D.H. HAMILTON:

COMPLEX VARIABLES

Department of Mathematics
University of Maryland
2009

A complex number is $z = x + \mathbf{i}y$, where x and y are real numbers, and $\mathbf{i}$ is the imaginary number $\sqrt{-1}$. This is just another way to write a point $(x,\ y)$ of the plane. With this notation the plane is the *complex plane* $\mathbb{C}$. Then with $z = x + \mathbf{i}y$ as above, $x = \Re(z)$ is called the real part of z, and $y = \Im(z)$ is called the imaginary part of z. The x axis is the real axis (which allows us to think of the real numbers $\mathbb{R}$ as a subset) and the y axis is the "imaginary axis." For example the real part of $3+\mathbf{i}2$ is 3 and the imaginary part 2. Often we write $y\mathbf{i}$ instead of $\mathbf{i}y$. There are other shortcut notations. For example, the complex number $3 + \mathbf{i}(-2)$ may be written as $3 - 2\mathbf{i}$. Also, every real number is a complex number; for example, $7 = 7 + \mathbf{i}(0)$. Furthermore, $z = 0$ means that the real part $x = 0$ and the imaginary part $y = 0$. However the complex plane is much more than being just different notation for the plane. This is because there is an addition and multiplication so we can do algebra like the ordinary numbers:

1 Algebra

The definitions of addition and multiplication of real numbers are extended to the complex numbers in the only reasonable way.

First, addition. Two complex numbers are added simply by adding together their real parts and imaginary parts: we define

$$(a + \mathbf{i}b) + (c + \mathbf{i}d) = (a + c) + \mathbf{i}(b + d).$$

For example, $(3 + 2\mathbf{i}) + (4 - 6\mathbf{i}) = (7 - 4\mathbf{i})$.

Next, multiplication. As we assume $\sqrt{-1}\sqrt{-1} = \mathbf{ii} = -1$

$$(2 + 3\mathbf{i})(4 + 5\mathbf{i}) = 2(4 + 5\mathbf{i}) + 3\mathbf{i}(4 + 5\mathbf{i}) = 8 + 10\mathbf{i} + 12\mathbf{i} + 15\mathbf{ii} = -7 + 22\mathbf{i}.$$

In general, the definition will be that

$$(a + \mathbf{i}b)(c + \mathbf{i}d) = (ac - bd) + \mathbf{i}(ad + bc)$$

With these definitions $\mathbb{C}$ enjoys all the usual arithmetical properties (e.g. addition and multiplication are commutative and associative; the distributive property holds; etc.).

Also we see that every $z = x + \mathbf{i}y \neq 0$ has a multiplicative inverse

$$z^{-1} = \frac{1}{x+\mathbf{i}y} = \frac{1}{x+\mathbf{i}y}\frac{x-\mathbf{i}y}{x-\mathbf{i}y} = \frac{x-\mathbf{i}y}{x^2+y^2},$$

i.e.

$$z^{-1} = \frac{x}{x^2+y^2} - \mathbf{i}\frac{y}{x^2+y^2}$$

For example

$$\frac{1}{2+\mathbf{i}} = \frac{2}{5} - \mathbf{i}\frac{1}{5}$$

In theory one complex equation for z can be converted into two real equations but this is not very effective, e.g. solve

$$(2+\mathbf{i})z = 1 - \mathbf{i}$$

which by sticking to complex notation has solution

$$z = \frac{1-\mathbf{i}}{2+\mathbf{i}} = (1-\mathbf{i})\left(\frac{2}{5} - \mathbf{i}\frac{1}{5}\right) = \frac{2}{5} - \frac{1}{5} - \mathbf{i}\frac{2}{5} - \mathbf{i}\frac{1}{5} = \frac{1}{5} - \mathbf{i}\frac{3}{5}$$

This is much easier than writing $z = x + \mathbf{i}y$ and converting the equation $(2+\mathbf{i})(x+\mathbf{i}y) = 1 - \mathbf{i}$ into two real equations

$$\begin{array}{rcrcr} 2x & - & y & = & 1 \\ x & + & 2y & = & -1 \end{array}$$

2 Geometry

Naturally the complex plane has a geometric side which is closely connected to its algebra. The "complex conjugate":

$$\overline{z} = \overline{x+\mathbf{i}y} = x - \mathbf{i}y\ ,$$

gives the reflection $\rho(z) = \overline{z}$ in the real axis. One has the two fundamental properties:

$$\overline{z+w} = \overline{z} + \overline{w},\ \overline{zw} = \overline{z}\,\overline{w}\ ,$$

furthermore $\Re(z) = (z+\overline{z})/2,\ \Im(z) = (z-\overline{z})/(2\mathbf{i})$.

