Equivariant intersections of integral hypersurfaces mod p

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Abstract

This is a short note proving a Lefschetz principle type result on the intersection locus of G-invariant hypersurfaces. The author thanks Sam Lidz for helpful discussions and suggestions.

Let G an algebraic group over $\mathbb{Z}$ act on $\mathbb{P}^n_{\mathbb{Z}}$. We prove the following result.

Theorem 1 (Invariance under reduction)**.** *Fix generic G-invariant hypersurfaces $f_1, \dots, f_n \in \mathbb{Z}[x_0, \dots, x_n]$. There is an integer N such that for any prime p not dividing N, the intersection locus over $\mathbb{C}$ and the intersection locus over $\overline{\mathbb{F}}_p$ are isomorphic as G-sets.*

Proof. Let $X = \operatorname{Proj}(\mathbb{Z}[x_0, \dots, x_n]/(f_1, \dots, f_n))$; this is a closed subscheme of $\mathbb{P}^n_{\mathbb{Z}}$. Note that for any field k, we have $X(k)$ in canonical bijection with $V_k(f_1, \dots, f_n) := V(\bar{f}_1, \dots, \bar{f}_n) \subseteq \mathbb{P}^n_k$, where $\bar{f}_i = f_i \otimes 1 \in \mathbb{Z}[x_0, \dots, x_n] \otimes_{\mathbb{Z}} k$. This bijection is G-equivariant. Let K be the field generated by the coordinates of the points $[p_0 : \cdots : p_n] \in V_{\overline{\mathbb{Q}}}(f_1, \dots, f_n)$ with some $p_i = 1$. Note that K is a number field, so we can consider the ring of integers $\mathcal{O}_K$.

We next show that there is $N \in \mathcal{O}_K$ such that

$$X_{\mathcal{O}_K[1/N]} = \operatorname{Proj}(\mathcal{O}_K[1/N][x_0, \dots, x_n]/(f_1, \dots, f_n)) \cong \coprod_{P \in V_{\mathbb{C}}(f_1, \dots, f_n)} (\operatorname{Spec} \mathcal{O}_K[1/N])_P,$$

where G acts by mapping $(\operatorname{Spec} \mathcal{O}_K[1/N])_P$ isomorphically onto $(\operatorname{Spec} \mathcal{O}_K[1/N])_{g.P}$.

Since all solutions are defined over K, then X_K is a disjoint union of points. Let $D(x_i)$ be the standard affine patch of $\mathbb{P}^n$ corresponding to $x_i \neq 0$. Set $x_{j/i} := x_j/x_i$ and $f_{k/i} := f_k(x_{1/i}, \dots, x_{n/i})$. Then

$$U_i := X_K \cap D(x_i)_K \cong \operatorname{Spec}(K[x_{0/i}, \dots, x_{n/i}]/(f_{1/i}, \dots, f_{n/i})).$$

The points $P = [a_0^P : \cdots : a_n^P]$ on this affine patch $U_i(K)$ are thus given by the maximal ideals $\mathfrak{m}_P = (x_{j/i} - a_{j/i}^P : j \neq i)$. These ideals satisfy

$$(f_{1/i}, \dots, f_{n/i}) = \prod_{P \in U_i(K)} \mathfrak{m}_P.$$

For $\alpha \colon U_i(K) \to [n] - \{i\}$, write $x_\alpha - a_\alpha$ for $\prod_{P \in U_i(K)} (x_{\alpha(P)/i} - a^P_{\alpha(P)/i})$, so all the $x_\alpha - a_\alpha$ generate the ideal $\prod_{P \in U_i(K)} \mathfrak{m}_P$. Thus we have

$$f_{j/i} = \sum_\alpha c^\alpha_{ji}(x_\alpha - a_\alpha) \tag{1}$$

and

$$x_\alpha - a_\alpha = \sum_j d^\alpha_{ji} f_{j/i} \tag{2}$$

where the c_{ji}^α and d_{ji}^α are polynomials over K. Now let $N \in \mathcal{O}_K$ be the product (ranging over all i, j, α) of all the denominators appearing in the coefficients of any c_{ji}^α or d_{ji}^α as well as the denominators of the $a_{j/i}^P$ and all differences $a_{j/i}^P - a_{j/i}^Q$ ranging over $P, Q \in U_i(K)$.

