LOCAL MODULI

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CONTENTS

1. INTRODUCTION

The goal of this article is to introduce some basic deformation theory and some particular aspects of its usage in moduli theory.

2. BASIC DEFORMATION THEORY

We begin by introducing some basic notions which will motivate some of our later definitions.

2.1. Artin rings.

An important part of the deformation theory we develop is Artin rings.

Definition 2.1. A ring A is an *Artin* ring if it is a Noetherian ring of dimension zero.

The spectrum of an Artin ring A consists of a single point and has reduced structure $\operatorname{Spec} k$, where k is the residue field of A. There is also a natural map $\operatorname{Spec} k \to \operatorname{Spec} A$. Morally, we think of $\operatorname{Spec} A$ as an infinitesimal thickening of $\operatorname{Spec} k$. One example of an Artin ring is $k[\varepsilon]/\varepsilon^2$. An example of the aforementioned philosophy is that the tangent space of a scheme X at a point x admits an interpretation as the space of maps $\operatorname{Hom}(\operatorname{Spec} k[\varepsilon]/\varepsilon^2, X)$ with target x. Here, the infinitesimal data associated to $\operatorname{Spec} k[\varepsilon]/\varepsilon^2$ is the tangent direction to x. Artin rings can also be used to give a criteria for smoothness (see [2, Tag 02H6]).

We make the following definition towards capturing the notion of local moduli.

Definition 2.2. Fix a complete, local ring W with maximal ideal $\mathfrak{m}_W$ and residue field k. The category Art_W consists of Artin local W-algebras R such that $W/\mathfrak{m}_W \to R/\mathfrak{m}_R$ is an isomorphism with maps given by local W-algebra homomorphisms.

Remark 2.3. Note Art_k is a full subcategory of Art_W since any k-algebra can be made into a W-algebra via $W \to W/\mathfrak{m}_W = k$. In particular, the dual numbers $k[\varepsilon]/\varepsilon^2$ is always in Art_W.

We will study local moduli problems, which are naturally interpreted as functors $\mathrm{Art}_W \to \mathrm{Set}$. Many classical moduli problems can be thought of as functors $\mathrm{Ring}_k \to \mathrm{Set}$, and thus give local moduli problems by restricting to the subcategory Art_k. Often, these local moduli problems are more tractable then the general ones.

2.2. **Deformations.** Now, we need to ask: What is a deformation? To make sense of what this should be, we first look to complex geometry for inspiration. There, a deformation of a complex manifold M is a proper holomorphic map from $X \to B$, where B is a small neighborhood of 0 contained in $\mathbb{C}^n$ and $\pi^{-1}(0) = M$. We can imitate this construction algebro-geometrically by thinking of a scheme X/S as a family of schemes paramerized by S.

Definition 2.4. Let $R' \to R$ be a surjective ring homomorphism, and X be a (blank) scheme over $S = \operatorname{Spec} R$. A *deformation* of X is a (blank) scheme X' over $S' = \operatorname{Spec} R'$ fitting into the diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \big\downarrow & \lrcorner & \big\downarrow \\ S & \longrightarrow & S' \end{array}$$

When R' is an Artin ring with residue field $R = k$, we say X' is an *infinitesimal deformation* of X.

We will concern ourselves only with infinitesimal deformations in this exposition. One reason for this is that arbitrary deformations can be very complicated, while infinitesimal deformations are simpler and more easily understood. Indeed, at first glance, it might even seem as though the infinitesimal deformations are too simple, since the spectrum of an Artin ring consists of a single point. However, we will see that we still gain valuable information under this restriction. We already saw some ways in which Artin rings produce useful data in the previous subsection.

As we are ultimately interested in using this theory to study moduli, we consider some immediate connections. First we single out a particular type of infinitesimal deformation.

Definition 2.5 (First order deformations)**.** Let X/k be a scheme. A deformation $X' \to k[\varepsilon]/\varepsilon^2$ of X is called a *first order deformation* of X.