Exercises

1. Compute $\overline{(1+2\mathbf{i})(1-2\mathbf{i})}$

2. Check the two fundamental properties of complex conjugate

3. Determine if it is true that

$$\overline{\left\{\frac{1}{z}\right\}} = \frac{1}{\overline{z}}$$

2.1 Modulus

To measure the distance between the points $(0,0),(x,y)$ we use Pythagorus:

$$|z| = |x+iy| = \sqrt{x^2+y^2}\ .$$

This has such special properties for complex numbers we call it the modulus or "mod" rather then distance. For obvious reasons the modulus is also sometimes called absolute value. As $|z|^2 = z\overline{z}$ we use the complex conjugates to show

$$|zw| = |z|\,|w|\ .$$

which gives an easy proof of the Triangle inequality:

$$|z-w| \leq |z| + |w|\ .$$

For by previous

$$\begin{aligned} |z-w|^2 &= (z-w)\overline{(z-w)} \\ &= z\overline{z} - z\overline{w} - w\overline{z} + w\overline{w} \\ &\leq |z|^2 + 2|z\overline{w}| + |w|^2 \\ &= |z|^2 + 2|z|\,|w| + |w|^2 \\ &= (|z|+|w|)^2\ , \end{aligned}$$

where we used the fact that

$$z\overline{w} + w\overline{z} = 2\Re(z\overline{w}) \leq 2|z\overline{w}|$$

Exercises

1. Compute
$$\left|\frac{(1-\mathbf{i})(1+2\mathbf{i})(5+5\mathbf{i})}{(1-2\mathbf{i})(1+\mathbf{i})(1+\mathbf{i})}\right|$$
2. Show that for any complex number z with $|z|=1$
$$\overline{z}=\frac{1}{z}$$
3. Sketch all points z so that $|z-(1+\mathbf{i})|=3$.

2.2 Arguments

This brings us to the geometric realization of a complex number z as a vector of length $r=|z|$ (its modulus) making an angle θ with the OX axis. θ is called the argument of z (or "arg"). We know that angles are real numbers regarded as equal if they differ by integer multiples of 2π. The argument of a nonzero complex number z is defined by considering $w=z/|z|=x+\mathbf{i}y$ which is a complex number of modulus 1. Now by considering the circle $x^2+y^2=1$ we see there is a number θ, in the half open interval $(-\pi,\pi]$:

$$\cos(\theta)=x,\ \sin(\theta)=y.$$

This defines the Argument, $\theta=\mathrm{Arg}(z)$, of z. The argument, $\arg(z)$, is the class of $\theta+2\pi n$, where n is an integer $0,\pm1,\pm2,\ldots.$

THEOREM 1 *Suppose that z,w are nonzero complex numbers with moduli s,t, and with arguments θ,ϕ respectively. Then* $\arg(zw)=\theta+\phi$.

Now by definition $z=s(\cos(\theta)+\mathbf{i}\sin(\theta)),\ w=t(\cos(\phi)+\mathbf{i}\sin(\phi))$. Hence

$$\begin{aligned} zw &= s(\cos(\theta)+\mathbf{i}\sin(\theta))\,t(\cos(\phi)+\mathbf{i}\sin(\phi))\ , \\ &= st\left\{(\cos(\theta)\cos(\phi)-\sin(\theta)\sin(\phi))+\mathbf{i}(\sin(\theta)\cos(\phi)+\cos(\theta)\sin(\phi))\right\}\ , \\ &= st(\cos(\theta+\phi)+\mathbf{i}\sin(\theta+\phi))\ , \end{aligned}$$

by trig identities.

As a corollary we have the very useful "de Moivre's formula":

$$(\cos(\theta)+\mathbf{i}\sin(\theta))^n=\cos(n\theta)+\mathbf{i}\sin(n\theta)\ ,$$

for any integer n.

