COMPLEX NUMBERS AND SERIES

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1. Complex Numbers

Definition 1.1. A complex number is a number z of the form $z = x + iy$, where x and y are real numbers, and i is another number such that $i^2 = -1$.

When $z = x + iy$ as above, x is called the *real part* of z, and y is called the *imaginary part* of z. We often write yi instead of iy. For example, $3 + i2 = 3 + 2i$, and the real part of $3 + 2i$ is 3 and the imaginary part of $3 + 2i$ is 2. There are some shortcut notations. For example, the complex number $3 + (-2)i$ is written as $3 - 2i$. Also, every real number is a complex number; for example, $7 = 7 + i(0)$.

The set of all real numbers is denoted by $\mathbb{R}$, and the set of all the complex numbers is denoted by $\mathbb{C}$.

2. The Complex Plane

Given real numbers a and b, we picture the complex number $a + ib$ as a point in the plane, with x coordinate a and y coordinate b (see Figure 1). The complex plane is just the usual, two dimensional plane, with the interpretation that a point (a, b) in the plane corresponds to the complex number $a + ib$.

Notice that the horizontal axis of the complex plane (the "real axis") corresponds to the set of real numbers. The vertical axis is called the "imaginary axis".

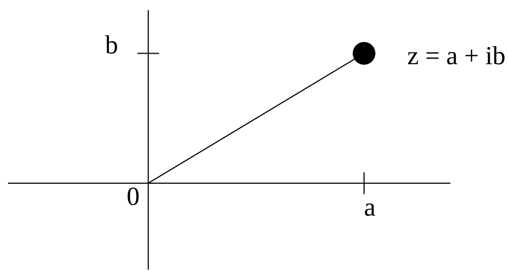

Figure 1. A complex number $a + ib$

3. Addition and Multiplication of Complex Numbers

The definitions of addition and multiplication of real numbers are extended to the complex numbers in the only reasonable way.

First, addition. Two complex numbers are added simply by adding together their real parts and imaginary parts: we define

$$(a + ib) + (c + id) = (a + c) + i(b + d) \ .$$

For example, $(3 + 2i) + (4 - 6i) = (7 - 4i)$.

Next, multiplication. For example, we will have

$$\begin{aligned}(2 + 3i)(4 + 5i) &= 2(4 + 5i) + 3i(4 + 5i) \\ &= 8 + 10i + 12i + 15i^2 \\ &= 8 + 10i + 12i - 15 \\ &= -7 + 22i \ .\end{aligned}$$

In general, the definition will be that for real numbers a, b, c, d

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc) \ .$$

With the definitions above of addition and multiplication, $\mathbb{C}$ enjoys all the good arithmetic properties of $\mathbb{R}$ (addition and multiplication are commutative and associative; the distributive property holds; etc.).

4. Why Complex Numbers Were Invented

In mathematics, we do a lot of solving of polynomial equations, which amounts to finding a root of a polynomial. For example, the solutions to the equations $x^2 = 1$ are the same as the solutions of $x^2 - 1 = 0$, that is, they are the roots of the polynomial $x^2 - 1$. These roots are $x = 1$ and $x = -1$. However, some polynomial equations have no real number solutions: for example, the equation

$$x^2 + 1 = 0$$

has no real number solutions, because $x^2 + 1 \geq 1$ if x is a real number.

The complex numbers were invented to provide solutions to polynomial equations. For example if we substitute i for x in the equation above, we get a solution:

$$i^2 + 1 = -1 + 1 = 0 \; .$$

In the early days, the boldness of simply defining a new number i as a solution was considered suspicious (hence the term "imaginary part"), just as the existence of $\sqrt{5}$ was a matter of religious controversy for the ancient Greeks. Today, the complex number system is so deeply rooted in physical theory (e.g. quantum mechanics) that one could argue the complex number system is a more "real" description of the world than the real number system. (The famous physicist Roger Penrose wrote an essay to this effect, "Nature is complex".) At any rate, students of today are expected to transcend in a blink the worries of past geniuses.

It is pretty easy (from the quadratic formula) to see that with complex numbers, we can find roots for *any* quadratic polynomial. For example, the two roots of $z^2 + 3z + 10$ are

$$z = \frac{-3 \; \pm \sqrt{3^2 - 4(10)}}{2} = \frac{-3 \; \pm \sqrt{-31}}{2} = \frac{-3}{2} \; \pm i\frac{\sqrt{31}}{2} \; .$$

However, since $\mathbb{C}$ is built from $\mathbb{R}$ basically by adding in just that one extra element i, and then just taking combinations $a + bi$, it is a rather amazing fact that ANY nonconstant polynomial with real coefficients (or even with complex coefficients) has a root which is a complex number. This fact (which we won't prove) is called the Fundamental Theorem of Algebra, stated next.

5. The Fundamental Theorem of Algebra

Theorem 5.1 (Fundamental Theorem of Algebra)**.** *Every nonconstant polynomial with coefficients in $\mathbb{C}$ (or $\mathbb{R}$) has a root in $\mathbb{C}$.*

There is an important corollary to the Fundamental Theorem of Algebra. For reference, we'll give a name (although often appeals to this corollary are expressed simply as appeals to the Fundamental Theorem):

Theorem 5.2 (Factorization Theorem)**.** *Suppose $p(z)$ is a polynomial of degree n at least 1, with complex coefficients, say $p(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_z a + c_0$, with $c_n \neq 0$.*

Then p can be factored as a product of linear terms

$$p(z) = c_n(z - z_1)(z - z_2) \cdots (z - z_n)$$

where the numbers $z_1, z_2, \ldots, z_n$ are the roots of $p(z)$. (Possibly some roots appear more than once.)

