Toric geometry and moduli space of sheaves

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12/24/2014

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$$e(X) = \text{alternating sum of of Betti numbers} = \int_X c_{top}(X).$$

  The last equality only true if X is complete (compact).

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E.g. From (3) we see easily that $e(\mathbb{P}^2) = 3$.

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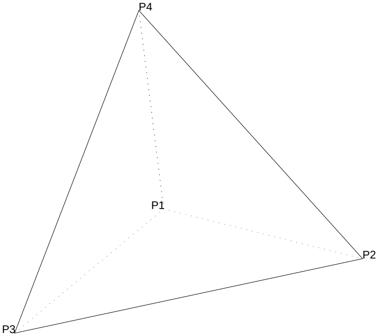
- X has a natural affine open covering by open subsets $U_1, \ldots, U_{e(X)}$, where $U_i \cong \mathbb{C}^n$ is T-invariant and is centered at the i-th fixed point.
- The coordinate axes in each U_i extend to T-invariant lines joining pairs of fixed points in X. Newton polyhedron $\Delta(X)$ is a polyhedron associated to X, whose vertices and edges correspond respectively to the fixed points and the invariants lines in X.

Example $\Delta(\mathbb{P}^3)$



$$(\mathbb{P}^3)^T = \{P1, P2, P3, P4\}$$

Six T-invariant lines $\{P1P2, \ldots\}$

Poincaré Polynomial

Let X be a variety over $\mathbb{C}$ of dimension n. The Poincaré polynomial of X is defined by

$$P_X(z) = \sum_{i=0}^{2n} b_i(X) z^i$$

where $b_i(X) = \text{rank } H_i(X)$ is the i-th Betti number (Borel-Moore homology if X is not compact).

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Note:

$$e(X) = P_X(-1).$$

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$T_{p_i}X$ is a $\mathbb{C}^*$-representation and hence splits into eigenspaces

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Then Bialynicki-Birula's theorem proves that X has a cell decomposition with the cells $C_1, \ldots, C_n$, and $T_{p_i}C_i = T_{p_i}^+ X$.

Corrolaries of BB decomposition

This means that there exists a filtration

$$X = X_n \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

by closed subschemes with each $X_i - X_{i-1}$ is a disjoint union of affine spaces called cells.

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- $P_X(z) = \sum_{p_i \in X^T} z^{2 \dim T^+_{p_i} X}$.

Example: $\mathbb{P}^2$

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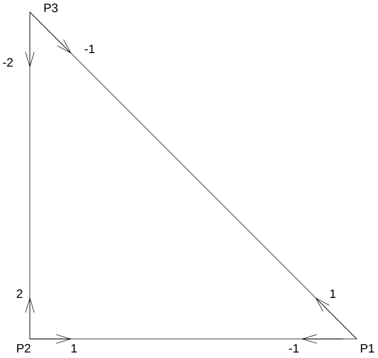

Cell decomposition: $P3 \coprod (P1P3 - P3) \coprod (\mathbb{P}^2 - P1P3)$.

$$P_{\mathbb{P}^2}(z) = 1 + z^2 + z^4.$$

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$$h^{p,q}(X) = \sum_F h^{p-n_F,q-n_F}(F).$$

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- If $Y \subset X$ is a codimension d T-invariant subvariety then it defines a class

$$[Y] = [(\mathbb{C}^\infty - \{0\})^k \times^T Y] \in H_T^{2d}(X).$$

Example

1. If $V_{(a_1,\ldots,a_k)} = \mathbb{C}$ is the representation of $T = \mathbb{C}^{*k}$ of weight $(a_1, \ldots, a_k)$ then it can be regarded as an equivariant line bundle over a point. Then, $c_i^T(V_{(a_1,\ldots,a_k)}) = a_1 s_1 + \cdots + a_k s_k$.

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2. The diagonal action of $T = \mathbb{C}^{*n}$ on $\mathbb{P}^{n-1}$ induces an action on the tautological line bundle $\mathcal{O}(-1)$. Let $\xi = c_1^T(\mathcal{O}(1))$. Then, it can be seen that

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Under this

$$[V_I] = \prod_{i \in I} (\xi + s_i)$$

where $V_I = \{x_i = 0\}_{i \in I} \subset \mathbb{P}^{n-1}$.

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Then $c_d^T(N_{F/X})$ is invertible in $S^{-1}H_T^*(X)$ for any T-fixed set F.
Suppose that now that the restriction map $H_T^*(X) \to H_T^*(X^T)$ becomes surjective after localizing at S.

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E.g. $X = \mathbb{P}^1$ then $S = \{1, s_1 - s_2, (s_1 - s_2)^2, \dots\}$ and we get

$$S^{-1}H_T^*(\mathbb{P}^1) = \frac{\mathbb{Z}[s_1, s_2, \xi]_{s_1 - s_2}}{((\xi + s_1)(\xi + s_2))} \cong \mathbb{Z}[s_1, s_2]_{s_1 - s_2} \oplus \mathbb{Z}[s_1, s_2]_{s_1 - s_2}$$

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LHS is the ordinary degree of the products of the cycles. $\int$ in RHS is the equivariant push-forward $S^{-1}H_T^*(F) \to S^{-1}H_T^*(pt)$ (X does not need to be compact, X^T compact is sufficient).

