PERIOD DOMAINS AND CLASSIFYING SPACES OF HODGE STRUCTURES

ADAM MELROD

ABSTRACT. Notes prepared for the 2023 Hodge theory RIT organized by Patrick Brosnan.

CONTENTS

1. PRELIMINARIES

We first recall some basic notions.

Definition 1.1. A (pure) Hodge structure of weight n, denoted $(H_{\mathbb{Z}}, H^{p,q})$, consists of a free $\mathbb{Z}$-module $H_{\mathbb{Z}}$ along with a decomposition $H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}$ where $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ and $H^{p,q} = \overline{H^{q,p}}$. The vector of Hodge numbers $h = (h^{p,q}) = (\dim H^{p,q})$ is the type of the Hodge structure.

Using the decomposition $H^{p,q}$ of a Hodge structure, we obtain a finite decreasing filtration F of $H_{\mathbb{C}}$ by taking $F^p := \bigoplus_{i \geq p} H^{i,n-i}$. This prompts the following (equivalent) definition.

Definition 1.2. A (pure) Hodge structure of weight n, denoted $(H_{\mathbb{Z}}, F)$, consists of a free $\mathbb{Z}$-module $H_{\mathbb{Z}}$ along with a finite decreasing filtration F on $H_{\mathbb{C}}$ so that $H = F^p \oplus \overline{F^{n-p+1}}$.

From a Hodge structure with filtration F on $H_{\mathbb{C}}$, we obtain a decomposition $H^{p,q} = F^p \cap \overline{F^q}$. The decomposition data and filtration data are thus equivalent.

Example 1.3. Throughout these notes we will use elliptic curves as a testing ground for the new ideas we develop. As such, let E be an elliptic curve. There is a natural Hodge structure H of weight 1 and type $(1,1)$ on the first cohomology $H^1(E, \mathbb{Z})$ arising from Hodge decomoposition. Concretely, we take $H_{\mathbb{Z}} := H^1(E, \mathbb{Z}) \cong \mathbb{Z}^2$ and the decomposition is

$$H_{\mathbb{C}} = H^1(E, \mathbb{C}) = H^{1,0} \oplus H^{0,1} = \{f(z)dz\} \oplus \{g(\bar{z})d\bar{z}\}$$

where $H^{1,0}$ are the holomorphic 1-forms and $H^{0,1}$ are the antiholomorphic 1-forms. The Hodge filtration F associated to this Hodge structure is

$$H^1(E, \mathbb{C}) \supseteq \{f(z)dz\} \supseteq 0.$$

2. Polarized Hodge Structures

We introduce the notion of a polarization on a Hodge structure.

Definition 2.1. A *polarization* for a Hodge structure $(H_{\mathbb{Z}}, H^{p,q})$ of weight n is a nondegenerate bilinear form $Q : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}$ which extends to $H_{\mathbb{C}}$ by linearity and is symmetric if n is even and alternating if n is odd. We also require that Q be subject to the following relations:

$$Q(H^{p,q}, H^{p',q'}) = 0 \text{ if } (p', q') \neq (q, p). \tag{2.1.1}$$

$$i^{p-q} Q(x, \bar{x}) > 0 \text{ for nonzero } x \in H^{p,q}. \tag{2.1.2}$$

These conditions are called the *Hodge-Riemann bilinear relations*. In terms of the filtration F, the relations become:

$$Q(F^p, F^{n-p+1}) = 0. \tag{2.1.1'}$$

$$Q(C_F x, \bar{x}) > 0 \text{ for nonzero } x. \tag{2.1.2'}$$

where C_F is the *Weil operator* defined by $C_F(x) := i^{p-q}x$ for $x \in H^{p,q}$.

Definition 2.2. A polarized Hodge structure of weight n, denoted $(H_{\mathbb{Z}}, F, Q)$, is a pure Hodge structure $(H_{\mathbb{Z}}, F)$ with a polarization Q for $(H_{\mathbb{Z}}, F)$.

Let us contextualize this definition. The additional data of a polarization will allow us to construct a *period domain* that classifies polarized Hodge structures. This polarization data is quite natural. In the following, we consider a large class of naturally occurring polarized Hodge structures.

Example 2.3. Fix a compact Kähler manifold X with Kähler form ω. Fix an integer $k \geq 0$. Let $(H_{\mathbb{Z}}, H^{p,q})$ be the pure Hodge structure obtained from the primitive kth cohomology of X. We use ω to define a nondegenerate bilinear form $Q\colon H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}$ by

$$Q(\alpha, \beta) := (-1)^{k(k-1)/2} \int_X \alpha \wedge \beta \wedge \omega^{\dim(X)-k}.$$

This form Q is a polarization for $(H_{\mathbb{Z}}, H^{p,q})$ so we get the polarized Hodge structure $(H_{\mathbb{Z}}, H^{p,q}, Q)$.