Here are some examples:

$$\begin{aligned} z^3 + 2z^2 + z &= z(z+1)^2 \ ; && \text{roots are } 0, -1, -1. \\ z^2 + 2z + 5 &= (z - [-1+2i])(z - [-1-2i]) \ ; && \text{roots are } -1 \pm 2i. \\ z^4 - 1 &= (z-1)(z+1)(z-i)(z+i) \ ; && \text{roots are } 1, -1, i, -i \ . \end{aligned}$$

For those interested, there is an appendix which explains how the Factorization Theorem follows from the Fundamental Theorem of Algebra and the polynomial long division algorithm.

Notice, no polynomial of degree n can have more than n distinct roots, because it cannot have more than n distinct linear factors.

6. The Geometry of Addition in $\mathbb{C}$

As in Figure 1, we view a complex number $a + ib$ as the point (a, b) in the plane. First we picture addition of complex numbers – it is just like vector addition in the plane. Addition of complex numbers is

$$(a + ib) + (c + id) \ = \ (a + c) + i(b + d)$$

and addition of vectors is

$$(a, b) + (c, d) \ = \ (a + c, b + d) \ .$$

A vector (a, b) can be thought of as an arrow or motion (go a to the right and b up). The addition of vectors (a, b) and (c, d) can be visualized as the addition of arrows by a head-to-tail rule (or as the motion you get by following the motion (a, b) with the motion (c, d)). See Figure 2.

7. Distance in the Complex Plane

If a and b are real numbers, then the absolute value of the complex number $z = a + ib$ is defined to be $|z| = \sqrt{a^2 + b^2}$. Notice, z is a real number when $b = 0$, and in this case the definition agrees with the usual definition of absolute value.

If you picture $z = a + ib$ in the complex plane as in Figure 1, then from the Pythagorean Theorem you can see that $|z|$ is the distance from z to the origin. (Put another way, $|z|$ is the length of the line segment between 0 and z.)

Similarly, if z_1 and z_2 are two complex numbers, then (just as with real numbers) the distance from z_1 to z_2 is $|z_2 - z_1|$. To see this, write z_1 and z_2 in the forms $z_1 = a + ib$ and $z_2 = (a + c) + i(b + d)$ (see Figure 2). Then $z_2 - z_1 = c + id$, and $|z_2 - z_1| = \sqrt{c^2 + d^2}$, the distance from z_1 to z_2.

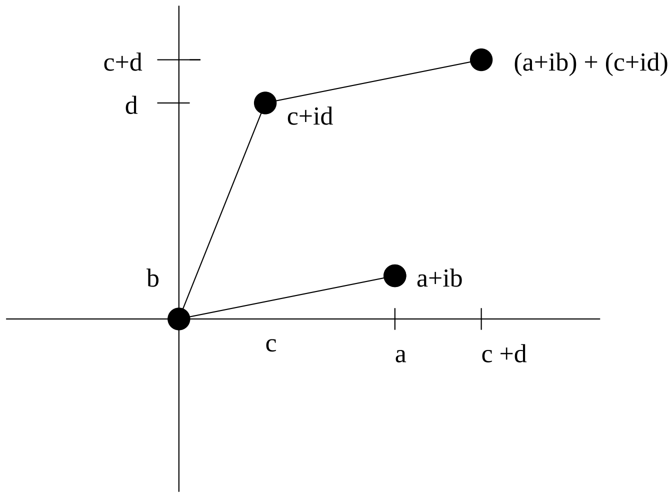

FIGURE 2. Addition of $a + ib$ and $c + id$

For example, the distance from $2 + 3i$ to $5 + 4i$ is $|3 + i| = \sqrt{10}$, and the distance from $2 + 3i$ to $-1 + 6i$ is $|(-1 - 2) + i(6 - 3)| = |-3 + 3i| = \sqrt{18}$.

8. Derivatives in $\mathbb{C}$

We can set up differential calculus for functions from $\mathbb{C}$ to $\mathbb{C}$ simply by making the natural definitions and then seeing that proofs for real-number calculus go over to complex-number calculus.

First we need to have an idea of what limits mean. We define that a sequence of complex numbers $z_1, z_2, z_3, \ldots$ converges to a complex number L when $\lim_{n\to\infty} |z_n - L| = 0$; in other words, when the distance from z_n to L goes to zero. See Figure 3 for a picture. Likewise, given a function f with complex number inputs and outputs, we define $\lim_{z\to z_0} f(z) = L$ if $\lim_{z\to z_0} |f(z) - L| = 0$.

Then we define $f'(z_0)$, the derivative of f at z_0, to be

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

whenever this limit exists. Notice this looks just like the real definition. The proofs in the real case for product rule, quotient rule, $(z^5)' = 5(z^4)$ and so on give the same facts for the complex case.

There is magic to complex number calculus which we won't reach in these notes, but setting up the basic definitions and initial facts for complex derivatives is just a matter of copying the real number ideas.

