of regular values of f.
(b) Compute the tangent hyperplane to $f^{-1}(1)$ at the point $(1, 1, 0, 0)$.
(c) Let S^3 be the unit sphere in $\mathbb{R}^4$. Show that $f^{-1}(1)$ is transverse to S^3.

11. [742] Let D be the closed unit disk in $\mathbb{R}^2$, and $S^1 = \partial D$. Let f and g be smooth embeddings of S^1 in $\mathbb{R}^3$ such that $f(S^1) \cap g(S^1) =$. Define $\lambda : S^1 \times S^1 \to S^2$ by

$$\lambda(x, y) = \frac{f(x) - g(y)}{||f(x) - g(y)||}.$$

(a) Show that λ is a smooth map.

(b) Suppose f extends to a smooth map $\hat{f} : D \to \mathbb{R}^3$ such that $\hat{f}(D) \cap g(S^1) =$ and $\hat{f}|S^1 = f$. Show that the degree of λ is 0.

(c) Prove a partial converse of (b) by showing that if the degree of λ is 0, then λ extends to a smooth map $\hat{\lambda} : D \times D^1 \to S^2$.

12. [742] Let D^k be the closed unit ball in $\mathbb{R}^k$, and $S^k = \partial D^{k+1}$.

(a) Let $h : D^k \to \mathbb{R}^n$, $k \leq n$, be an embedding with $h(0) = 0$. Show that the map $H : D^k \times I \to \mathbb{R}^n$ defined by

$$H(x, t) = \begin{cases} t^{-1}h(tx), & 0 < t \leq 1 \\ Dh(0)x, & t = 0 \end{cases}$$

is an isotopy from h to a linear map.

(b) Let $f, g : D^n \to M^n$ be embeddings. Suppose M^n is orientable and f and g preserve orientation. Show f is isotopic to g.

(c) Let $\mathrm{Diff}_+(S^n)$ denote the group of smooth, orientation preserving diffeomorphisms of S^n, and G the subgroup consisting of restrictions of orientation preserving diffeomorphisms of D^{n+1}. Let $f \in \mathrm{Diff}_+(S^n)$. Show that $f \in G$ if and only if f is isotopic to the identity.

TOPOLOGY
MASTER'S QUALIFYING EXAMINATION

January 6, 1995

1.

Let K be a 4-dimensional simplicial complex which has 8 0-simplices, 12 1-simplices, 9 2-simplices, 10 3-simplices and 6 4-simplices. Suppose that

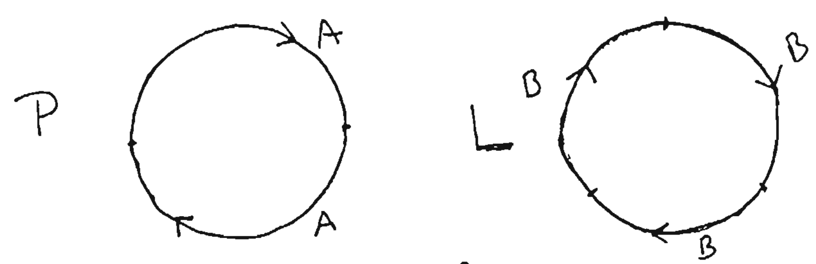
$$H_0(K) = \mathbb{Z}, \quad H_1(K) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2, \quad H_2(K) = \mathbb{Z} \oplus \mathbb{Z}/3, \quad H_3(K) = \mathbb{Z} \oplus \mathbb{Z}/4$$

What is $H_4(K)$?

2.

Let X be a topological space and let $Y \subset X$ be a subspace. Then Y is a *retract* of X if and only if there exists a continuous map $r : X \longrightarrow Y$ such that $r(y) = y$ for all $y \in Y$.

(1) Let Y be a retract of X. Show that if X is contractible, then Y is contractible.
(2) Let Y be a retract of X. Show that if X is connected, then Y is connected.
(3) Show that every map $f : [0,1] \longrightarrow [0,1]$ has a fixed point.



3.

Let P and L respectively be the identification spaces of the 2-disc as indicated in the pictures. Let A and B be the indicated arcs on their respective boundaries.

