Complex Numbers and Series

Here are the central concepts and results in our unit on complex numbers and series, which can be found on the webpage with url

`http://www.math.umd.edu/undergraduate/courses/dcp/141/141cxnotes.pdf`

Definition 1.1: A complex number is a number z of the form $z = x + iy$ (or equivalently, $z = x + yi$), where x and y are real numbers, and where $i^2 = -1$.

Comment: The set of all complex numbers is denoted **C** and the set of real numbers (that is, all z such that $z = x$) is denoted **R**.

Comment: Each complex number $a + ib$ can be considered as a point in the plane, with a representing the x coordinate and b the y coordinate.

Addition of complex numbers: $(a + ib) + (c + id) = (a + c) + i(b + d)$
Multiplication of complex numbers: $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$

Distance between $z = a + ib$ and 0: $|z| = \sqrt{a^2 + b^2}$. The number $|z|$ is the *modulus* of z.

Distance between $z_1 = a + ib$ and $z_2 = c + id$: $|z_1 - z_2| = \sqrt{(c - a)^2 + (d - b)^2}$

Theorem 5.1 – Fundamental Theorem of Algebra: Every nonconstant polynomial with coefficients in **C** (or **R**) has a root in **C**.

Theorem 5.2 – Factorization Theorem: Suppose p is a polynomial of degree $n \geq 1$ and with complex coefficients, so $p(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0$, with $c_n \neq 0$. Then p can be factored as a product of linear terms

$$p(z) = c_n(z - z_1)(z - z_2) \cdots (z - z_n)$$

where the numbers $z_1, z_2, \ldots, z_n$ are the roots of p. (Possibly some roots appear more than once.)

Derivative of f: If the limit exists, then $\displaystyle f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$.

Comment: Derivatives for complex polynomials and many other complex functions are obtained just like the corresponding ones for real functions. Thus if $f(z) = z^5$, then $df/dz = 5z^4$.

Infinite series: A complex infinite series has the form $\sum_{n=1}^{\infty} b_n$, where the terms $b_1, b_2, \ldots$ are complex numbers. The series converges to the complex number L provided that the partial sums s_n converge to L in the complex plane, that is, if

$$\lim_{n \to \infty} (b_1 + b_2 + \cdots + b_n) = \lim_{n \to \infty} s_n = L$$

Power series: A complex power series has the form $\sum_{n=0}^{\infty} a_n z^n$, where the terms a_n are complex numbers and z is a (complex) variable.

Power series for exponential, sine, cosine functions:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots$$
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots$$

Euler's Formula: For any complex number z, $e^{iz} = \cos z + i \sin z$

Polar form of z: $z = x + iy = R\cos\theta + iR\sin\theta = Re^{i\theta}$

Products in polar form: If $z_1 = R_1 e^{i\theta_1}$ and $z_2 = R_2 e^{i\theta_2}$, then

$$z_1 z_2 = R_1 e^{i\theta_1} R_2 e^{i\theta_2} = R_1 R_2 e^{i\theta_1} e^{i\theta_2} = (R_1 R_2) e^{i(\theta_1 + \theta_2)}$$

Thus for products, moduli are multiplied and angles are added.

Non-WebAssign Exercises:

1. Plot in the complex plane the following:
   (a) The points i, $2 - i, 2 + i$, -5
   (b) The collection of points z such that $|z| = 1/2$
   (c) The collection of points z such that $|z - 1 + 3i| = 2$
   (d) The points z, w, and zw, where $z = 3e^{i\pi/4}$ and $w = 2e^{4i\pi/3}$

2. Find all (complex) solutions of the given equation.
   (a) $4z^2 + 27 = 0$
   (b) $z^2 - z + 1 = 0$
   (c) $z^2 + 3z + 3 = 0$

3. Let $z = 1 - i$ and $w = \sqrt{3} - i$. Find the polar form of zw and z/w, and plot them in the complex plane.

4. Let $f(z) = -4z^5 + 3z^2 - 2z + 1/z$. Find the derivative of the function f.

5. Write out the first 5 terms of
   (a) $\{i^n\}_{n=1}^{\infty}$ (b) $\{(\frac{1+i}{2})^n\}_{n=1}^{\infty}$

6. Like for real power series, a complex power series $\sum_{n=0}^{\infty} c_n z^n$ either converges for all complex numbers z or converges only for $z = 0$, or else there is a positive (real) number R such that $\sum_{n=0}^{\infty} c_n z^n$ converges for $|z| < R$ and diverges for $|z| > R$. We write $R = \infty$ if the power series converges for all z, and $R = 0$ if the power series converges only for $z = 0$. We call R the *radius of convergence* of the power series. Find the radius of convergence R for

   (a) $e^z = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$ (b) $\displaystyle\sum_{n=0}^{\infty} z^n$ (c) $\displaystyle\sum_{n=0}^{\infty} (1+i)^n z^n$ (d) $\displaystyle\sum_{n=0}^{\infty} 3^n z^{2n}$