Problem Set D, Problem 1.

```
> restart:
```

We consider the second order differential equation

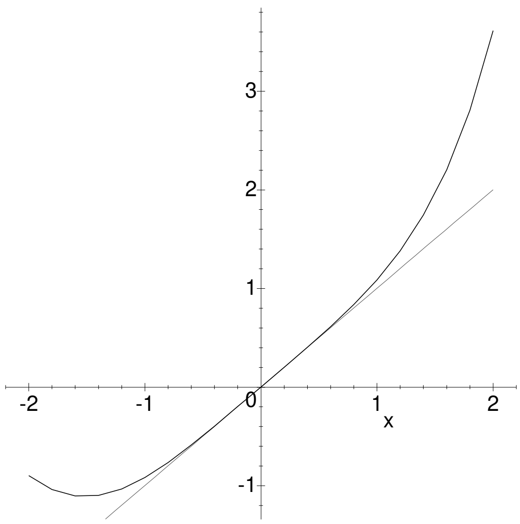
```
> airy := diff(y(x), x$2) = x*y(x);
```

$$airy := \frac{\partial^2}{\partial x^2} y(x) = x\, y(x)$$

(a)

When x is close to 0, we want to compare the solution of Airy's equation with the solution of the IVP y''(x) = 0, y'(0) = 1, y(0) = 0. The solution of this IVP is y = x. Here is a plot of the numerical solution of Airy's equation together with the function y = x.

```
> with(plots): with(DEtools):
> fac1plot := plot(x, x = -2..2):
> airy1plot := DEplot(airy, y(x), x = -2..2, {[y(0) = 0, D(y)(0) = 
  1]},
  method = rkf45, linecolor = black):
> display({fac1plot, airy1plot});
```



The facsimile solution agrees well with the actual solution in a neighborhood of 0.

(b)

For x close to $-16 = -4^2$, we want to compare the solution of Airy's equation to the facsimile solution

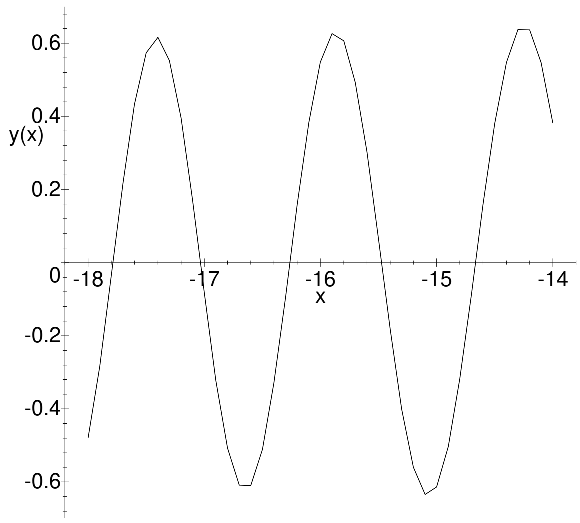
```
> fac2 := (c1, c2) -> c1*sin(4*x + c2);
```

$$fac2 := (c1, c2) \rightarrow c1 \sin(4\,x + c2)$$

First we'll plot a numerical solution of Airy's equation.

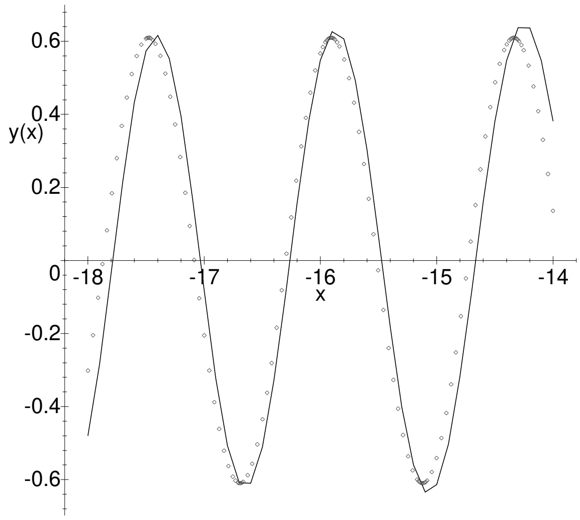
```
> airy2plot := DEplot(airy, y(x), -18..-14, {[y(0) = 0, D(y)(0) = 
  1]},
```

```
  method = rkf45, stepsize = 0.1, linecolor = black):
> airy2plot;
```



This certainly looks like a sine wave. Let's see how well it matches up with an appropriate sine wave. We have to choose constants c_1 and c_2 in the facsimile solution to make it match up. Note first that c_1 is the amplitude of the facsimile solution, and we can see from the graph that the amplitude of the solution of Airy's equation is about .61 on this interval. The constant c_2 determines the phase shift, and can be read off from the zeros of the solution. In particular, the solution of Airy's equation has a zero at about -16.3, so we should choose c_2 to satisfy $4\,(-16.3) + c_2 = 0$.

```
> fac2plot := plot(fac2(0.61, 4*16.3), x = -18..-14, style = POINT):
> display({fac2plot, airy2plot});
```



The plots match up well. Why did we have to choose the values of c_1 and c_2 by hand? Our analysis suggests that the facsimile solution should approximate the actual solution near $x = -K^2$. But the initial condition that we used for Airy's equation is at the point $x = 0$, where the facsimile solution is not a good approximation. Thus, the undetermined constants in the facsimile solution are unrelated to the initial conditions in Airy's equation, and we needed to choose them by hand to get a good match. Nevertheless, the frequency of the facsimile solution is determined by K, and is independent of the undetermined constants. Thus, we can at least conclude that the frequency of the solutions of Airy's equation in a neighborhood of $x = -K^2$ will be almost proportional to K.