(1) Compute $\pi_1(L)$ and $\pi_1(P)$.
(2) The respective images $\overline{A}$ and $\overline{B}$ of the intervals A and B are homeomorphic to circles. Let $f : \overline{A} \longrightarrow \overline{B}$ be a homeomorphism. Let X be the identification space $X = P \cup_f L$. Compute $\pi_1(X)$.

4.

(1) Let W be Hausdorff. For all n, let K_n be a compact subset of W. Prove that $\bigcap_{n=1}^{\infty} K_n$ is compact.
(2) Let $f : K \to \mathbb{R}$ be continuous, where K is compact. Prove that there exists $x_0 \in K$ such that for all $x \in K$, we have $f(x) \geq f(x_0)$.

5.

(1) Let T be a closed orientable surface of genus 2 and $t_0 \in T$ a basepoint. Fix an $n > 1$. Find a normal subgroup G of $\pi_1(T, t_0)$ of index n.
(2) Corresponding to G, there is a covering space $p : S \to T$ and some $s_0 \in S$ such that $p_\#(\pi_1(S, s_0)) = G$. What is the Euler characteristic of S?
(3) Show that every closed orientable surface of genus $g > 1$ is a covering space of a surface of genus 2.

6.

Let X be a topological space and consider the map

$$\begin{aligned} \Delta : X &\longrightarrow X \times X \\ x &\longmapsto (x, x) \end{aligned}$$

Show that X is Hausdorff if and only if $\Delta(X)$ is closed in $X \times X$.

Geometry / Topology

TOPOLOGY (730)
PH. D. QUALIFYING EXAMINATION

January, 1995

1.

Let K be a 4-dimensional simplicial complex which has 8 0-simplices, 12 1-simplices, 9 2-simplices, 10 3-simplices and 6 4-simplices. Suppose that

$$H_0(K) = \mathbb{Z}, \quad H_1(K) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2, \quad H_2(K) = \mathbb{Z} \oplus \mathbb{Z}/3, \quad H_3(K) = \mathbb{Z} \oplus \mathbb{Z}/4$$

What is $H_4(K)$?

2.

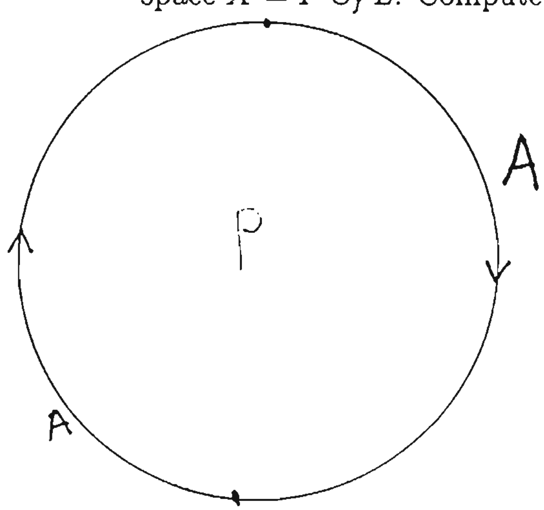
Let X be a topological space and let $Y \subset X$ be a subspace. Then Y is a *retract* of X if and only if there exists a continuous map $r : X \longrightarrow Y$ such that $r(y) = y$ for all $y \in Y$.

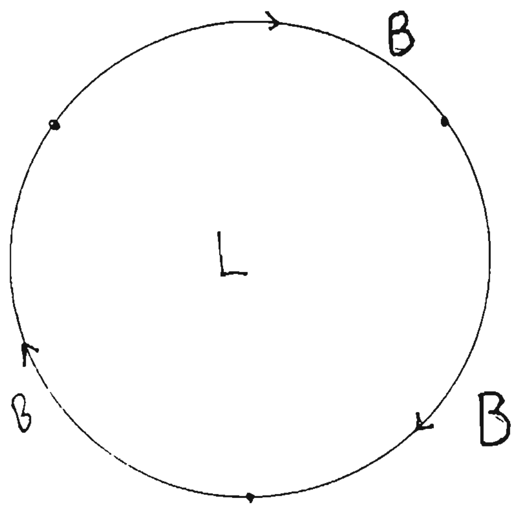
(1) Let Y be a retract of X. Show that if X is contractible, then Y is contractible.
(2) Let Y be a retract of X. Show that if X is connected, then Y is connected.
(3) Show that every map $f : [0,1] \longrightarrow [0,1]$ has a fixed point.