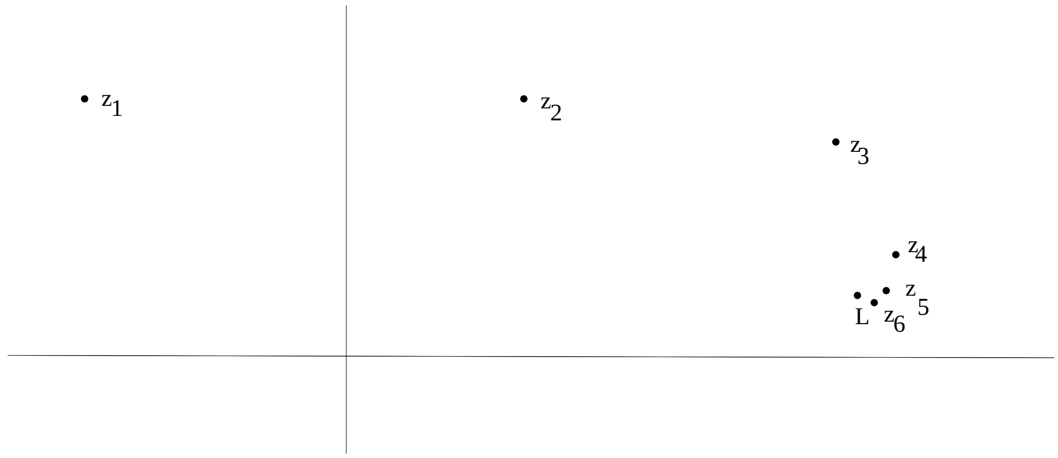

FIGURE 3. Part of a sequence of points z_n converging to L

9. Complex power series

A complex infinite series is an expression of the form

$$\sum_{n=1}^{\infty} b_n$$

where the terms $b_1, b_2, b_3 \ldots$ are complex numbers. (A special case is where the terms b_n are real numbers.) We say such a series converges to a complex number L if the sequence of partial sums s_n converges to L:

$$\lim_{n\to\infty} (b_1 + b_2 + b_3 + \cdots + b_n) \ = \ L \ .$$

The definitions are just like those for the real numbers. Only the picture looks different: instead of numbers s_n on a line getting closer and closer to a point, we have numbers s_n in the complex plane getting closer and closer to a point.

Power series with complex coefficients are defined just as you'd expect. A power series has the form

$$\sum_{n=0}^{\infty} a_n z^n$$

where the terms a_n are complex numbers and z is a variable. For complex series, just as for real series, absolute convergence implies convergence, and terms a_n must converge to zero if the series is to converge.

What we know about the geometric series goes over perfectly to the complex case:

$$\begin{aligned}\sum_{n=0}^{\infty} z^n &= \lim_{n\to\infty}(1+z+z^2+\cdots+z^{n-1}) \\ &= \lim_{n\to\infty}\frac{1-z^n}{1-z} \\ &= \frac{1}{1-z}\ , \quad \text{if } |z|<1\ .\end{aligned}$$

Just as in the real case, if $|z| \geq 1$, then the geometric series diverges.

For each complex power series, there is a radius of convergence R, just as in the real case. We have $0 \leq R \leq \infty$, and the series diverges if $|z| > R$, and converges if $|z| < R$. The arguments are essentially the same as in the real number case. Notice, the points z such that $|z| < R$ are the points in the plane with distance to the origin less than R. When $0 < R < \infty$, these are the points inside a circle of radius R. The radius of convergence, for complex series, is actually the radius of a circle: the circle of radius R centered at the origin.

10. Exp, cos, sin

We use the power series we found for the exponential, cosine and sine functions as real functions to give a definition of functions (still called exp, cos, sin) when the inputs are complex numbers:

$$\begin{aligned}e^z &= 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots+\frac{z^n}{n!}+\cdots \\ \cos(z) &= 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots \\ \sin(z) &= z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots\end{aligned}$$

The radius of convergence of all three of these series is infinity— they converge for every complex number z. Notice that the cosine series takes the even-degree terms from the exponential series and then alternates signs plus/minus, while the sine series takes the odd-degree terms from the exponential series and then alternates signs plus/minus.

Mathematicians like to say that good theorems become definitions. Above, the theorems are that certain familiar functions (cos, exp, sin in the real number case) can be given by power series. The definitions for the complex plane give us new tools for the "real" world.

It is not hard to see, after substituting iz for z in the exponential series, that a remarkable identity holds (DeMoivre's Formula): for any complex number z,

$$e^{iz} = \cos(z) + i\sin(z)\ .$$

When z is a real number θ, we have a complex number $\cos(\theta) + i\sin(\theta)$. This is a point in the complex plane with x-coordinate $\cos(\theta)$ and y-coordinate $\sin(\theta)$. That is, it is a point on the unit circle: $e^{i\theta}$ is a point on the unit circle. See Figure 4.