Exercises

1. Compute $\arg(-1+\mathbf{i})^5$.

2. Find complex numbers z, w so

$$\mathrm{Arg}(zw) \neq \mathrm{Arg}(z) + \mathrm{Arg}(w)\ .$$

3. Use De Moivre to show that for $n = 1, 2...$ $\cos(n\theta) = P_n(\cos(\theta))$ where P is a polynomial of degree n.

3 Roots of Polynomials

Polynomials over the complex numbers z of degree n may be defined as

$$p(z) = a_n\, z^n + ... + a_1\, z + a_0$$

where the coefficients $a_k \in \mathbf{C}$ and $a_n \neq 0$. Finding the roots of polynomials, i.e. ζ so that $p(\zeta) = 0$ has been of importance in science for hundreds of years. We have already discussed the linear case $a_1 z + a_0 = 0$ and know that it has a single solution $-a_0/a_1$. Of course we introduced the complex numbers to solve $z^2 + 1 = 0$ so we'd like to know what other equations can be solved by complex numbers. The simplest are the power equations

$$z^n = w$$

for given complex w and number n. Using polar form

$$w = R(\cos(\phi) + \mathbf{i}\sin(\phi))$$

with R, ϕ known and

$$z = r(\cos(\theta) + \mathbf{i}\sin(\theta))$$

for unknown r, θ. Then d'Moivre gives

$$r^n(\cos(n\theta) + \mathbf{i}\sin(n\theta)) = R(\cos(\phi) + \mathbf{i}\sin(\phi))$$

so hence for some integer k

$$r^n = R,\ n\theta = \phi + 2\pi k$$

Therefore we find

$$r = R^{1/n},\ \theta = \frac{\phi}{n} + \frac{2\pi k}{n}$$

where $k = 0, 1, ...n - 1$ (as other values of k just repeat). For example we can solve

$$z^3 = -\sqrt{2} + \mathbf{i}\sqrt{2}$$

Now $|-\sqrt{2} + \mathbf{i}\sqrt{2}| = 2, \arg(-\sqrt{2} + \mathbf{i}\sqrt{2}) = 3\pi/4 + 2\pi k.$

So if $z = r(\cos(\theta) + i\sin(\theta))$ we have

$$r^3 = 2, 3\theta = 3\pi/4 + 2\pi k$$

so $r = 2^{1/3}, \theta = \pi/4 + 2\pi k/3$. Therefore the three roots of $-\sqrt{2} + \mathbf{i}\sqrt{2}$ are

$$\begin{array}{lll} z_0 & = 2^{1/3}(\cos(\pi/4) + \mathbf{i}\sin(\pi/4)) & = 2^{-1/6}(1 + \mathbf{i}) \\ z_1 & = 2^{1/3}(\cos(\pi/4 + 2\pi/3) + \mathbf{i}\sin(\pi/4 + 2\pi/3)) & = 2^{-7/6}(-(1 + \sqrt{3}) + \mathbf{i}(-1 + \sqrt{3})) \\ z_2 & = 2^{1/3}(\cos(\pi/4 + 4\pi/3) + \mathbf{i}\sin(\pi/4 + 4\pi/3)) & = 2^{-7/6}((-1 + \sqrt{3}) - \mathbf{i}(1 + \sqrt{3})) \end{array}$$

In particular we see that we can find two complex square roots $\pm\sqrt{w}$ for any complex $w \neq 0$. Therefore by completing the square as usual we find that any *complex* quadratic equation $az^2 + bz + c = 0$ has two roots

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Back in the 16^{th} century similar formulas were discovered for 3^{rd} and 4^{th} degree polynomials. This lead to the conjecture that every n^{th} degree polynomial has n (counting multiplicity) roots. We discuss this history in the appendix. But the short answer is the Fundamental Theorem of Algebra:

THEOREM 2 (FTA)
Every polynomial p of degree n has n roots ζ_k in the complex plane, i.e.

$$p(z) = a(z - \zeta_1)(z - \zeta_2)..(z - \zeta_n)$$

Exercises

1. Find the roots of the equation $z^2 + 2z - \mathbf{i} = 0$.
2. Find the roots of the equation $(z - 1)^4 = z^4$.
3. Factorize $z^5 + 1$ into linear terms.