Example 2.4. Let us specialize to elliptic curves and work out the details. We have the Hodge structure on E as in Example 1.3. In the context of example 2.3, we have $k = 1$ and E is a 1-dimensional manifold so the bilinear form Q is $Q(\alpha, \beta) = \int_E \alpha \wedge \beta$. Observe Q is alternating since $Q(\alpha, \beta) = \int_E \alpha \wedge \beta = -\int_E \beta \wedge \alpha = -Q(\beta, \alpha)$. We need only check the Hodge-Riemann bilinear relations to verify that Q is a polarization. We first check (2.1.1). As there are only two nonzero $H^{p,q}$ in the decomposition, it suffices to show that $Q(\alpha, \beta) = 0$ where α and β are both $(1, 0)$ forms or both $(0, 1)$ forms. Thus $\alpha \wedge \beta$ is locally of the form $f dz \wedge dz$ or $f d\bar{z} \wedge d\bar{z}$ which are both zero so $Q(\alpha, \beta) = \int_E \alpha \wedge \beta = 0$. Now we need to check (2.1.2). Take any $\alpha \in H^{1,0}$ a nonzero holomorphic 1-form. Locally $\alpha = f dz$ and $\bar{\alpha} = \bar{f} d\bar{z}$ so $i\alpha \wedge \bar{\alpha}$ is locally $i|f|^2 dz \wedge d\bar{z} = 2|f^2| dx \wedge dy$ which always has positive integral locally. Thus $i^{1-0} Q(\alpha, \bar{\alpha}) = i \int_E \alpha \wedge \bar{\alpha} > 0$. Any nonzero $\beta \in H^{0,1}$ is the conjugate of a holomorphic 1-form α so $i^{0-1} Q(\beta, \bar{\beta}) = -iQ(\bar{\alpha}, \alpha) = iQ(\alpha, \bar{\alpha}) > 0$. We have verified that Q satisfies the Hodge-Riemann bilinear relations so Q is a polarization. We identify $H_{\mathbb{Z}} = H^1(E, \mathbb{Z})$ and $\mathbb{Z}^2$ with canonical basis α, β by taking the Poincaré dual of the canonical basis of $H_1(E, \mathbb{Z})$ so that the matrix of Q with respect to this basis is

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

3. Digression on Grassmanians and Flag Varieties

Before we finally construct the period domain of a polarized Hodge structure, we must first discuss Grassmanians and flag varieties which will be important in constructing our desired classifying space. The Grassmanian parameterizes fixed dimensional subspaces of a complex vector space and that notion is captured as follows:

Definition 3.1. Let V be a complex vector space. Let $\mathrm{Grass}(k, V)$ denote the set of complex vector subspaces of V of dimension k. This space carries the structure of a projective manifold. This is realized by the Plücker embedding $\mathrm{Grass}(k, V) \hookrightarrow \mathbb{P}(\bigwedge^k W)$ which associates to a k-dimensional subspace $W \subseteq V$ the line $[\alpha_1 \wedge \cdots \wedge \alpha_k] \in P(\bigwedge^k W)$ where α_i form a basis of W in V. This is well-defined since two bases of W differ by a change of basis matrix which will induce the same wedge product up to a nonzero constant so they determine the same line.

We are also interested in abstract flag varieties as they correspond to spaces of filtrations.

Definition 3.2. Fix a sequence $a_1 < a_2 < \cdots < a_k < n$. The *variety of flags* is

$$\mathrm{F}(a_1, \ldots, a_k; V) := \{W_1, \ldots, W_k : W_1 \subset \cdots \subset W_k\} \subset \mathrm{Grass}(a_1, V) \times \cdots \times G(a_k, V).$$

This is a smooth subvariety of the product of Grassmanians.

4. Period Domains

The notion of a variation of Hodge structures will motivate our definition of a period domain. Consider the following scenario: Let $f\colon \mathcal{X} \to \Delta$ be a proper smooth surjective morphism onto a complex polydisc Δ where each fiber F is a compact Kähler manifold. Each such F has a natural polarized Hodge structure on cohomology for each k. Ehresmann's theorem implies that the underlying lattice of these polarized Hodge structures are uniquely isomorphic and that the hodge numbers are all equal. However the Hodge filtration (equivalently the Hodge decomoposition) is not necessarily preserved under the natural isomorphism. We thus introduce a space that classifies the filtrations.

