COHOMOLOGY COMPARISON THEOREMS VIA HOMOLOGICAL ALGEBRA

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ABSTRACT. This paper was prepared as my final project for MATH602 Homological Algebra taught by Lutian Zhao.

CONTENTS

1. INTRODUCTION

The goal of this paper is to develop the ideas of hypercohomology, hyper-derived functors, and acyclicity and demonstrate how these ideas can be used to connect various cohomology theories. To illustrate the power of this framework, we will apply this theory in the setting of sheaves over manifolds and use it to recover some classical theorems regarding comparisons of various cohomology theories. We will use homological techniques and fundamental geometry results to prove:

Theorem 1.1 (De Rham's Theorem)**.** *Let X be a smooth manifold. There is an isomorphism*

$$H^k_{\mathrm{dR}}(X) \cong H^k_{\mathrm{sing}}(X, \mathbb{R}).$$

Theorem 1.2 (Dolbeault's Theorem)**.** *Let X be a complex manifold. There is an isomorphism*

$$H^{p,q}_{\bar{\partial}}(X) \cong H^q_{\mathrm{sh}}(X, \Omega^p_{\mathrm{an}})$$

where Ω^p_{an} is the complex of sheaves of holomorphic forms.

Theorem 1.3 (Analytic de Rham Theorem)**.** *Let X be a complex manifold. There is an isomorphism*

$$H^k_{\mathrm{sing}}(X, \mathbb{C}) \cong \mathbb{H}^k(X, \Omega^\bullet_{\mathrm{an}})$$

where $\Omega^\bullet_{\mathrm{an}}$ is the complex of sheaves holomorphic forms.

Theorem 1.4 (Algebraic de Rham Theorem)**.** *Let X be a smooth complex projective variety with Zariski topology and X_{an} the underlying set of X with the complex manifold topology. There is an isomorphism*

$$H^k_{\mathrm{sing}}(X_{\mathrm{an}}, \mathbb{C}) \cong \mathbb{H}^k(X, \Omega^\bullet_{\mathrm{alg}})$$

where $\Omega^\bullet_{\mathrm{alg}}$ is the complex of sheaves of algebraic differential forms.

We loosely follow the presentation of [3] but develop the material in a slightly more general language.

2. Background

2.1. **Hyper-derived functors.** We recall some standard facts. Fix a left exact additive functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories. Assume that $\mathcal{A}$ has enough injectives.

Definition 2.1. The right hyper-derived functors of F are functors $\mathbb{R}^i F\colon\ \mathbf{Ch}(\mathcal{A}) \to \mathcal{B}$.

For the purposes of this paper, the construction of the hyper-derived functors is superfluous so we omit it. The interested reader can consult [2]. The main fact about hyper-derived functors that we rely on is the following.

Fact 2.2 ([2, p. 150]). Let $A^\bullet$ be a chain complex. There are two spectral sequences associated with the right hyper-derived functors of F:

$$ {}^{\mathrm{II}}E_2^{pq} = (R^pF)(H^q(A^\bullet)) \Rightarrow \mathbb{R}^{p+q}F(A^\bullet) \qquad \text{(weakly convergent)} \tag{2.3}$$

$$ {}^{\mathrm{I}}E_2^{pq} = H^p(R^qF(A^\bullet)) \Rightarrow \mathbb{R}^{p+q}F(A^\bullet) \qquad \text{(if } A^\bullet \text{ is bounded below)} \tag{2.4}$$

This gives that $\mathbb{R}^\bullet F$ vanishes on exact complexes and sends quasi-isomorphisms to isomorphisms.

2.2. **Acyclicity.** In the primary part of this paper, we will want to compute cohomology using objects that are not necessarily injective. Those objects will have nice properties that allow us to compute cohomology regardless. In this section, we make this notion precise.

Definition 2.5 (F-acyclicity)**.** Fix $F\colon \mathcal{A} \to \mathcal{B}$ a left exact additive functor between abelian categories. An object $A \in \mathcal{A}$ is called F-acyclic if $R^iF(A) = 0$ for all $i > 0$.