4 Sequences

Once we have the distance between points we can talk about limits and convergence of sequences etc. A sequence of complex numbers can be written $z_n = x_n + \mathbf{i}y_n$. So $\lim_{n\to\infty} z_n = w = u + \mathbf{i}v$ means $\lim_{n\to\infty} |z_n - w| = 0$ which is an ordinary real limit of calculus. Of course this is just the same as $\lim_{n\to\infty} x_n = u$, $\lim_{n\to\infty} y_n = v$, i.e two real limits.

For example with fixed $a + \mathbf{i}b \in \mathbb{C}$ we define the sequence

$$z_n = \left(1 + \frac{a + \mathbf{i}b}{n}\right)^n .$$

Let us write

$$1 + \frac{a + ib}{n} = r_n(\cos(\theta_n) + \mathbf{i}\sin(\theta_n))$$

where

$$r_n = \sqrt{(1 + \frac{a}{n})^2 + (\frac{b}{n})^2} = \sqrt{1 + 2\frac{a}{n} + \frac{a^2 + b^2}{n^2}}$$

and

$$\theta_n = \tan^{-1}\left(\frac{b}{n + a}\right)$$

Therefore if $z_n = R_n(\cos(\phi_n) + i\sin(\phi_n))$ by de Moivre we find

$$R_n = (r_n)^n = \left(1 + 2\frac{a}{n} + \frac{a^2 + b^2}{n^2}\right)^{n/2} \to e^a$$

and

$$\phi_n = n\theta_n = n\tan^{-1}\left(\frac{b}{n + a}\right) \to b$$

Therefore

$$\left(1 + \frac{a + \mathbf{i}b}{n}\right)^n \to e^a\left(\cos(b) + \mathbf{i}\sin(b)\right)$$

Exercises

For which complex numbers z does the sequence $\omega_n = z^n$ converge?

4.1 Series

Series are just sequences of partial sums of $\sum_{n=0}^{\infty} \omega_n$. As with sequences the whole question of convergence comes down the convergence of the real and imaginary parts. So it is no surprise that complex series have exactly the same criteria for convergence that series of real (nonpositive) numbers have:

A complex series $\sum_{n=0}^{\infty} \omega_n$ converges if

1. $|\omega_n| < a_n$ for convergent positive series $\sum_{n=0}^{\infty} a_n$ "absolute convergence"
2. $|\omega_{n+1}|/|\omega_n| \leq r < 1, n > N$ "ratio test"
3. $|\omega_n|^{1/n} \leq r < 1, n > N$ "radical test"

Just like real series, we usually do not have an explicit formula for the limit. Again an exception is the geometric series:

$$\sum_{n=0}^{\infty} \omega^n \ ,$$

this time defined for fixed complex number $\omega = a + \mathbf{i}b$. Following exactly the same derivation as the real case

$$\sum_{k=0}^{n} \omega^k = \frac{1 - \omega^{n+1}}{1 - \omega}$$

provided $\omega \neq 1$. This converges to $1/(1-\omega)$ if and only if $|\omega| < 1$. Therefore

$$\sum_{n=0}^{\infty} \omega^n = \frac{1}{1-\omega}, \ |\omega| < 1 \ .$$

For real ω this is just the usual power series but we can have some fun by choosing some complex values.

Exercises

1. For what values of z does the series $\sum_{n=0}^{\infty} nz^n$ converge.
2. Prove that the infinite series

$$1+\frac{1}{2}\cos(\theta)+\frac{1}{2^2}\cos(2\theta)+\frac{1}{2^3}\cos(3\theta)+.. = \Re\left\{\frac{1}{1-\frac{1}{2}(\cos(\theta)+\mathbf{i}\sin(\theta))}\right\}$$

5 Complex Valued Functions

In calculus we have considered functions f defined on an interval $[a, b]$ of real numbers with real values. It is not much of a leap of the imagination to allow f to take complex values: $f(t) = x(t) + \mathbf{i}y(t)$ where $x(t), y(t)$ are ordinary real valued functions.