Definition 4.1. Fix a polarized Hodge structure $(H_{\mathbb{Z}}, F, Q)$ of weight n and type $(h^{p,q})$. The period domain $\mathcal{D}$ of this data is the space of all pure Hodge structures with underlying lattice $H_{\mathbb{Z}}$ of weight n and type $(h^{p,q})$ for which Q is a polarization. Formulated in terms of filtrations, the period domain may be defined to be the space of all filtrations

$$H_{\mathbb{C}} = F^0 \subset F^1 \subset \cdots \subset F^n \subset \{0\},$$

where $\dim(F^p) = \sum_{i \geq p} h^{i,n-i}$, on which Q satisfies (2.1.1) and (2.1.2).

The latter formulation in terms of filtrations is the one we will primarily use to describe this space geometrically. We associate another manifold to the period domain $\mathcal{D}$.

Definition 4.2. The compact dual $\check{\mathcal{D}}$ is the same underlying set as the period domain except that we remove the requirement that the second Hodge-Riemann bilinear relation is satisfied.

The first Hodge-Riemann bilinear relation is a closed condition. In this way, the compact dual $\check{\mathcal{D}}$ is a closed submanifold of a flag variety and thus is a compact complex manifold. The second Hodge-Riemann bilinear relation is an open condition and realizes the period domain $\mathcal{D}$ as a open subset of $\check{\mathcal{D}}$.

Remark 4.3. Both $\mathcal{D}$ and $\check{\mathcal{D}}$ are homogeneous spaces. Fix a polarized Hodge structure $(H_0, F_0, Q_0) \in \mathcal{D}$. Let G be the automorphisms of $H_{0,\mathbb{R}}$ preserving Q_0 and $G_{\mathbb{C}}$ the complexification of G. The group G acts transitively on $\mathcal{D}$ and $G_{\mathbb{C}}$ acts transitively on $\check{\mathcal{D}}$.

Example 4.4. Again lets continue from Example 2.4. We identify the corresponding period domain and compact dual of an elliptic curve. The underlying lattice is $\mathbb{Z}^2$ and the form Q on the lattice is

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The set of relevant filtrations in $\mathcal{D}$ are those of the form

$$\mathbb{C}^2 \supset F^1 \supset \{0\}$$

where $\dim F^1 = 1$ and on which Q satisfies (2.1.1) and (2.1.2). A point in $\mathcal{D}$ is thus determined by $\lambda \in \mathbb{C}^2$ spanning F^1 on which $Q(\lambda, \lambda) = 0$ and $iQ(\lambda, \bar{\lambda}) > 0$. We write $\lambda = z_1\alpha + z_2\beta$ where $z_1, z_2 \in \mathbb{C}$ in terms of the canonical basis α, β so that the Hodge-Riemann relations become $z_1 z_2 - z_2 z_1 = 0$ and $i(z_1\bar{z}_2 - z_2\bar{z}_1) > 0$. Note that the first relation always holds in this case so the compact dual $\check{\mathcal{D}}$ is identifiable with $\mathbb{CP}^1$ as every line in $\mathbb{C}^2$ yields a valid point of $\check{\mathcal{D}}$. The second condition implies that $z_1 \neq 0$ so we may scale λ to be $\alpha + z_2\beta$. Then the final condition become $\operatorname{Im}(z_2) > 0$. Specifying z_2 determines λ so the period domain is the complex upper half plane $\mathfrak{h}$.

We are also interested in classifying isomorphism classes of polarized Hodge structures which leads to the notion of a period space.

Definition 4.5. Let Γ be the group of automorphisms of $H_{\mathbb{Z}}$ preserving Q, that is,

$$\Gamma := \{g \colon H_{\mathbb{Z}} \to H_{\mathbb{Z}} : Q(gx, gy) = Q(x, y) \text{ for all } x, y \in H_{\mathbb{Z}}\}.$$

The group Γ is a group of matrices with integral coefficients which acts on the period domain $\mathcal{D}$. The quotient $\Gamma\backslash\mathcal{D}$ is called the *period space* and classifies the isomorphism classes of Hodge structures.

Example 4.6. We determine the period space of elliptic curves with the period domain $\mathcal{D} = \mathfrak{h}$ identified in Example 4.4. The group Γ of linear transformations $\mathbb{Z}^2 \to \mathbb{Z}^2$ that preserve the bilinear form Q is exactly the group $\mathrm{SL}(2, \mathbb{Z})$. The action of Γ on $\mathfrak{h}$ is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{c + dz}{a + bz}.$$

The negative identity matrix acts trivially so we get an induced action by $\mathrm{PSL}(2, \mathbb{Z})$ on $\mathcal{D}$ and the period domain $\Gamma\backslash\mathcal{D} \cong \mathrm{PSL}(2, \mathbb{Z})$ which is the classical moduli space of elliptic curves. From this we can see that an elliptic curve is completely characterized by it's polarized Hodge structure up to isometry.

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