We now describe $X_{\mathcal{O}_K[1/N]}$ over affine patches. Fix an affine patch $U_i = X_{\mathcal{O}_K[1/N]} \cap D(x_i)_{\mathcal{O}_K[1/N]}$. Then the cooordinate ring of U_i is

$$\Gamma_i := \mathcal{O}_K[1/N][x_{0/i}, \ldots, x_{n/i}]/(f_{1/i}, \ldots, f_{n/i}).$$

Note that $(x_{0/i}, \ldots, x_{n/i})$ is a prime ideal of $\mathcal{O}_K[1/N][x_{0/i}, \ldots, x_{n/i}]$, as the quotient is $\mathcal{O}_K[1/N]$. Thus so is $\mathfrak{p}_P = (x_{0/i} - a_{j/i}^P, \ldots, x_{n/i} - a_{j/i}^P)$ for all $P \in U_i(K)$. Notice that $\mathfrak{p}_P$ and $\mathfrak{p}_Q$ are comaximal for all distinct $P, Q \in U_i(K)$, since there is some index j so that $a_{j/i}^P - a_{j/i}^Q = (x_{j/i} - a_{j/i}^Q) - (x_{j/i} - a_{j/i}^P) \in \mathfrak{p}_P + \mathfrak{p}_Q$ is nonzero, and thus invertible, as otherwise P and Q would be the same point in $U_i(K)$. Moreover, by (1) and (2), we have

$$(f_{1/i}, \ldots, f_{n/i}) = \prod_{P \in U_i(K)} \mathfrak{p}_P.$$

By the Chinese remainder theorem

$$\Gamma_i \cong \prod_{P \in U_i(K)} \mathcal{O}_K[1/N][x_{0/i}, \ldots, x_{n/i}]/\mathfrak{p}_P \cong \prod_{P \in U_i(K)} \mathcal{O}_K[1/N].$$

It follows that

$$U_i \cong \operatorname{Spec} \Gamma_i \cong \coprod_{P \in U_i(K)} (\operatorname{Spec} \mathcal{O}_K[1/N])_P.$$

To get $X_{\mathcal{O}_K[1/N]}$, we glue all the affine parts U_i along their intersections U_{ij}. Note $U_{ij} = \operatorname{Spec} \Gamma_i[x_{j/i}^{-1}]$ and $U_{ji} = \operatorname{Spec} \Gamma_j[x_{i/j}^{-1}]$. In $\Gamma_i[x_{j/i}^{-1}]$, we have that $(\mathfrak{p}_P)_{x_{i/j}} = (1)$ if and only if $x_{j/i} \in \mathfrak{p}_P$ if and only if $a_{j/i}^P = 0$. Thus

$$\Gamma_i[x_{j/i}^{-1}] = \prod_{\{P \in U_i(K) : a_{j/i}^P \neq 0\}} \operatorname{Spec} O_K[1/N]$$

and

$$\Gamma_j[x_{i/j}^{-1}] = \prod_{\{P \in U_j(K) : a_{i/j}^P \neq 0\}} \operatorname{Spec} O_K[1/N].$$

The isomorphism $U_{ij} \to U_{ji}$ is induced by the map $\Gamma_i[x_{j/i}^{-1}] \to \Gamma_j[x_{i/j}^{-1}]$ given by $x_{k/i} \mapsto x_{k/j}$ for $k \neq j$ and $x_{j/i} \mapsto x_{i/j}^{-1}$. This sends

$$\begin{aligned}(x_{0/i} - a_{0/i}^P, \ldots, x_{j/i} - a_{j/i}^P, \ldots x_{n/i} - a_{n/i}^P) &\mapsto (x_{0/j} - a_{0/j}^P, \ldots, x_{i/j}^{-1} - a_{j/i}^P, \ldots x_{n/j} - a_{n/j}^P) \\ &= (x_{0/j} - a_{0/j}^P, \ldots, x_{i/j} - a_{i/j}^P, \ldots x_{n/j} - a_{n/j}^P).\end{aligned}$$