For example $f(t) = \cos(t) + \mathbf{i}\sin(t)$ which we already know gives a parametrization of the circle $x^2 + y^2 = 1$. In fact the whole theory of parametrized curves in the plane can be written in terms of complex variables. The derivative of such a function poses no problems:

$$z(t) = x(t) + \mathbf{i}y(t) \Rightarrow \frac{dz}{dt} = x'(t) + \mathbf{i}y'(t)$$

so for example the function $z(t) = 2\cos(t) + \mathbf{i}\sin(t)$ which parametrizes an elipse with major axis 4 and minor axis 2 has derivative

$$z'(t) = -2\sin(t) + \mathbf{i}\cos(t)$$

The usual rules of derivatives hold, and you can even integrate.

Many of the uses of complex variables in science and engineering involves complex valued functions of real variables, e.g. in electrical engineering or quantum theory one uses complex valued functions of (real) time t. However one also considers functions of a complex variable:

5.1 Functions of a Complex Variable

So now we are talking about functions $f(z)$ defined on a domain $D \subset \mathbb{C}$ taking complex values. Our polynomials $p(z)$ are examples of such functions. Other examples are rational functions like

$$f(z) = \frac{3}{z^2 + 1}$$

defined for all $z \neq \pm\mathbf{i}$. Complex valued functions can be added, multiplied and divided like ordinary functions. [1]

[1] Sometimes complex functions can be differentiated wrt z. But we do not deal with complex derivatives in this course.

5.2 Power series

Power Series generalize polynomials and provide a ready supply of functions. Allowing complex coefficients a_n:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

First we have to deal with convergence. All the usual tests for convergence work. Using these we find there is a radius of convergence R, i.e. the Power Series converges absolutely inside circle $|z - z_0| = R$ and diverges for outside circle $|z - z_0| = R$. The prototypical Power Series is the Geometric Series $\sum_{n=0}^{\infty} z^n$ which we know converges for $|z| < 1$. Thus it has radius of convergence $R = 1$. Another famous Power Series is

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \ .$$

Now taking the ratio of successive terms

$$\frac{|z|^{n+1}}{(n+1)!} \frac{n!}{|z|^n} = \frac{|z|}{n+1}$$

which converges to 0 as $n \to \infty$. Thus by the ratio test the series converges for all complex z and so the radius of convergence is $R = \infty$.

Of course we want to do the all the usual operations to Power Series: addition, multiplication, composition and division (not by 0 of course).

Exercises

1. By multiplying the Power Series show that

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n, \ |z| < 1$$

2. What is the radius of convergence of the Power series

$$(a) \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} z^{2n-1} \ , \ (b) \ \ \sum_{n=1}^{\infty} \frac{1}{n} z^n$$

6 Complex Exponential

The highlight of our theory is the definition of complex exponential, using the familar power series

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

which we know converges for all z. Using the Power Series for e^x we see that $\exp(x) = e^x$ for real numbers x, i.e. $\exp(z)$ extends e^x to the complex plane. Also we see some hitherto unexpected connections with the trig functions. For if we put $z = \mathbf{i}y$

$$\exp(\mathbf{i}y) = \sum_{n=0}^{\infty} \frac{\mathbf{i}^n y^n}{n!}$$

Now $\mathbf{i}^n$ takes the values $\mathbf{i}, -1, -\mathbf{i}, 1$ in turn so taking the real and imaginary parts of the the series gives

$$\exp(\mathbf{i}y) = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} + \mathbf{i} \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!}$$

You may recognize that the first Power Series is $\cos(y)$ and the second one is $\sin(y)$.

Thus we have proved

$$\exp(\mathbf{i}y) = \cos(y) + \mathbf{i}\sin(y)$$

the famous "Euler's formula" which shows $\exp, \cos, \sin$ are different sides of the same thing. In particular putting $y = \pi$ we get the identity

$$e^{\mathbf{i}\pi} + 1 = 0$$

which Euler declared "proves God exists".

Consider the complex valued function $f(t) = \exp(t\zeta)$ for fixed $\zeta \in \mathbb{C}$ and real t. Now differentiating wrt t we find $f'(t)$ is given by

$$\sum_{n=0}^{\infty} \frac{d}{dt}\frac{t^n\zeta^n}{n!} = \sum_{n=1}^{\infty} n\frac{t^{n-1}\zeta^n}{n!} = \zeta \sum_{n=1}^{\infty} \frac{t^{n-1}\zeta^{n-1}}{(n-1)!} = \zeta \sum_{n=0}^{\infty} \frac{(t\zeta)^n}{n!} = \zeta \exp(t\zeta) \ ,$$