Definition 2.6. A resolution $0 \to A \to B^\bullet$ is called F-acyclic if B^i is F-acyclic for all $i \geq 0$.

We will show that acyclic resolutions can be used to compute cohomology. Towards that end, we need a few lemmas.

Lemma 2.7. *If $A^\bullet$ is a complex of F-acyclic objects bounded below, then $\mathbb{R}^nF(A) = H^n(F(A^\bullet))$.*

Proof. The acyclicity assumptions on $A^\bullet$ yield $R^qF(A^\bullet) = 0$ for $q > 0$ and $R^0F(A^\bullet) = F(A^\bullet)$. Hence the spectral sequence (2.4) collapses at the E_2 page, so

$$\mathbb{R}^nF(A) = H^n(R^0F(A^\bullet)) = H^n(F(A^\bullet)).$$

□

Lemma 2.8. *For any $A \in \mathcal{A}$, we have $\mathbb{R}^nF(A[0]) = R^nF(A)$.*

Proof. Since $A[0]$ is concentrated in degree zero, then $H^q(A[0]) = 0$ for $q \neq 0$ and $H^0(A[0]) = A$. Thus the spectral sequence (2.3) collapses at the E_2 page so

$$\mathbb{R}^nF(A[0]) = (R^nF)(H^0(A[0])) = R^nF(A).$$

□

Lemma 2.9. *Let $0 \to A \xrightarrow{d} B^\bullet$ be a resolution. The natural map $A[0] \to B^\bullet$ of chain complexes is a quasi-isomorphism.*

Proof. The natural map $A[0] \to B^\bullet$ is depicted below.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "d"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & B^0 \arrow[r] & B^1 \arrow[r] & B^2 \arrow[r] & \cdots
\end{tikzcd}$$

Since $0 \to A \to B^\bullet$ is exact, then $H^i(B^\bullet) = 0 = 0_* H^i(A[0])$ and $d_* H^0(A[0]) = d(A) = \ker(B^0 \to B^1) = H^0(B^\bullet)$, so $A[0] \to B^\bullet$ is a quasi-isomorphism. □

With these lemmas collected, we can put them together to get the following theorem that tells us how we can use acyclic resolutions to compute cohomology.

Theorem 2.10. *Let $0 \to A \to B^\bullet$ be an F-acyclic resolution. There is an isomorphism*

$$R^n F(A) \cong H^n(F(B^\bullet)).$$

Proof. Applying lemma 2.9 together with the fact that hyperderived functors take quasi-isomorphisms to isomorphisms yields $\mathbb{R}^n F(A[0]) \cong \mathbb{R}^n F(B^\bullet)$. Then lemmas 2.7 and 2.8 give

$$R^n F(A) = \mathbb{R}^n F(A[0]) \cong \mathbb{R}^n F(B^\bullet) = H^n(F(B^\bullet)).$$

□

2.3. **Sheaf Cohomology.** Let us contextualize the preliminaries we have developed so far. Our aim is to investigate classical cohomology theories on geometric spaces using the language of homological algebra. To do this, we need an appropiate abelian category in which to conduct our investigation. The category that we will use is that of abelian sheaves over X, denoted $\mathbf{Sh}(X)$. This category has enough injectives so will allows us to do cohomology. There is a natural functor on $\mathbf{Sh}(X)$ which we recall below.

Definition 2.11. The global sections functor $\Gamma\colon \mathbf{Sh}(X) \to \mathbf{Ab}$ is the functor that takes $\mathcal{F}$ to its global sections $\Gamma(X, \mathcal{F})$ and takes maps of sheaves to the induced map on global sections. It is additive and left exact.

Convention 2.12. Throughout the rest of the paper, we will have X be some type of topological space. We will always be considering the global sections functor $\Gamma\colon \mathbf{Sh}(X) \to \mathbf{Ab}$, so we will suppress mention of Γ (i.e. we will use acyclic to mean Γ-acyclic etc.).