Let $\mathcal{M}$ be a moduli space parameterizing families of schemes over a field k. The tangent space of $\mathcal{M}$ at a point X corresponds to the space of maps $\mathrm{Hom}(k[\varepsilon]/\varepsilon^2, \mathcal{M})$ whose image is X. Since $\mathcal{M}$ is a moduli space, these naturally correspond to the first order deformations of X.

On the other hand, we can imitate this construction in reverse to get a local moduli problem. Fix a smooth scheme X/W.

Definition 2.6. The *functor of deformations* $\mathrm{Def}_X \colon \mathrm{Art}_W \to \mathrm{Set}$ associates to a ring $A \in \mathrm{Art}_W$, the set of deformations of X over $\operatorname{Spec} A$.

Our primary goal will be to understand this "local moduli" functor.

3. PRO-REPRESENTABLE FUNCTORS

Given any moduli problem, arguably the most important question to ask is if the moduli problem is representable. In the case of local moduli, we will not be able to get full representability, but we can often get something very close.

3.1. **Pro-representability.** First, we define an enlargement of the category Art_W.

Definition 3.1. The category $\widehat{\mathrm{Art}}_W$ consists of all complete local Noetherian W-algebras $\mathcal{O}$ such that $\mathcal{O}/\mathfrak{m}_{\mathcal{O}}^n \in \mathrm{Art}_W$ for all n, with morphisms the local W-algebra homomorphisms.

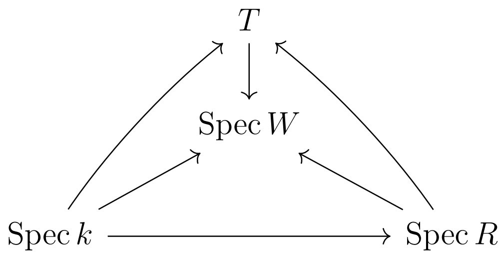
We formalize what it means to be almost representable as follows:

Definition 3.2. A functor $F\colon \mathrm{Art}_W \to \mathrm{Set}$ is *pro-representable* if there exists an $\mathcal{O} \in \widehat{\mathrm{Art}}_W$ and an isomorphism of functors

$$\mathrm{Hom}_{\widehat{\mathrm{Art}}_W}(\mathcal{O}, -)|_{\mathrm{Art}_W} \xrightarrow{\sim} F(-).$$

This notion of pro-representability turns out to be what we care about.

Example 3.3. Let $T \to \operatorname{Spec} W$ be a Noetherian scheme over W, with a k-rational point $t \in T(k)$. We define the functor $F\colon \mathrm{Art}_W \to \mathrm{Set}$, by taking $F(R)$ to be the set of the R-points of T such that the following diagram commutes:



This functor is pro-representable, because

$$\mathrm{Hom}_{\widehat{\mathrm{Art}}_W}(\widehat{\mathcal{O}_{T,t}}, R)|_{\mathrm{Art}_W} \xrightarrow{\sim} F(R).$$

To see this, fix an Artin local ring R. Any morphism $\operatorname{Spec} R \to T \in F(R)$ factors through $\operatorname{Spec} \mathcal{O}_{T,t}$ so we have a map $\mathcal{O}_{T,t} \to R$. This induces a map on completions $\widehat{\mathcal{O}_{T,t}} \to \widehat{R} = R$. Also, given a map $\widehat{\mathcal{O}_{T,t}} \to R$, we get a map $\mathcal{O}_{T,t} \to R$ via the natural map $\mathcal{O}_{T,t} \to \widehat{\mathcal{O}_{T,t}}$. There is a canonical morphism $\operatorname{Spec} \mathcal{O}_{T,t} \to T$, so we compose to get a map $\operatorname{Spec} R \to T \in F(R)$. These processes are mutually inverse, so we establish the bijection.

Interestingly, what this shows, is that given knowledge of $F(R)$, without knowing T itself, we can predict some of the properties of $\mathcal{O}_{T,t}$ that can be read from its completion, like regularity or dimension.