Thus the glueing identifies the $(\operatorname{Spec} \mathcal{O}_K[1/N])_P$ in U_i with $(\operatorname{Spec} \mathcal{O}_K[1/N])_P$ in U_j. This glueing is canonically associated to how the points of $(U_i)_K$ and $(U_j)_K$ are glued to get X_K, so we have that

$$X_{\mathcal{O}_K[1/N]} \cong \coprod_{P \in X_K(K)} (\operatorname{Spec} \mathcal{O}_K[1/N])_P = \coprod_{P \in V_{\mathbb{C}}(f_1, \ldots, f_n)} (\operatorname{Spec} \mathcal{O}_K[1/N])_P.$$

Now we consider the G-action on $X_{\mathcal{O}_K[1/N]}$. Since it is affine, then multiplication by g is induced by a map of $\mathcal{O}_K[1/N]$-algebras $g \colon \prod_{P \in V_{\mathbb{C}}(f_1, \ldots, f_n)} \mathcal{O}_K[1/N] \to \prod_{P \in V_{\mathbb{C}}(f_1, \ldots, f_n)} \mathcal{O}_K[1/N]$. The map must send 1 to

1, and thus permutes the copies of $\operatorname{Spec} \mathcal{O}_K[1/N]$. Observe that the action by g on the $\mathbb{C}$-points is given by base changing the map g. Thus the way in which g permutes the copy $(\operatorname{Spec} \mathcal{O}_K[1/N])_P$ is the same as how g permutes the actual point P in $V_{\mathbb{C}}(f_1, \ldots, f_n)$. More succinctly, the G-action on $X_{\mathcal{O}_K[1/N]}$ acts by mapping $(\operatorname{Spec} \mathcal{O}_K[1/N])_P$ isomorphically onto $(\operatorname{Spec} \mathcal{O}_K[1/N])_{g.P}$.

Now let $\mathfrak{p} \in \mathcal{O}_K$ be any nonzero prime which does not divide N and lies over p. Then $\mathfrak{p}$ is maximal in $\mathcal{O}_K[1/N]$, so the quotient $F = \mathcal{O}_K[1/N]/\mathfrak{p}$ is a finite field of characteristic p. Then base changing $X_{\mathcal{O}_K[1/N]}$ to F identifies the copy of $(\operatorname{Spec} \mathcal{O}_K[1/n])_P$ with an F-point which further corresponds to a solution $\overline{P} \in V_F(f_1, \ldots, f_n)$. In fact, every solution in $V_F(f_1, \ldots, f_n)$ comes from the base change of an $\mathcal{O}_K[1/N]$-point of $X_{\mathcal{O}_K[1/N]}$ as there are d $\mathcal{O}_K[1/N]$-points which give rise to d different F-points, and $|X_F(F)| = |V_F(f_1, \ldots, f_n)| \leq d$ by Bezout's theorem for $\overline{\mathbb{F}}_p$. Identifying $\mathbb{C}$-points P with the corresponding point $\overline{P}$, the action of g on $V_F(f_1, \ldots, f_n)$ is $g.\overline{P} = \overline{g.P}$, since the action comes from base changing the action over $\mathcal{O}_K[1/N]$, which further corresponds to the aciton on $V_{\mathbb{C}}(f_1, \ldots, f_n)$. Thus, as G-sets, we have the isomorphism

$$V_{\overline{\mathbb{F}}_p}(f_1, \ldots, f_n) = V_F(f_1, \ldots, f_n) \cong V_{\mathbb{C}}(f_1, \ldots, f_n).$$

We conclude the theorem. □