In the following example, we see how our current setup provides a good setting in which to interpret classical cohomology theories.

Example 2.13. Let X be a smooth manifold. Classical de Rham cohomology, denoted $H^k_{\mathrm{dR}}(X)$, is the cohomology of the chain complex of abelian groups

$$0 \to \mathcal{A}^0(X) \to \mathcal{A}^1(X) \to \mathcal{A}^2(X) \to \cdots$$

where $\mathcal{A}^k(X)$ denotes the smooth k-forms on X and the differentials are the exterior derivative maps. The smooth k-forms form a sheaf $\mathcal{A}^k$ on X, namely the sheaf that associates to an open set U the smooth k-forms defined on U. These sheaves fit neatly into a chain complex $\mathcal{A}^\bullet$ of abelian sheaves on X

$$0 \to \mathcal{A}^0 \to \mathcal{A}^1 \to \mathcal{A}^2 \to \cdots$$

where the differentials are the exterior derivatives on the level of open sets. Importantly, we can recover the classical de Rham complex by applying the global section functor and even recover the de Rham cohomology as $H^k_{\mathrm{dR}}(X) = H^n(\Gamma(\mathcal{A}^\bullet))$.

The last remark in the example hints at the result of Theorem 2.10 and indeed this is foreshadowing for later, but first we need to discuss sheaf cohomology.

Definition 2.14 (Sheaf cohomology)**.** Fix X a topological space and $\mathcal{F}$ an abelian sheaf over X. We define the sheaf cohomology of X with coefficients in $\mathcal{F}$, denoted $H^n_{\mathrm{sh}}(X, \mathcal{F})$, to be the abelian group $R^n\Gamma(\mathcal{F})$, the image of $\mathcal{F}$ under the right derived functor of Γ. Similary, for a complex $\mathcal{F}^\bullet$ of abelian sheaves over X, we define the sheaf hypercohomology $\mathbb{H}^n(X, \mathcal{F}^\bullet)$ to be $\mathbb{R}^n\Gamma(\mathcal{F}^\bullet)$.

Our notation and terminology for sheaf cohomology is suggestive of a correspondence between sheaf cohomology and singular cohomology. Indeed, this occurs when X is nice (for example, if X is a manifold). To make sense of such a correspondence, we need some sheaf analogue of a group of coefficients that will correspond to singular cohomology. This prompts the following.

Definition 2.15 (Constant sheaf)**.** Let A be an abelian group. The constant sheaf $\underline{A}$ on X is the sheaf of continuous functions $X \to A$ where A is equipped with the discrete topology. Note this is equivalent to locally constant functions on X with values in A.

We now make the correspondence precise:

Fact 2.16 ([3, Remark 2.2.16]). If X is a manifold and $\underline{A}$ is a constant sheaf on X, then there is an isomorphism

$$H^k_{\mathrm{sh}}(X, \underline{A}) \cong H^k_{\mathrm{sing}}(X, A).$$

2.4. **Acyclic Sheaves and a Caveat.** We have now established the bulk of the homological algebra we will need to prove our desired results. However, we have not discussed any methods to show certain sheaves are actually acyclic, which we will inevitably need if we are to have any chance of using the machinery developed so far. There is a particular class of sheaves that arises in the scenarios we consider, called fine sheaves, which are acyclic when the base space is a manifold. These sheaves are roughly those that support partitions of unity. We will not explore this notion in any depth beyond using the fact that the sheaves we will consider are fine, as that would distract from the purpose of the paper. We will however remark when we use these assumptions so that the rigorous reader may investigate the idea on their own time.