3.2. **Schlessinger's Criterion.** In this subsection, we develop the necessary theory required to state a criterion which characterizes the pro-representability of a local moduli problem satisfying certain conditions.

Definition 3.4. A functor $F\colon \mathrm{Art}_W \to \mathrm{Set}$ is *left exact* if $F(W) = \{*\}$ and F commutes with fibred products, i.e. the natural map

$$F(X \times_Y Z) \xrightarrow{\sim} F(X) \times_{F(Y)} F(Z)$$

is bijective.

Definition 3.5. Let $F\colon \mathrm{Art}_W \to \mathrm{Set}$ be left exact. The *tangent space* to F, which we denote by T_F, is the set $F(k[\varepsilon]/\varepsilon^2)$.

Remark 3.6. The tangent space to F is equipped with the structure of a vector space over k.

Proof. The zero element is the image of $* \to T_F$ induced by $k \to k[\varepsilon]/\varepsilon^2$. An element $\lambda \in k$ induces a ring homomorphism $\lambda\colon k[\varepsilon]/\varepsilon^2 \to k[\varepsilon]/\varepsilon^2$ with the map $a + b\varepsilon \mapsto a+\lambda b\varepsilon$, so $F(\lambda)\colon T_F \to T_F$ is the scalar multiplication. We have the map $+\colon k[\varepsilon]/\varepsilon^2 \times_k k[\varepsilon]/\varepsilon^2 \to k[\varepsilon]/\varepsilon^2$ given by $(a + b\varepsilon, a + b'\varepsilon) \mapsto a + (b + b')\varepsilon$. By left exactness $F(+)$ is a map $T_F \times T_F \to T_F$ which is the addition operation on the tangent space. It is easy to see that these operations make T_F a vector space. ☐

Definition 3.7. A functor $F\colon \mathrm{Art} \to \mathrm{Set}$ is *formally smooth* if for every surjection $\pi\colon R \to R'$ in Art_W, the map $F(\pi)\colon F(R) \to F(R')$ is surjective.

Definition 3.8. A surjection $R \to R'$ in Art_W is *small* if $I := \ker(R \to R')$ satisfies $I\mathfrak{m}_R = 0$.

Remark 3.9. Any surjection in Art is the composition of a finite numnber of small surjections.

Proof. Take $R \to R'$ a surjection in Art_W. The map factors as $R \to R/I \to R'$ and the last map is small, so it suffices to consider quotient maps $R \to R/I$. Since R is Artin, the maximal ideal of R is nilpotent, so let n be such that $\mathfrak{m}_R^n = 0$. Then the quotient map $R \to R/I$ factors as

$$R \to R/\mathfrak{m}_R^n I \to R/\mathfrak{m}_R^{n-1} I \to \cdots \to R/\mathfrak{m}_R I \to RI.$$

☐

We can now state the characterization.

Theorem 3.10 (Schessinger's Criterion [1])**.** *A functor $F\colon \mathrm{Art}_W \to \mathrm{Set}$ is pro-representable if and only if F is left exact and*

$$\dim_k(T_F) < \infty.$$

It suffices to check left exactness in the case $R_1 \to R_2 \leftarrow R_3$, where the first map is a small surjection. If F is pro-represented by $\mathcal{O}$, is formally smooth, and $\dim(T_F) = d$, then there is an isomorphism $\mathcal{O} \cong W[[t_1, \dots, t_d]]$.

One can think of this result as a generalization of Cohen's structure theorem. For the sake of comparison, we recall Cohen's structure theorem.

Theorem 3.11 (Cohen's structure theorem [2, Tag 0C0S])**.** *If $(A, \mathfrak{m}, K)$ is a complete regular local ring containing a field, then $A \cong K[[X_1, ..., X_d]]$, where $d = \dim(A)$.*

References

[1] Frans Oort. Finite group schemes, local moduli for abelian varieties, and lifting problems. *Compositio Mathematica*, 23(3):265–296, 1971.

[2] The Stacks project authors. The stacks project. https://stacks.math.columbia.edu, 2024.