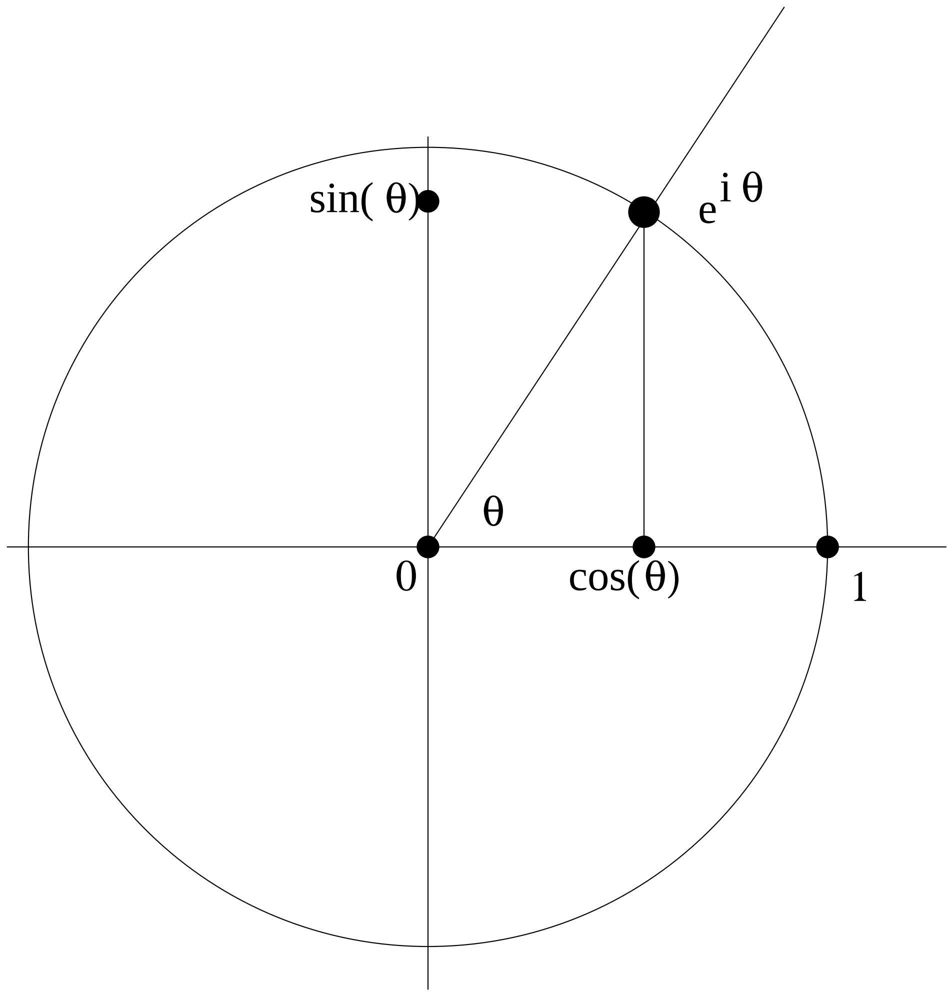

Figure 4. $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

For example: $e^{2\pi i} = 1$, $e^{i(\pi/2)} = i$, $e^{i\pi} = -1$, and

$$e^{\pi i} + 1 = 0 \ .$$

We regard the last equation as rather cool, because it is a single simple equation using only the fundamental numbers $e, \pi, i, 1$ and 0, and using each of these numbers exactly once.

The addition formula for the exponential function is valid for complex numbers as well as real numbers: for any complex numbers w and z,

$$e^w e^z \;=\; e^{w+z} \;.$$

The main point at this stage is simply to believe this. For the curious, there is an explanation in an Appendix.

11. Polar Decomposition

A point in the plane corresponds to an ordered pair (x, y). The distance to the origin is $\sqrt{x^2 + y^2}$; call this R. The point is on a circle of radius R centered at the origin. The corresponding complex number $z = x + iy$ has absolute value $|z| = R$. If the point is not the origin, then it makes some angle θ with the horizontal axis, and we have

$$x = R\cos(\theta), \qquad y = R\sin(\theta) \;\;.$$

Writing this in terms of the complex plane, we get

$$z = x + iy = R\cos(\theta) + iR\sin(\theta) \; = R(\cos(\theta) + i\sin(\theta)) = Re^{i\theta} \;.$$

This $Re^{i\theta}$ is the *polar form*, or *polar decomposition*, of z. The number (angle) θ is called the *argument* of z. Of course, any of the numbers $\theta, \theta \pm 2\pi, \theta \pm 2(2\pi), \theta \pm 3(2\pi), \ldots$ will serve as an angle for the point. We can always insist on a definite choice if needed (e.g., from the interval $[0, 2\pi]$, or from $(-\pi/2, 3\pi/2]$); but if no definite choice is specified, then any of these angles is acceptable as the argument.

For an example consider the complex number $z = -2 + 2i$. The absolute value of z is $\sqrt{(-2)^2 + 2^2} = \sqrt{8}$. The argument of z is $3\pi/4$. Therefore the polar form of z is $Re^{i\theta} = \sqrt{8}e^{i3\pi/4}$. See Figure 5.

12. The Geometry of Multiplication in $\mathbb{C}$

To picture multiplication geometrically, we use the polar form. Suppose we have two complex numbers z_1 and z_2. Consider their polar forms,

$$z_1 = R_1 e^{i\theta_1} \;, \quad z_2 = R_1 e^{i\theta_2} \;.$$

Then

$$z_1 z_2 = R_1 e^{i\theta_1} R_2 e^{i\theta_2} = R_1 R_2 e^{i\theta_1} e^{i\theta_2} = (R_1 R_2) e^{i(\theta_1 + \theta_2)} \;.$$

Consequently the polar form of $z_1 z_2$ is $(R_1 R_2)e^{i(\theta_1+\theta_2)}$. Therefore $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$, and $|z_1 z_2| = |z_1||z_2|$.