i.e. $f(t) = \exp(t\zeta)$ satisfies the usual differential equation $f'(t) = \zeta f(t)$. We use this to derive the addition formula for complex z, w:

$$\exp(z + w) = \exp(z)\exp(w)$$

First consider the function $g(t) = \exp(tz)\exp(\omega - tz)$ (for complex z, ω):

$$\frac{dg}{dt} = z\exp(tz)\exp(\omega - tz) - z\exp(tz)\exp(\omega - tz) = 0$$

by derivative rules. Therefore $g(t)$ is the constant $g(0)$, i.e

$$g(1) = \exp(z)\exp(\omega - z) = \exp(\omega)$$

for all z. Putting $\omega = z + w$ gives the addition formula.

The addition formula gives us an explicit formula for $\exp(z)$ for

$$\exp(x + \mathbf{i}y) = \exp(x)\exp(\mathbf{i}y) = e^x\{\cos(y) + \mathbf{i}\sin(y)\}$$

This gives De Moivre's formula as well as all the trig identities as special cases of the addition formula for complex exponential.

Exercises

1. Compute $\exp(1 - \mathbf{i}\frac{\pi}{4})$
2. From the addition formula for complex exp derive the trig addition formula for $\sin(a + b)$ and $\cos(a + b)$
3. Show that for all complex z, $\exp(z) \neq 0$.
4. By defining complex

   $$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}\ ,\ \sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

   show that

   $$\cos(z) = \frac{1}{2}\{\exp(\mathbf{i}z) + \exp(-\mathbf{i}z)\}\ ,\ \sin(z) = \frac{1}{2\mathbf{i}}\{\exp(\mathbf{i}z) - \exp(-\mathbf{i}z)\}$$

5. Use the previous formula to show that for all complex z

   $$\cos^2(z) + \sin^2(z) = 1$$

7 Applications to integration

Complex variables has fundamental applications throughout science. Here we give an easier way of integrating trig expressions. In the course we considered the problem of finding antiderivatives for expressions of the form

$$t^n e^{at} \sin^k(bt) \cos^j(ct) \ , \ \text{where } n, k, j = 0, 1, 2, 3..$$

This can be treated by Euler's formula to yield

$$t^n e^{at} \frac{1}{(2\mathbf{i})^k} \{e^{b\mathbf{i}t} - e^{-b\mathbf{i}t}\}^k \frac{1}{2^j} \{e^{c\mathbf{i}t} + e^{-c\mathbf{i}t}\}^j$$

Expanding one obtains the sum of terms of the form $t^n e^{\alpha t}$ where α is complex. These terms can then be integrated by parts.

In many cases the integrals have the form $\int_0^{2\pi} F(\sin(t), \cos(t))dt$. It saves a lot of time to know the formula

$$\int_0^{2\pi} e^{n\mathbf{i}t} dt = \begin{cases} 0 & n \neq 0 \\ 2\pi & n = 0 \end{cases} \ , \ \text{for } n = 0, \pm 1, \pm 2... \ .$$

For example consider the problem of computing $\int_0^{2\pi} \sin^8(t)dt$. Now $\sin^8(t)$ may be rewritten via Euler's formula to give

$$\frac{1}{(2\mathbf{i})^8} \{e^{\mathbf{i}t} - e^{-\mathbf{i}t}\}^8 = \frac{1}{(2\mathbf{i})^8} \{e^{8\mathbf{i}t} - 8e^{6\mathbf{i}t} + 28e^{4\mathbf{i}t} - 56e^{2\mathbf{i}t} + 70 - .. - 8e^{-6\mathbf{i}t} + e^{-8\mathbf{i}t}\} \ ,$$

by the binomial formula (also true for complex numbers). Now integrating from 0 to 2π most of these expressions give zero, except for the power zero term, ie the middle term. Thus the integral is

$$\int_0^{2\pi} \frac{1}{2(\mathbf{i})^8} \{e^{\mathbf{i}t} - e^{-\mathbf{i}t}\}^8 dt = \frac{1}{2^8(-1)^4} \int_0^{2\pi} 70 \, dt = \frac{35\pi}{64} \ .$$