3. Cohomology Comparison Theorems

3.1. **De Rham's Theorem.** Let X be a smooth manifold and $\mathcal{A}^\bullet$ the sheaf complex of differential forms as described in Example 2.13. As a manifold supports smooth partitions of unity, then $\mathcal{A}^\bullet$ is a complex of fine, and thus acyclic, sheaves. We focus for now on the degree zero part of the complex which is $0 \to \mathcal{A}^0 \to \mathcal{A}^1$. Note that on an open set U, the sections of $\mathcal{A}^0$ are the smooth functions on U. Thus the kernel of derivative map on U are exactly the locally constant functions on U with values in $\mathbb{R}$, so we can identify $\ker(\mathcal{A}^0 \to \mathcal{A}^1)$ with the constant sheaf $\underline{\mathbb{R}}$. Also recall the Poincaré lemma which states that every closed k-forms is exact on open balls of $\mathbb{R}^n$ for $1 \leq k \leq n$. This then implies that $\mathcal{A}^1 \to \mathcal{A}^2 \to \cdots$ is exact. Hence, we have the exact sequence

$$0 \to \underline{\mathbb{R}} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \mathcal{A}^2 \to \cdots$$

Thus $\mathcal{A}^\bullet$ is an acyclic resolution of $\underline{\mathbb{R}}$. Applying theorem 2.10, fact 2.16, and the last remark of example 2.13 gives an isomorphism

$$H^k_{\mathrm{sing}}(X, \mathbb{R}) \cong H^k_{\mathrm{sh}}(X, \underline{\mathbb{R}}) \cong H^k(\Gamma(\mathcal{A}^\bullet)) = H^k_{\mathrm{dR}}(X).$$

We have thus proven de Rham's theorem.

3.2. **Dolbeault Theorem.** Let X be a complex manifold. We recall the basic notions of Dolbeault cohomology. Let $\mathcal{A}^{1,0}$ denote the sheaf of one-forms that contain only dz locally and $\mathcal{A}^{0,1}$ denote the one-forms that contain only $d\bar{z}$ locally. The sheaf of (p,q)-forms on X is the sheaf $\bigwedge_1^p \mathcal{A}^{1,0} \wedge \bigwedge_1^q \mathcal{A}^{0,1}$. The Dolbeault operator is a map $\overline{\partial}\colon \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}$ which makes the following a complex

$$0 \to \mathcal{A}^{p,0} \to \mathcal{A}^{p,1} \to \mathcal{A}^{p,2} \to \cdots$$

We denote this complex by $\mathcal{A}^{p,\bullet}$. We also remark that each of these sheaves is fine, since they support partitions of unity, so are acyclic. The Dolbeault cohomology groups $H^{p,q}(X)$ are the cohomology groups of the complex $H^q(\Gamma(\mathcal{A}^{p,\bullet}))$. We denote the sheaf of holomorphic p-forms on X by Ω^p_{an}. To obtain our desired comparison result, we need a complex analogue of the Poincaré lemma. This manifests as the $\overline{\partial}$-Poincaré lemma [1, p. 25], which gives the exactness of

$$0 \to \Omega^p_{\text{an}} \to \mathcal{A}^{p,0} \to \mathcal{A}^{p,1} \to \mathcal{A}^{p,2} \to \cdots$$

Thus $\mathcal{A}^{p,\bullet}$ is an acyclic resolution of Ω^p_{an}. Applying theorem 2.10, we obtain an isomorphism

$$H^q_{\text{sh}}(X, \Omega^p_{\text{an}}) \cong H^q(\Gamma(\mathcal{A}^{p,\bullet})) = H^{p,q}_{\overline{\partial}}(X).$$

We thus have Dolbeault's theorem.

3.3. **Analytic de Rham Thoerem.** Let X be a complex manifold and let Ω^k_{an} be the sheaf of holomorphic k-forms on X. The sheaves assemble into a complex

$$0 \to \Omega^0_{\text{an}} \to \Omega^1_{\text{an}} \to \Omega^2_{\text{an}} \to \cdots$$

where the differentials are the maps $d = \partial + \overline{\partial}$. We denote this complex by $\Omega^\bullet_{\text{an}}$. The suitable analogue of the Poincaré lemma in this case is the holomorphic Poincaré lemma [3, Theorem 2.5.1] which tells us that the sequence

$$0 \to \underline{\mathbb{C}} \to \Omega^0_{\text{an}} \to \Omega^1_{\text{an}} \to \cdots$$

is exact.