Let $Re^{i\theta}$ be a complex number in polar form. Multiplication by this complex number defines a function from $\mathbb{C}$ to $\mathbb{C}$: an input z goes to the output $Re^{i\theta}z$. This function can be understood geometrically in two steps. First, multiplication by $e^{i\theta}$ increases the argument of z by θ, which means that the function defined by multiplication by $e^{i\theta}$ rotates every point in the plane through an angle θ. Second, multiplication by the positive number R multiplies the absolute value (distance to the origin) by R.

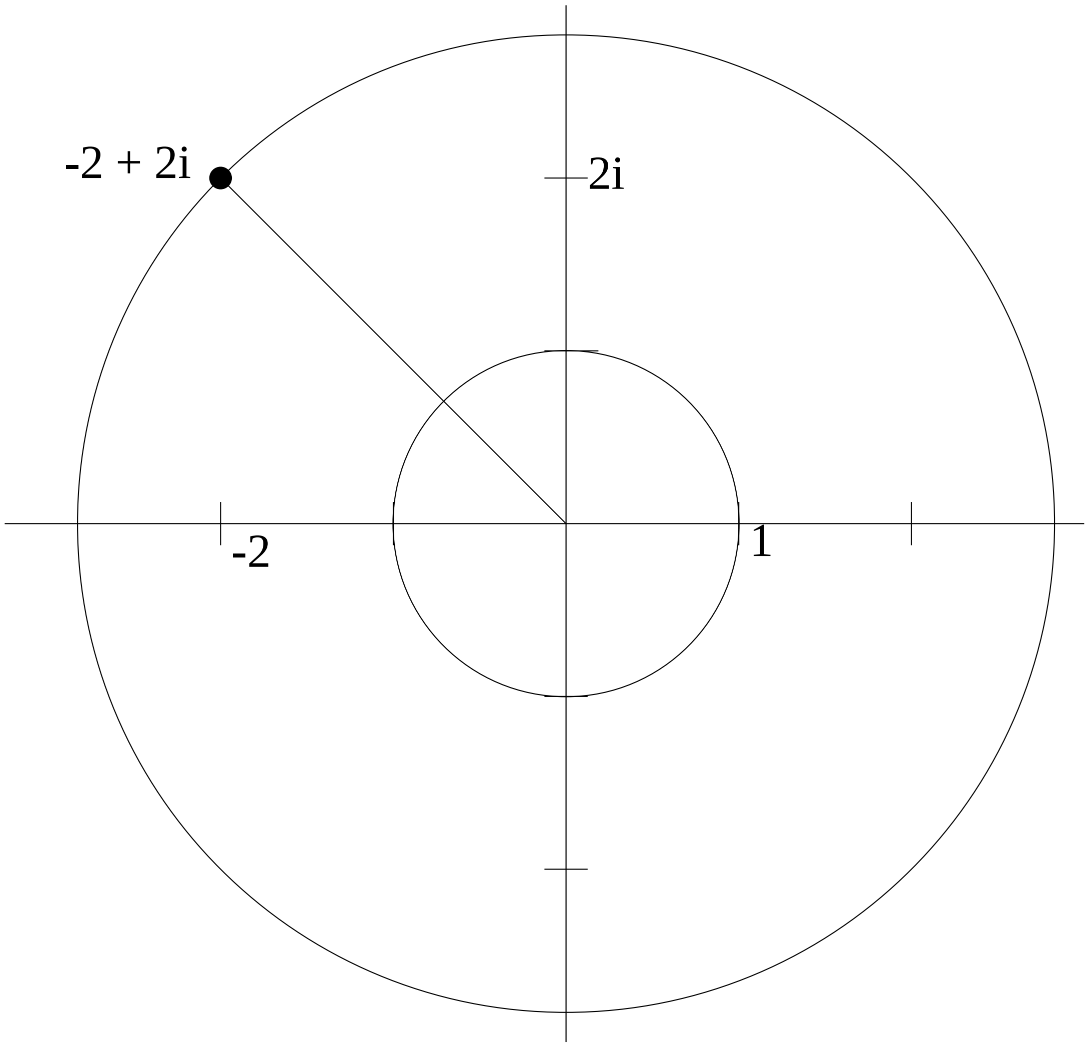

FIGURE 5. Polar form of $-2+2i$ is $\sqrt{8}e^{i3\pi/4}$

13. Trig identities

Let us prove the addition formulas for cosine and sine using complex numbers and power series. Suppose a and b are real numbers. We have

$$\begin{aligned} e^{i(a+b)} &= [e^{ia}][e^{ib}] \\ \cos(a+b)+i\sin(a+b) &= [\cos(a)+i\sin(a)][\cos(b)+i\sin(b)] \end{aligned}$$

If we multiply out the right hand side above, we get

$$[\cos(a)\cos(b)-\sin(a)\sin(b)] \ + \ i[\cos(a)\sin(b)+\sin(a)\cos(b)] \ .$$

When two complex numbers are equal, their real parts are equal and their imaginary parts are equal. Because a and b are real, the above equation then implies the addition formulas for cosine and sine:

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\sin(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b)$$

Many more trig identities can be found in a similar way (you will find one in one of the exercises). This is an example of how understanding in the complex plane gives us better understanding in the real number case.

14. Exercises

Below, when a complex number is written in a form like $a + ib$, then we are assuming that the numbers a, b are real numbers.