3.

Let P and L respectively be the identification spaces of the 2-disc as indicated in the pictures. Let A and B be the indicated arcs on their respective boundaries.

(1) Compute $\pi_1(L)$ and $\pi_1(P)$.
(2) The respective images $\overline{A}$ and $\overline{B}$ of the intervals A and B are homeomorphic to circles. Let $f : \overline{A} \longrightarrow \overline{B}$ be a homeomorphism. Let X be the identification space $X = P \cup_f L$. Compute $\pi_1(X)$.





Geometry / Topology

734 PH.D. QUALIFYING EXAMINATION

January 1995

1. Let X_p denote the space obtained by attaching an n-cell to S^{n-1} by a map of degree p, where p is a prime.

(a) Compute $H_*(X_p \times X_q, \mathbb{Z})$ where $\mathbb{Z}$ denotes the integers.
(b) Calculate $H^*(X_p \times X_p, \mathbb{Z})$.
(c) If r is a prime and $\mathbb{Z}_r$ is the cyclic group of order r, compute $H_*(X_p \times X_p, \mathbb{Z}_r)$.

2. The embedding $\mathbb{C}^k - \{0\} \hookrightarrow \mathbb{C}^n - \{0\}$ given by $(z_1, \ldots, z_k) \mapsto (z_1, \ldots, z_k, 0, \ldots, 0)$ induces an inclusion of $\mathbb{C}P^k \hookrightarrow \mathbb{C}P^n$ where $\mathbb{C}P^t$ denotes complex projective space of (complex) dimension t. Let $\pi : \mathbb{C}P^n \to \mathbb{C}P^n/\mathbb{C}P^k$ denote the natural projection.

(a) Show π induces a monomorphism $\pi^* : H^*(\mathbb{C}P^n/\mathbb{C}P^k, \mathbb{Z}) \to H^*(\mathbb{C}P^n, \mathbb{Z})$.
(b) Show $\mathbb{C}P^2/\mathbb{C}P^1$ is not a retract of $\mathbb{C}P^4/\mathbb{C}P^1$.

3. Suppose X is a finite 3-dimensional CW-complex with 1 zero cell, 2 one cells, 2 two cells and 2 three cells. Moreover suppose $\pi_1(X)$ is the non-abelian quaternion group of order 8 and the universal cover $\tilde{X}$ of X has $H_2(\tilde{X}) = 0$. (Homology is with integer coefficients.)

(a) Compute $H_*(\tilde{X})$.
(b) Show $H_1(X)$ and $H_2(X)$ are finite groups and determine $H_3(X)$.

January 1995

Geometry / Topology

Math 742 Exam

Note: In the following problems, all manifolds should be assumed to be smooth and without boundary.

1. Let M be an oriented codimension one submanifold of an oriented manifold N. Show that there is a neighborhood U of M in N so that U is diffeomorphic to $M \times \mathbb{R}$.

2. Let M^7 and N^{18} be compact connected manifolds of dimension 7 and 18 respectively. Suppose that $f: M \to N$ is continuous. Show that for any $\epsilon > 0$ there are smooth imbeddings $f_1: M \to N$ and $f_2: M \to N$ so that $|f(x) - f_i(x)| < \epsilon$ for all $x \in M$ and so that $f_1(M) \cap f_2(M)$ is empty.

3. Let $M \subset \mathbb{R}^9$ be a compact 3 dimensional submanifold of $\mathbb{R}^9$. Show that there is a linear map $L: \mathbb{R}^9 \to \mathbb{R}^8$ so that the restriction $L|_M$ is an embedding of M into $\mathbb{R}^8$.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
AUGUST 1994

Geometry /

TOPOLOGY (Ph.D. version)

Instructions to the student

1. Select two of the four courses listed (730, 734, 740, 742) and answer all three questions in each category. Only these six questions will be graded.