Remark 3.1. A neat application of spectral sequences can be used to prove the holomorphic Poincaré lemma by using a double complex in combination with the smooth and $\overline{\partial}$ Poincaré lemmas (this is the method of proof taken in the cited source).

In contrast to the previous two cases, we cannot now apply theorem 2.10 since the sheaves Ω^k_{an} are not fine as they do not admit partitions of unity. We thus apply lemma 2.9 together with fact 2.2 to get an isomorphism

$$H^k_{\text{sing}}(X, \mathbb{C}) \cong H^k_{\text{sh}}(X, \underline{\mathbb{C}}) \cong \mathbb{H}^k(X, \Omega^\bullet_{\text{an}}).$$

This is the Analytic de Rham Theorem.

3.4. **Algebraic de Rham Theorem.** Our final goal is the algebraic de Rham theorem which relates a purely algebraic cohomology theory to a topological one. Let X be a smooth complex projective variety with Zariski topology and X_{an} the underlying set of X with the complex manifold topology. Denote the complex of algebraic k-forms on X by $\Omega^\bullet_{\text{alg}}$. As we are working in the algebraic category, there is not much hope for a Poincaré lemma similar to that which has arisen in each of our arguments so far. We are thus forced to seek another approach. Our saving grace is Serre's GAGA principle, which establishes an equivalence between the category of sheaves over X and the category of sheaves over X_{an} via the analytification functor $(-)_{\text{an}}$. Moreover, we have the following.

Fact 3.2 ([3, p. 91]). For all k, there are isomorphisms

$$H^k(X, \mathcal{F}) \cong H^k(X_{\mathrm{an}}, \mathcal{F}_{\mathrm{an}}).$$

Additionally, we have

$$(\Omega^p_{\mathrm{alg}})_{\mathrm{an}} \cong \Omega^p_{\mathrm{an}}.$$

Hence

$$H^p(X, \Omega^q_{\mathrm{alg}}) \cong H^p(X_{\mathrm{an}}, \Omega^q_{\mathrm{an}}).$$

We consider the two spectral sequences converging to $\mathbb{H}^k(X_{\mathrm{an}}, \Omega^\bullet_{\mathrm{an}})$ and $\mathbb{H}^k(X, \Omega^\bullet_{\mathrm{alg}})$ coming from fact 2.2. We are list them below.

$$(E_2^{pq})_{\mathrm{an}} = H^p(X, H^q(\Omega^\bullet_{\mathrm{an}})) \Rightarrow \mathbb{H}^{p+q}(X_{\mathrm{an}}, \Omega^\bullet_{\mathrm{an}}) \tag{3.3}$$

$$(E_2^{pq})_{\mathrm{alg}} = H^p(X, H^q(\Omega^\bullet_{\mathrm{alg}})) \Rightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_{\mathrm{alg}}) \tag{3.4}$$

Fact 3.2 gives an isomorphism of the E_2 pages of these two spectral sequences which ascends to an isomorphism at the E_∞ level by the mapping lemma for E_∞. This then implies that

$$\mathbb{H}^k(X_{\mathrm{an}}, \Omega^\bullet_{\mathrm{an}}) \cong \mathbb{H}^k(X, \Omega^\bullet_{\mathrm{alg}}).$$

Combining this with the Analytic de Rham Theorem, we secure an isomorphism

$$H^k_{\mathrm{sing}}(X_{\mathrm{an}}, \mathbb{C}) \cong \mathbb{H}^k(X, \Omega^\bullet_{\mathrm{alg}}).$$

Whence we have proven the Algebraic de Rham Theorem.

References

[1] Phillip Griffiths and Joseph Harris. *Principles of Algebraic Geometry*, chapter 0, pages 1–127. John Wiley & Sons, Ltd, 1994.

[2] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994.

[3] Fouad El Zein and Loring W. Tu. *From Sheaf Cohomology to the Algebraic de Rham Theorem*, pages 70–122. Princeton University Press, 2014.