1. Plot the following points in the complex plane: $i, 2-i, 2+i, -5$.

2. (i) Compute $|7-2i|$.
(ii) Compute the distance from $7-2i$ to 0 in the complex plane.
(iii) Compute $|2e^{6i}|$.
(iv) Compute $|(2e^{6i})(7-2i)|$.

3. (Multiplicative inverse) Let $z = a + ib$ be a nonzero complex number (so at least one of the real numbers a, b is nonzero). Then $1/z$ is the number such that $(z)(1/z) = 1$, and there is only one such number.
(i) Show that

$$1/z = \frac{a - ib}{a^2 + b^2} \ .$$

[Hint: Just show that multiplying the right hand side by $a+ib$ produces the number 1; then the right hand side must be a correct formula for $1/z$.]
(ii) Compute real numbers a, b such that $1/(2+3i) = a + ib$.
(iii) Compute real numbers a, b such that $(1-2i)/(2+3i) = a + ib$.
(iv) If the polar form of z is $Re^{i\theta}$, then what is the polar form of $1/z$?

4. Let $z = 2 + i2$.
(i) Find the polar form of z (i.e., find real numbers R and θ such that $R > 0$ and $2 + i2 = Re^{i\theta}$).
(ii) Find real numbers c and d such that $1/z = c + id$.
(iii) Find the polar form of $1/z$.

5. Now consider a complex number z written in various forms: $z = x + iy = e^{a+ib} = Re^{i\theta}$, where x, y, a, b, R and θ are real numbers.
(i) Give formulas using x and y for R and $\tan(\theta)$. For which z are the formulas valid?
(ii) Give formulas for R and θ in terms of x and y.
(iii) Compute the polar form of e^{2-3i}.

6. (i)Find a trig identity which for any real numbers θ expresses $\cos(3\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$. [Hint: $e^{i3\theta} = e^{i\theta}e^{i\theta}e^{i\theta}$, so $\cos(3\theta) + i\sin(3\theta) = (\cos(\theta) + i\sin\theta)(\cos(\theta) + i\sin\theta)(\cos(\theta) + i\sin\theta)$. Compare real parts.]

(ii) Then use the identity $\cos^2(\theta)+\sin^2(\theta) = 1$ to find an identity involving only cosine: find numbers a and b such that $\cos(3\theta) = a\cos(\theta) + b\cos^3\theta$.

(iii) Check that your formula in (ii) is true at $\theta = \pi/4$ and $\theta = \pi$.

7. (Complex conjugates) Let $z = a + ib$; then the complex conjugate $\overline{z}$ is defined to be $\overline{z} = a - ib$.

(i)How are the locations of z and $\overline{z}$ in the complex plane related?

(ii) Check that $z\overline{z} = |z|^2 = a^2 + b^2$.

(iii) Show that if z is nonzero, then $1/z = (\overline{z})/(|z|^2)$. (Multiply z by this expression and check that the product is 1.)

(iv) Use the formula in (iii) to find $1/z$ if $z = 2 + 3i$.

8. (Complex conjugation respects arithmetic)

(i) Show that $(\overline{w})(\overline{z}) = \overline{wz}$. To do this, given real numbers a, b, c, d, simply compute to check that $(a - ib)(c - id) = \overline{(a + ib)(c + id)}$.

(ii) Similarly check that $\overline{w} + \overline{z} = \overline{w + z}$.

(iii) Use (i) to give an elementary proof that for any complex numbers w and z, we have $|wz| = |w||z|$. (HINT: $|wz|^2 = (wz)(\overline{wz})$, and $|w|^2|z|^2 = w\overline{w}z\overline{z}$.)

9. From the last problem, it follows that if $p(z)$ is a polynomial with real coefficients and w is a complex number and $p(w) = 0$, then also $p(\overline{w}) = 0$.

(i) Check that any polynomial of the form $q(z) = (z - w)(z - \overline{w})$ is a polynomial with real coefficients.

(ii) (Real Factorization Theorem) Deduce using the Factorization Theorem that any nonconstant polynomial with only real coefficients can be factored as a product of polynomials with only real coefficients and with degree one or two.

10. Consider the complex numbers $w = 3e^{i\pi/4}$ and $z = 2e^{i7\pi/8}$. Plot these two numbers and also the number wz in the complex plane. Also plot $2e^{6i}$.

11. (Roots of unity) Let n be a positive integer. The complex numbers $e^{2\pi i/n}$ has its nth power equal to 1. Likewise, if k is a nonnegative integer in the set $0, 1, \ldots, n-1$, then $e^{2\pi ik/n}$ also has its nth power equal to 1. Such a number is called an *nth root of unity.* These numbers can be drawn on the unit circle in the complex plane.

(i) Draw all the fourth roots of unity on the unit circle.

(ii) Draw (in another picture) all the eighth roots of unity.

12. Let $z = -1 + i$.

(i)Write z in polar form.

(ii) Use the polar form to compute z^{16}.

13. If n is a positive integer and M is a positive real number, then the equation $z^n = M$ has exactly the following n solutions: $M^{1/n}e^{2\pi ki/n}$, $k = 0, 1, 2, \ldots, n - 1$.

Find all solutions of the equation $z^8 = 16$, and plot these solutions in the complex plane.