2. Use a different booklet for each question. Complete the top page of the booklet and also write your code number on each page of the booklet. Do not write your name anywhere.

3. Keep scratch work on separate pages in the same booklet.

Geometry / Topology

MATH 730 WRITTEN EXAM, AUGUST, 1994

PH.D EXAM

1)

a) Prove that every finite covering space of a torus $T = S^1 \times S^1$ is again a torus.

b) Find two four-fold covering maps $f : T \to T$ and $g : T \to T$ such that there do *not* exist homeomorphisms $\alpha : T \to T$ and $\beta : T \to T$ with $\alpha \circ f = g \circ \beta$.

2) Let $x_0 \in \mathbb{RP}^2$. Then let $\mathbb{RP}^2 \vee \mathbb{RP}^2 = \mathbb{RP}^2 \times \{x_0\} \cup \{x_0\} \times \mathbb{RP}^2 \subset \mathbb{RP}^2 \times \mathbb{RP}^2$.

a) Compute the fundamental groups of $\mathbb{RP}^2 \vee \mathbb{RP}^2$ and $\mathbb{RP}^2 \times \mathbb{RP}^2$.

b) Prove that $\mathbb{RP}^2 \vee \mathbb{RP}^2$ is not a retract of $\mathbb{RP}^2 \times \mathbb{RP}^2$.

3) Let the simplicial complex K be a triangulation of a solid torus with one 3-simplex (but none of its faces) removed.

a) Compute the homology of $|K|$, the realization of K.

b) Prove that any homotopy equivalence from $|K|$ to itself has a fixed point.

August 1994

Geometry / Topology

734 Ph.D. Qualifying Examination

1. Let $p : S^n \to RP^n$ be the covering map and let $\phi_k : S^n \to S^n$ be a map of degree k. Let $X_k = RP^n \cup_{p\phi_k} e^{n+1}$. I.e, attach an $n+1$ cell to RP^n by the map $p\phi_k$.

   a) Determine the integral homology of X_k.

   b) Show there exists a map $f : X_k \to RP^{n+1}$ with $f|RP^n = \text{id}$ and determine the induced map in homology.

2. Let (X, A) denote a pair of topological spaces.

   a) Suppose $H_*(X, A; \mathbf{Z}) = \sum_{n=0}^{\infty} H_n(X, A; \mathbf{Z})$ is finitely generated (as an abelian group) so that $\chi(X, A) = \sum_{n=0}^{\infty} (-1)^n \text{rank} H_n(X, A; \mathbf{Z})$ is defined. Let $\mathbf{F} = \mathbf{Q}$, the rational numbers or $\mathbf{F}_p$, the field of p elements. Show $H_*(X, A; \mathbf{F})$ is a finitely generated $\mathbf{F}$-module and

$$\chi(X, A) = \sum_{n=0}^{\infty} (-1)^n \dim_{\mathbf{F}} H_n(X, A; \mathbf{F})$$

   b) Suppose (X, A) is a pair of topological spaces and that two of the three pairs $A = (A, \emptyset)$, $X = (X, \emptyset)$ or (X, A) has $H_*(-; \mathbf{Z})$ finitely generated. Show the third pair is also finitely generated homology and $\chi(X) = \chi(A) + \chi(X, A)$.

   c) Suppose M is a manifold which is the boundary of an odd dimensional manifold W. Show $\chi(M)$ is even.

3. Recall the connected sum $M \# N$ of two oriented n-manifolds M and N is obtained by removing an embedded open disc D^n from each obtaining manifolds with boundary M' and N' and then identifying the two boundary spheres by an orientation reversing diffeomorphism. It is known that $M \# N$ can be oriented so as to induce the given orientations on M' and N'.

   a) Let G be any abelian group. Show $H_*(CP^2 \# CP^2; G) \simeq H_*(S^2 \times S^2; G)$.

   b) Show $CP^2 \# CP^2$ is not homotopy equivalent to $S^2 \times S^2$ by using the cohomology ring structure.

