MATH 498P: INTRODUCTION TO FRACTAL GEOMETRY AND DYNAMICAL SYSTEMS

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Course Objectives: Fractals are strange but beautiful objects that appear in nature and arts as results of self-organization and self-similarity. In dynamics they are "responsible" for the presence of highly-irregular, chaotic motions. The course is an introduction to a circle of topics in fractal geometry and chaotic dynamics.

Syllabus

This course is an invitation to Fractal Geometry and Dynamical Systems. These two areas have long history of development and are associated with such great names as Poincare, Kolmogorov, Smale and Cantor, Hausdorff, Besicovich to name a few. A unifying factor for merging dynamics with fractal geometry is self-similarity, which is a crucial feature of fractal sets and at the same time is realted to various symmetries in dynamics (such as rescaling of time and space). This is important as symmetry is an attribute of many physical laws. Currently the two areas are subject of intensive mathematical study and have much to offer to gifted students and help them to come closer to contemporary ideas in mathematical research. Therefore, the focus of the course is on ideas rather than complicated mathematical techniques and the goal is to make these ideas accessible to students.

I will discuss some basic concepts of Fractal Geometry and provide several core examples of fractal sets. A special emphasize will be given to fractal sets associated with dynamical systems with chaotic behavior.

Prerequisites: Calculus II and Linear Algebra.

Mode of Instruction: Lectures, research projects.

Readings: Lecture Notes, text-books.

Text-book: Ya. Pesin and V. Climenhaga, "Lectures on Fractal Geometry and Dynamical Systems", Student Mathematical library, 52, AMS, Providence, RI, 2009

Topics Covered

Fractal Geometry.

1. Concept of fractals. Self-similarity. Einstein's proof of the Pithagorian theorem. The von Koch curve.
2. Metric spaces: definition of the metric, examples of metric spaces, topological structure of metric spaces.
3. The middle-third Cantor set: construction, symbolic representation, the cardinality of the set, relations to the 3-adic representation of numbers.
4. The Sir-Pinski game and the Sierpinski gasket. Symbolic representations of the Sierpinski gasket.
5. The Hausdorff measure and Hausdorff dimension. Basic properties of the Hausdorff dimension.
6. The lower and upper box dimensions. Basic properties of the box dimension.
7. Generalized Cantor sets: construction and symbolic representation. Calculation of the Hausdorff and box dimensions of a generalized Cantor set (with the proof of the formula for the Hausdorff and box dimensions).
8. The Hausdorff and box dimensions of the middle-third Cantor set and the Sierpinski gasket.
9. The Mass Distribution Principle. The estimate from below of the Hausdorff dimension of a generalized Cantor set with the use of the Mass Distribution Principle.
10. The lower and upper pointwise dimensions of a measure. Relations between Hausdorff, box and pointwise dimensions.

Dynamical systems.

1. Dynamical systems with discrete time. Types of trajectories, fixed and periodic points. Attracting, repelling and saddle fixed and periodic points. Population models.
2. Affine transformations of the plane: geometric properties and algebraic description. Isometries. Similarity transformations.
3. The symbolic dynamical systems (Σ_p^+, σ) and (Σ_p, σ): the metric, the dynamics of the full shift (including the number of periodic points). Invariant measures for the full shift σ, Bernoulli and Markov measures.
4. One-dimensional Markov maps: definition, construction of the repeller, symbolic representation. Description of periodic points

and calculation of the number of periodic orbits of a given period n. Calculation of the Hausdorff and box dimensions of a linear one-dimensional Markov map.

(5) Bernoulli and Markov measures for a linear one-dimensional Markov map. Calculation of the pointwise dimension of Bernoulli and Markov measures for a linear one-dimensional Markov map, entropy and Lyapunov exponents.

(6) The Smale-Williams solenoid: definition of the map (by the formula) and the geometric structure of the attarctor. The Hausdorff and box dimensions of the attractor.

(7) The Smale horseshoe: definition of the horseshoe map and the geometric structure of the invariant set; symbolic representation of the horseshoe. The Hausdorff and box dimensions of the linear Smale horseshoe. Homoclinic orbits and their relation to horseshoes.

(8) The Sharkovski theorem. Prove that "period three implies chaos".

(9) The saddle-node and period doubling bifurcations. The dynamics and bifurcations of the quadratic map $f(x) = x^2 + c$ for $c > -\frac{5}{4}$.

(10) (time permitting) The FitzHugh-Nagumo map: definition of the map (by the formula), fixed points and their stability. The phase portrait for small values of the parameter A. Description of the chaotic behavior of the map for large values of the parameter A. The trapping region, horseshoe and attractor for the FitzHugh-Nagumo map.