14. (Differentiation) All the usual formulas for differentiation work for polynomials, cosine, sine and the exponential function considered over complex numbers. For example, since $e^{iz} = \cos(z) + i\sin(z)$, we can say the derivative of e^{iz} with respect to z is $ie^{iz} = -\sin(z) + i\cos(z)$. (You can check this one by differentiating the power series.) These formulas in particular are true if we restrict inputs to real numbers (in this case one often writes t in place of z).

Compute the second derivative of e^{it} at the input $t = \pi/2$.

15. (DeMoivre) To understand why $e^{iz} = \cos(\mathrm{z}) + i\sin(z)$, compute by hand the first eight terms of these series, and compare.

16. (Geometric series) Let z be the complex number $z = 1/2 + i/2$.
(i) Verify that $|z| < 1$.
(ii) Find the real numbers a and b such that

$$\sum_{n=0}^{\infty} z^n = a + ib \ .$$

Appendix A. Explanation of the Factorization Theorem

Suppose $p(z) = c_n z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0$ is a polynomial, of degree n at least 1, with complex coefficients (which, again, as a special case could be just real numbers). Polynomial long division works just as well with complex coefficients as with real coefficients. So, given a particular complex number z_0, we could use the polynomial long division to find

$$\frac{p(z)}{z - z_0} = q(z) + \frac{w}{z - z_0}$$

for some polynomial $q(z)$ and some complex number w. By multiplying both sides by $(z - z_0)$, we see this equation givs us

$$p(z) = q(z)(z - z_0) + w \ .$$

By substituting z_0 for z in this last line, you can see that $w = p(z_0)$. For example,

$$\frac{z^4 - 1}{z - 2} = z^3 + 2z^2 + 4z + 8 + \frac{15}{z - 2}$$

and this produces

$$z^4 - 1 = (z^3 + 2x^2 + 4z + 8)(z - 2) + 15 \ .$$

In the example above, where $p(2) = 15$, the number 2 is not a root of $p(z)$ and we got the proper remainder 15 in the last line. If instead of $z - 2$ we use a $z - z_0$ where z_0 is a root of $p(z)$, then we will have zero remainder, and $z - z_0$ will be a factor of $p(z)$. The point: if $p(w) = 0$, then $(z - w)$ is a factor of $p(z)$.

For example, using $p(z) = z^4 - 1$, we see that the complex number i is a root. Doing the polynomial long division, we could find

$$\frac{z^4 - 1}{z - i} = z^3 + iz^2 - z + i \text{ , and} z^4 - 1 = (z^3 + iz^2 - z + i)(z - i)$$

Applying the same procedure to the polynomial $z^3 + iz^2 - z + i$ and one of its roots, we could factor out another linear term, and then another, to end up with

$$z^4 - 1 = (z - 1)(z + 1)(z - i)(z + i) \ ,$$

the factorization which corresponds to the four roots $1, -1, i, -i$ of the polynomial $z^4 - 1$. This approach works on any polynomial to produce a factorization into linear terms, as stated in the Factorization Theorem.

Appendix B. The addition formula for the exponential function

Let w and z be any two complex numbers. We will explain the addition formula,

$$e^w e^z = e^{w+z} \ .$$

Our approach will be to look at the two sides as infinite series, and check that the same terms are being added up on each side.

First, because we are defining the exponential function with power series, we have

$$e^w e^z \ = \ \Big(\lim_{n\to\infty} 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots + \frac{w^n}{n!}\Big)\Big(\lim_{n\to\infty} 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!}\Big)$$

Next, because the limit of the product is the product of the limits, we have

$$e^w e^z \ = \ \lim_{n\to\infty} \left[\Big(1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots + \frac{w^n}{n!}\Big)\Big(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!}\Big)\right]$$

Next, we multiply out the terms in the product above,

$$\begin{aligned} e^w e^z \ = \ \lim_{n\to\infty} \Bigg[& 1 + \Big(w + z\Big) \ + \ \Big(\frac{w^2}{2!} + wz + \frac{z^2}{2!}\Big) \ + \ \Big(\frac{w^3}{3!} + \frac{w^2}{2!}z + w\frac{z^2}{2!} + \frac{z^3}{3!}\Big) \\ & + \cdots + \ \Big(\cdots + \frac{w^k}{k!}\frac{z^\ell}{\ell!} \ + \cdots\Big) + \cdots \Bigg] \end{aligned}$$

Similarly we have

$$\begin{aligned} e^{w+z} \ &= \ 1 + \Big(w + z\Big) \ + \ \frac{(w+z)^2}{2!} \ + \ \frac{(w+z)^3}{3!} \ + \ \cdots + (w+z)^n \ + \ \cdots \\ &= \ 1 \ + \ \Big(w + z\Big) \ + \ \Big(\frac{w^2}{2!} + \frac{2wz}{2!} + \frac{z^2}{2!}\Big) \ + \ \Big(\frac{w^3}{3!} + \frac{3w^2z}{3!} + \frac{3wz^2}{3!} + \frac{z^3}{3!}\Big) \\ & \quad + \ \Big(\cdots + \binom{n}{k}\frac{w^k z^{n-k}}{n!} + \cdots\Big) + \cdots \end{aligned}$$

In the last expression, that expression $\binom{n}{k}$ is called "n-choose-k" and it is given by the formula

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!} .$$

When we multiply out $(w+z)^n$ and consolidate terms, the Binomial Theorem tells us that we get $\binom{n}{k}$ as the coefficient of $w^k z^{n-k}$.