Geometry / Topology

740 PH.D. QUALIFYING EXAMINATION
FALL, 1994

1. Let M be a Riemannian manifold with metric $\langle\, ,\, \rangle$ and Levi-Civita connection ∇. If $f \in C^\infty(M)$, we define a vector field $\operatorname{grad} f$ by the formula $\langle (\operatorname{grad} f)_p, X\rangle = X(f)$, for all $X \in T_pM$.

   (a) Show that $\operatorname{grad}(fg) = f \operatorname{grad} g + g \operatorname{grad} f$.

   (b) Let $\phi \in C^\infty(M)$ be a positive function. Define a new metric $\langle\, ,\, \rangle' = \phi \langle\, ,\, \rangle$. Show that the Levi-Civita connection of $\langle\, ,\, \rangle'$ is

$$\nabla'_X Y = \nabla_X Y + \frac{1}{2\phi}\{X(\phi)Y + Y(\phi)X - \langle X, Y\rangle \operatorname{grad}\phi\},$$

   where $\operatorname{grad}\phi$ is taken with respect to the original metric.

2.

   (a) Let $\gamma : [0,1] \to M$ be a geodesic. Suppose that for some $t_0 \in (0,1)$, there is a minimizing geodesic α from $\gamma(0)$ to $\gamma(t_0)$ which is distinct from γ (i.e., α is not a reparametrization of a segment of γ.) Show that for $t > t_0$, γ is not minimizing between $\gamma(0)$ and $\gamma(t)$.

   (b) Let S^n be the unit sphere in $\mathbb{R}^{n+1}$, with the metric inherited from the Euclidean metric on $\mathbb{R}^{n+1}$. Let $\gamma : [a,b] \to S^n$ be a smooth function. Show that γ is a unit speed geodesic in S^n if and only if $\gamma''(t) = -\gamma(t)$ (considered as function into $\mathbb{R}^{n+1}$).

   (c) Write down the general form of a unit speed geodesic on S^n (give an expression in coordinates in $\mathbb{R}^{n+1}$).

3. Let M be a manifold of dimension $2n+1$, and α a 1-form on M. We say that α is a *contact* form if $\alpha \wedge (d\alpha)^n = \alpha \wedge d\alpha \wedge \cdots \wedge d\alpha \neq 0$ at each point of M.

   (a) Write down a contact form on $\mathbb{R}^3$.

   (b) Fix a point $x \in M$. The *kernel* of $d\alpha$ at x is the subspace of T_xM defined by

$$\ker d\alpha = \{v \in T_xM \mid d\alpha(v,w) = 0 \quad \forall\, w \in T_xM\}.$$

   Show that if α is a contact form,

$$\ker d\alpha = \{v \in T_xM \mid d\alpha(v,w) = 0 \quad \forall w \in \ker\alpha\},$$

   where $\ker\alpha = \{w \in T_xM \mid \alpha(w) = 0\}$. (*Hint:* Use the fact that $\alpha \wedge (d\alpha)^n \neq 0$ to show that if $v \neq 0$ and $d\alpha(v,w) = 0$ for all $w \in \ker\alpha$, then $v \notin \ker\alpha$.)

   (c) Show that if α is a contact form, then there is a unique vector field X on M such that $\alpha(X) = 1$, and X is in the kernel of $d\alpha$ at each point of M.

Geometry / Topology

WRITTEN GRADUATE QUALIFYING EXAM
DIFFERENTIAL TOPOLOGY (MATHEMATICS 742)

August, 1994

1. Prove that if f is a smooth embedding of S^n into S^{n+k} with $k \geq 3$, then $N^{n+k} = S^{n+k} \setminus f(S^n)$ is simply connected (or in other words, that every continuous map of S^1 into N extends to a continuous map of the 2-disk D^2 into N). Be sure to explain where the condition on k is used in your proof.

2. Let $f : M^7 \to N^4$ be a smooth map of smooth manifolds (without boundaries). Let C be a connected closed curve in N (in other words, a compact connected 1-dimensional submanifold, also without boundary) and assume f is transverse to C. Suppose L^4 is a submanifold of M, also without boundary, and that $f(L) \subseteq C$.