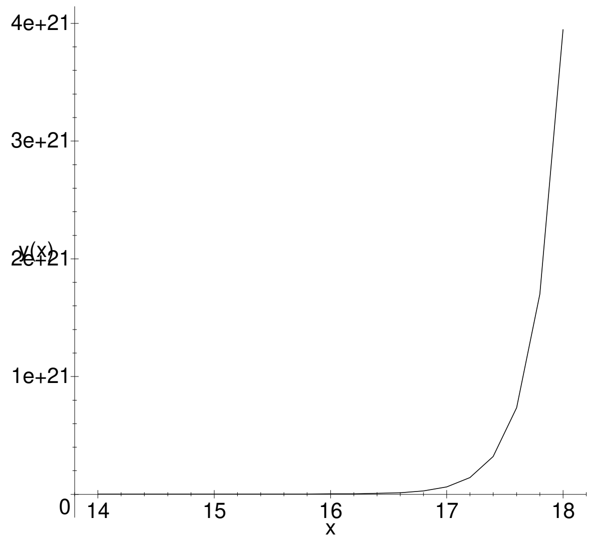
(c)

```
> fac3 := (c1, c2) -> c1*sinh(4*x + c2);
```

$$fac3 := (c1, c2) \to c1 \sinh(4\,x + c2)$$

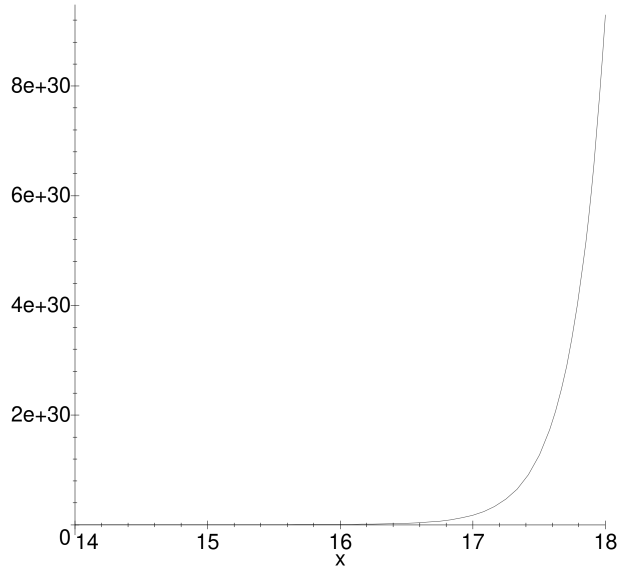
```
> airy3plot := DEplot(airy, y(x), 14..18, {[y(0) = 0, D(y)(0) = 1]},

  method = rkf45, linecolor = black):
> airy3plot;
```



We'd like to compare this with the graph of the hyperbolic sine function. Again, we have to choose c_1 and c_2 by hand. We'll start with an arbitrary choice: $c_1 = 1$, $c_2 = 0$.

```
> plot(fac3(1, 0), x = 14..18);
```

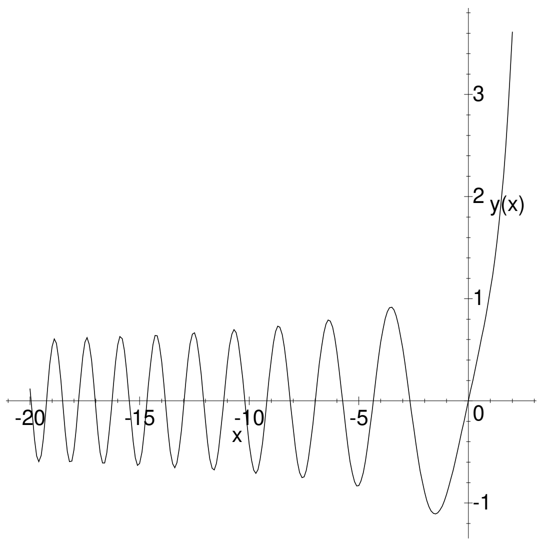


The graphs are remarkably similar. Note, however, that the values in the second graph are about 10^9 greater than in the first, which means that we should have chosen c_1 to be about $10^{(-9)}$.

(d)

Finally, we want to produce a graph of the numerical solution on the interval from -20 to 2.

```
> DEplot(airy, y(x), -20..2, {[y(0) = 0, D(y)(0) = 1]}, method =
  rkf45, stepsize = 0.1, linecolor = black);
```



This graph indicates that the frequency increases and the amplitude decreases as x goes to $-\infty$. The increasing frequency is predicted by our facsimile analysis; the decreasing amplitude is harder to

explain.
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