Now consider a given pair of nonnegative integers k, ℓ. In the infinite series defining $e^w e^z$, there is exactly one term of the form constant times $w^k z^\ell$, and the constant is $1/(k!\ell!)$. On the other hand, in the infinite series defining e^{w+z}, there is also exactly one term of the form constant times $w^k z^\ell$. This term occurs as part of $\frac{(w+z)^n}{n!}$, for exactly one n, which is $n = k + \ell$. In this case, $n - k = \ell$, and in the expansion of $\frac{(w+z)^n}{n!}$ the coefficient of $w^k z^\ell$ will be

$$\Big[\frac{1}{n!}\Big]\binom{n}{k} = \Big[\frac{1}{n!}\Big]\Big[\frac{n!}{k!(n-k)!}\Big] = \frac{1}{k!(n-k)!} = \frac{1}{k!\ell!} .$$

This matches the coefficient of $w^k z^\ell$ in the series defining $e^w e^z$.

Therefore the series for $e^w e^z$ and the series for e^{w+z} are defined by adding up the same terms. As you would expect, this tells us that $e^w e^z = e^{w+z}$.

(The very alert student will notice that the order of appearance of the terms $w^k z^\ell/(k!\ell!)$ is not the same in the two series. The order of appearance, it turns out, will not affect the sum of a series if the series converges absolutely, which is the situation we have here. We won't go into this technical point.)

Having proved the addition formula by direct examination from the definitions, let us consider proving it in a different way, with a slick trick. We are regarding the complex numbers w, z as constants. Define the function $f(t) = e^{zt}e^{z+w-zt}$. This is a function defined for any complex number inputs. The product formula for differentiation, with the chain rule, tells us that for any input t,

$$f'(t) = ze^{zt}e^{z+w-zt} - ze^{zt}e^{z+w-zt} = 0 .$$

Just as for the case of a function from $\mathbb{R}$ to $\mathbb{R}$, if a function from $\mathbb{C}$ to $\mathbb{C}$ has derivative identically zero, then that function must be a constant function. (We won't prove this fact here.) In particular, $f(0) = f(1)$, and therefore $e^{z+w} = e^z e^w$.

Appendix C. Complex notation for parametrized curves

Here we will consider how complex number notation can be used to describe parametrized curves (discussed in Chapter 10 in the Ellis and Gulick text).

Suppose an object moves in the plane, and at time t the object is at position $P(t)$. This is a parametrization of a curve. For example, if $P(t) =$

$(\cos(t), \sin(t))$, then the curve is the unit circle, and the parametrization describes a motion counterclockwise around the unit circle. Here there is a complete trip around the circle as t goes from 0 to 2π.

If we regard the plane as the complex plane, then we can describe the same motion in complex-number notation: the position at time t is $\cos(t)+i\sin(t)$, which is e^{it}. Similarly, to describe a counterclockwise motion around a circle of radius 3, we could say the position at time t is $3e^{it}$, rather than $(3\cos(t), 3\sin(t))$. If the motion is the same but four times as fast, we'd use $3e^{i4t}$, rather than $(3\cos(4t), 3\sin(4t))$.

It is often useful to employ such notation, replacing the trig functions with complex exponentials, because the exponentials tend to be more concise and more simple for computations. Here are some examples.

Velocity.

We have $P(t) = (x(t), y(t))$. The velocity vector $v(t)$ for the motion described by $P(t)$ is defined to be

$$v(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) .$$

Let us follow the example $P(t) = (3\cos(4t), 3\sin(4t))$. Clearly, $v(t) = (-12\sin(4t), 12\cos(4t))$. If we present the parametrization in complex notation as

$$P(t) = 3\cos(4t) + i3\sin(4t)$$

and compute the derivatives, then we get

$$v(t) = -12\sin(4t) + i12\cos(4t) .$$

You can see we're basically just doing the same thing as before, but letting the "i" separate terms instead of using a comma.

HOWEVER! we know that as functions of a complex variable

$$3e^{i4t} = 3\cos(4t) + i3\sin(4t) .$$

Therefore the derivatives of both sides will be equal (as functions of a complex variable), and it follows that

$$12ie^{i4t} = -12\sin(4t) + i12\cos(4t) .$$

(You can doublecheck this from knowing $12ie^{i4t} = 12i(\cos(4t) + i\sin(4t).)$ In sum, we can differentiate $P(t)$ just by differentiating $3e^{i4t}$, going back to the (x, y) notation for points whenever we want to.

Speed.

Given $P(t) = (x(t), y(t))$, the speed for the motion is defined to be

$$|v(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} .$$

With the same example $P(t) = 3e^{i4t}$, we have $|v(t)| = |12ie^{i4t}| = 12$. This is even easier than the computation with sines and cosines.

A more general example.

Let $P(t) = e^{(2+3i)t}$. (In real number notation this would be written as $P(t) = (e^{2t}\cos(3t), e^{2t}\sin(3t).)$ Then $v(t) = (2+3i)e^{2+3it}$, and the speed at time t is

$$|v(t)| \; = \; |(2+3i)e^{2+3it}| \; = \; |2+3i| \cdot |e^2| \cdot |e^{3it}| \; = \; \sqrt{13}e^2 \; .$$

Also, the acceleration vector is the derivative (studied in MATH 241) is the derivative of the velocity vector and for this example it would be $(2+3i)^2e^{2+3it} = (-5+12i)e^{2+3it}$.