(a) Show that L has a tubular neighborhood U in M such that $L = U \cap f^{-1}(C)$.

(b) Suppose that $f(L) \subsetneqq C$. Show that one can choose U to be diffeomorphic to $L \times \mathbb{R}^3$.

(c) Show by example that if $f(L) = C$, it may *not* be possible to choose U to be diffeomeorphic to $L \times \mathbb{R}^3$.

3. **True/False.** For each statement, state whether it is true or false and give a brief justification.

(a) Let ξ and η be two non-trivial (real) vector bundles over S^1. Then the Whitney sum $\xi \oplus \eta$ is a trivial bundle.

(b) Let $f_1(x_1, \ldots, x_n) = 0, \ldots, f_m(x_1, \ldots, x_n) = 0$ be m equations in n real variables. After arbitrarily small perturbations of the f_i's, the set of solutions of these equations is an $(n-m)$-dimensional smooth submanifold of $\mathbb{R}^n$ with trivial normal bundle. or is empty.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
GRADUATE WRITTEN EXAMINATION
JANUARY 1994

Geometry/
TOPOLOGY (Ph.D. version)

Instructions to the student

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2. Use a different booklet for each question. Complete the top page of the booklet and also write your code number on each page of the booklet. Do not write your name anywhere.

3. Keep scratch work on separate pages in the same booklet.

1. Let $\mathbf{R}$ denote the real numbers with the usual (interval) topology. Let $\mathbf{Z}$ denote the integers. Let

$$X = (0 \cup (1,2)) \times \mathbf{Z}, \quad \text{and} \quad Y = (0 \cup (1,2] \times \mathbf{Z},$$

topologized as subspaces of $\mathbf{R} \times \mathbf{R}$.

a. Prove that X and Y are *not* homeomorphic.

b. Prove that there exists a continuous bijection, $f : X \to Y$.

2. Describe all compact surfaces without boundary which have Euler characteristic -4, *e.g.* by giving a maximal set of pairwise nonhomeomorphic examples. Suppose that, for such a surface X, the continuous map $f : X \to X$ is homotopic to the identity. Show that f has a fixed point. Show by example that the hypothesis that f is homotopic to the identity is essential.

3. Let S^1 be the unit circle, realized as the unimodular complex numbers. Let X and Y be the subsets of $S^1 \times S^1 \times S^1$ defined by

$$X = (S^1 \times S^1 \times \{1\}) \cup (S^1 \times \{1\} \times S^1); \quad Y = X \cup (\{1\} \times S^1 \times S^1).$$

Compute the fundamental groups of X and Y.

734 Ph.D. Qualifying Examination: January 1994

1.

a) Give an explicit decomposition of $S^2 \times S^4$ as a cell complex.

b) Compute the cohomology ring of $S^2 \times S^4$.

c) Show that $S^2 \vee S^4$ is not a retract of $S^2 \times S^4$.

2. Let C_* and D_* be chain complexes and let f and g be chain maps from C_* to D_*. Define a chain homotopy between f and g, and prove that if f and g are chain homotopic then

$$f_* = g_* : H_*(C_*) \to H_*(D_*).$$

3. Let S^3 be the unit sphere in $\mathbf{R}^4$, and let S^1 be embedded in S^3 as the intersection of S^3 with a two dimensional plane through the origin. Let M be the space obtained from S^3 by identifying the embedded S^1 to a point. Let Λ be a non-zero Abelian group.

a) Compute the homology groups of M with coefficients in Λ.

b) Compute the Euler characteristic of M.

c) Prove that M is not a manifold.

d) Let $\Pi : S^3 \to M$ denote the quotient map. Show that $\Pi_* : H_3(S^3, \Lambda) \to H_3(M, \Lambda)$ is not zero.

RIEMANNIAN GEOMETRY (MATH 740)
GRADUATE QUALIFYING EXAM

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND, COLLEGE PARK

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Problem 1

Let k, l be positive real numbers. Let X be the Riemannian manifold whose underlying manifold is the interval $\{x \in \mathbb{R} \mid -l < x < l\}$ and whose Riemannian metric is the tensor field

$$g = \frac{k^2 dx^2}{(l^2 - x^2)^2}$$

(1) For $a, b \in X$ compute their distance with respect to g.
(2) Prove that (X, g) is geodesically complete.
(3) Compute the curvature tensor of g.