Exercises

Use complex methods to compute the integrals:

$$\int_0^{2\pi} \cos^6(2t)dt \ , \ \int \sin^4(3t) \cos^2(t)dt$$

Appendix: Fundamental Theorem of Algebra

The problem of finding roots of polynomials was very much tied up with the invention of the complex numbers themselves. Cardan (16^{th} century) discovered a formula for the roots of the equation $z^3 = 15z + 4$ which gave an answer involving $\sqrt{-121}$. Cardan knew that the equation had $z = 4$ as a solution and was able to manipulate 'complex numbers' to obtain the right answer without understanding what they were. Bombelli, in 1572, produced rules for manipulating these "complex numbers". Yet a 'proof' that the FTA was false was given by Leibniz in 1702 when he asserted that $z^4 + t^4$ could never be written as a product of two real quadratic factors. His mistake came in not realising that $\sqrt{\mathbf{i}}$ could be written in the form $a + b\mathbf{i}$.

D'Alembert in 1746 gives a sequence converging to a zero of the polynomial. His proof has the weakness that he did not have the necessary theory to prove convergence. In fact such a theory was not developed until the late 19^{th} century, never the less one sees his proof as being essentially correct. Indeed in France the FTA is called D'Alembert-Gauss.

Gauss in 1799 presented his first proof and also his objections to the other proofs. He is undoubtedly the first to spot the flaws in the earlier proofs. Actually Gauss does not claim to give the first proper proof. He merely calls his proof new and says of d'Alembert's proof, that despite his objections a rigorous proof could be constructed on the same basis. In fact Gauss's proof of 1799 also does not meet our present day standards.

In 1814 the Swiss accountant Jean Robert Argand published a proof based on d'Alembert's 1746 idea. Two years after Argand's proof appeared Gauss published a second proof of the FTA. This proof is complete and correct. Then he gave a third proof. Gauss introduced in 1831 the term 'complex number'. In 1849 (on the 50th anniversary of his first proof!) Gauss produced the first proof that a polynomial equation of degree n with complex coefficients has n complex roots. Despite the many proofs given by Gauss it seems fundamentally unfair to give to him the credit for FTA. A proof using calculus might be attributed to D'Alembert-Gauss, with important tidying up done by Argand.

A Proof in the style of D'Alembert-Argand

Consider a nonconstant polynomial $p = a_0 + a_1 z + .. + a_n z^n$, where $a_n \neq 0$. We argue by contradiction, i.e assume p does not have a root. Thus the function $u(z) = |p(z)|^2 > 0$. Now as $|p(z)| = |a_n||z|^n|1 + a_{n-1}/(a_n z) + ..|$ we have $\lim_{|z|\to\infty} u(z) = \infty$ so, by basic calculus, the function $u(z)$ achieves its minimum somewhere, say at z_0.

To make computations easier we make some simple transformations. So *without loss of generality* we may assume $z_0 = 0$ (otherwise just use translation $z - z_0$). Thus $|p(0)|^2 = |a_0|^2$ is the minimum of $u(z)$. Also since $p(0) \neq 0$, $p = a_0 + a_m z^m + .. + a_n z^n$ where $a_0 \neq 0$, and a_m is the next nonzero term. So without loss of generality (again) we may assume $a_0 = 1$ (by considering $p(z)/a_0$ instead), i.e. the minimum is now $u(0) = 1$. Finally we suppose that $a_m = \rho e^{i\phi}$ and let $\beta = \rho^{1/m} e^{i\phi/m}$ be an *mth* root. By making the substitution $p(z/\beta)$ we may assume $p(z) = 1 + z^m + ... + a_n z^n$.

So to obtain a contradiction we have only to show that $u = |p|^2$ cannot have its minimum at $z_0 = 0$. Expanding

$$u(z) = |1 + z^m + ...|^2 = \{1 + z^m + ...\}\overline{\{1 + z^m + ...\}}$$

So with $z = re^{i\theta}$,

$$u(z) = 1 + 2r^m \cos(m\theta) + \varepsilon r^m,$$

where the error $\varepsilon \to 0$ as $z \to 0$. But for $\theta = \pi/m$

$$u(z) = 1 - 2r^m + \varepsilon r^m,$$

so as $-2 < \varepsilon$ there is $z \to 0$ so that $u(z) < 1 = u(0)$. This means $z_0 = 0$ cannot give the minimum. Therefore p has no roots gives a contradiction. The only logical conclusion is that p has a root ζ.