Problem 2

Let $E = \mathbb{R}^4$ with the coordinates x, y, z, t.

(1) Let $h : E \longrightarrow \mathbb{R}$ denote the function

$$h(x, y, z, t) = x^2 + y^2 + z^2 - t^2$$

Show that $H = h^{-1}(-1)$ is a submanifold of E.

(2) For $p = (x, y, z, t) \in H$, compute the tangent space $T_p H$.

(3) Show that the restriction of the symmetric 2-tensor field on E

$$dxdx + dydy + dzdz - dtdt$$

to H is a Riemannian metric g on H.

(4) Show that if A is an 4×4-matrix which leaves invariant h, for example

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh(t) & \sinh(t) \\ 0 & 0 & \sinh(t) & \cosh(t) \end{bmatrix}$$

then A induces an isometry of (H, g).

Problem 3

Let (u, v) be coordinates on $\mathbb{R}^2$ and (x, y, z) be coordinates on $\mathbb{R}^3$. Consider the projection

$$P : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$(x, y, z) \longmapsto (x, y)$$

(1) Let $\omega = u dv$. Compute the exterior derivative $d\omega$ and the pull-back $P^*\omega$.
(2) Verify that $dP^*\omega = P^*(d\omega)$.
(3) Consider the 1-form $\tilde{\omega}$ on $\mathbb{R}^3$ defined by:

$$\tilde{\omega} = dz - x dy$$

Let $\gamma(t) = (u(t), v(t))$ be a smooth curve for $0 \le t \le 1$ and $z_0 \in \mathbb{R}$. Show that there is a unique smooth curve $\tilde{\gamma}(t) = (x(t), y(t), z(t))$; $0 \le t \le 1$ in $\mathbb{R}^3$ such that

- $x(0) = u(0)$, $y(0) = v(0)$, $z(0) = z_0$;
- $P \circ \tilde{\gamma} = \gamma$;
- $\int_{\gamma} \tilde{\omega} = 0$

742 Qualifying Examination: January 1994

1. Let $f : M \to N$ be a differentiable map between compact connected orientable differentiable manifolds of the same dimension.

   a. Let $Q \subseteq N$ be the set of points $q \in N$ such that $f^{-1}(q)$ is finite. Show that Q is dense in N.

   b. Show that Q has an open dense subset $\hat{Q}$ such that the cardinality of $f^{-1}(q)$ has constant parity on $\hat{Q}$.

   c. Give an example for which the containments $\hat{Q} \subset Q \subset N$ are strict.

2. Recall that complex projective n-space, $\mathbf{CP}^n$ is given by $(\mathbf{C}^{n+1} \setminus \{0\})/\sim$, where $\sim$ is defined by $(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n)$ for all $(z_0, \ldots z_n) \in \mathbf{C}^{n+1} \setminus \{0\}$ and all $\lambda \in \mathbf{C} \setminus \{0\}$. The image of $(z_0, \ldots, z_n)$ in $\mathbf{CP}^n$ is customarily denoted $[z_0, \ldots, z_n]$. Let α be a complex number and let

$$M = \{[z_0, z_1, z_2] \in \mathbf{CP}^2 : z_0^3 + z_1 z_2^2 + \alpha z_1^3 = 0\}.$$

Show that M is not a manifold if $\alpha = 0$, but that if $\alpha \neq 0$, M is a smooth orientable submanifold of $\mathbf{CP}^2$ of dimension 2 (complex dimension 1).

3. Let M be a differentiable manifold of dimension n and let

$$f : B^n \to M$$

be a differentiable embedding of the unit ball into M. Let M' be obtained from M and f by removing $f(B^n \setminus S^{n-1})$ and identifying $f(x)$ with $f(-x)$ for all $x \in S^{n-1}$. Prove that M' is (*i.e.* admits the structure of) a differentiable manifold, and determine under what conditions M' is orientable.