Example

$\mathbb{C}^* \curvearrowright \mathbb{P}^2$ by $t \cdot (x_0 : x_1 : x_2) = (tx_0 : t^2 x_1 : x_2)$.
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Macdonald's formula:

$$\sum_m P_X(z) q^m = \frac{(1+zq)^{2g}}{(1-q)(1-z^2q)}.$$

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E.g. $\mathrm{Hilb}^2(X) \cong (\mathrm{Bl}_\Delta X \times X)/S_2$.

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More precisely, we have

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So we need to understand $\mathrm{Hilb}^m\mathbb{C}^2$ first...

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Or $(x^4 - s, y) \in \mathrm{Hilb}^4\mathbb{C}^2$, as $s \to 0$ approaches to $(x^4, y) \in \mathrm{Hilb}^4\mathbb{C}^2$ again completely supported at the origin.

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By stability of $\mathcal{I}$ and Serre duality and a RR calculation:

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In fact we know that $\mathrm{Hilb}^m X$ is connected and smooth of dimension $2n$ (Fogarty).

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2-dimensional partitions

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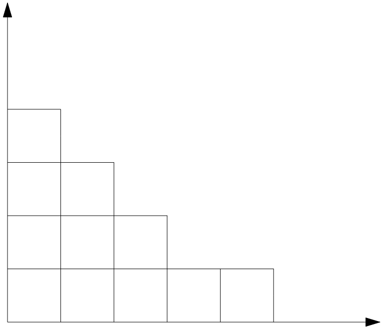

$I = (y^4, y^3x, y^2x^2, yx^3, x^5)$ of colength 11.
$\lambda = 4 + 3 + 2 + 1 + 1 \vdash 11$

Euler characteristics

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$$\sum_{m\geq 0} e(\mathrm{Hilb}^m X)q^{m-e(X)/24} = \eta(\tau)^{-e(X)},$$

where $\eta(-)$ is Dedekind eta function (modular of weight $1/2$).

Modular forms

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$$f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k f(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2, \mathbb{Z}).$$

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i.e. $T_I \text{Hilb}^m \mathbb{C}^2 = R - \chi(I, I)$ as a virtual T-representation.

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Therefore,

$$\begin{aligned}\mathrm{tr}_{T_I \mathrm{Hilb}^m \mathbb{C}^2} &= \frac{1 - P_I(t_1,t_2)P_I(t_1^{-1},t_2^{-1})}{(1-t_1)(1-t_2)} \\ &= Q + \frac{\overline{Q}}{t_1 t_2} - Q\overline{Q}\frac{(1-t_1)(1-t_2)}{t_1 t_2}\end{aligned}$$

where $\overline{Q}(t_1,t_2) = Q(t_1^{-1},t_2^{-1})$.

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It is proven by Ellingsrud and Strømme, Cheah

$$\mathrm{tr}_{T_I \mathrm{Hilb}^m \mathbb{C}^2} = \sum_{\square \in \lambda} t_1^{l(\square)} t_2^{-a(\square)-1} + t_1^{-l(\square)-1} t_2^{a(\square)}.$$

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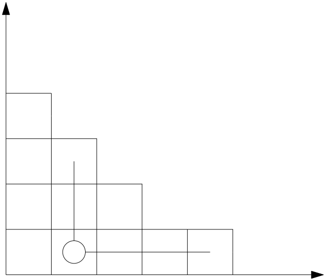

$$a(\square) = 2, \quad l(\square) = 3.$$

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$$\begin{aligned} b_{2k} \mathrm{Hilb}^m \mathbb{C}^2 &= \#\text{ of cells of dimension } k \\ &= \#\{\lambda \vdash m\} \text{ such that the largest part of } \lambda \text{ is } k - m \\ &= \#\{\mu \vdash m - (k - m) = 2m - k | \\ &\qquad \text{with parts of sizes at most } k - m\}. \end{aligned}$$

Betti numbers

Using this and BB decomposition they arrived at the following formulas:

$$b_{2k}\mathrm{Hilb}^m\mathbb{C}^2 = P(2m-k, k-m),$$

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The formula for Betti numbers can be put together:

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Other proofs were given by others...

Nakajima operators $\alpha_{-n}(\gamma)$

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$$Z(\gamma) = \{(\mathcal{I}, \mathcal{I}') | \mathcal{I}' \subset \mathcal{I}, \text{ Supp } \mathcal{I}/\mathcal{I}' = \{P\}, \text{ for some } P \in \gamma\}.$$

$$\text{Definition:} \quad (\alpha_{n-n'}(\gamma)\eta, \eta') = \int_{Z(\gamma)} \eta\eta' \quad \forall \eta \in \mathbb{H}_n, \eta' \in \mathbb{H}_{n'}.$$

For $m > 0$ define $\alpha_m(\gamma) = (-1)^m \alpha_{-m}(\gamma)^\dagger$. Then

$$[\alpha_l(\gamma), \alpha_k(\epsilon)] = -n\delta_{l+k} \int_X \gamma^{PD} \cup \epsilon^{PD}.$$

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$\mathbb{H}$ is an irreducible representation of the Heisenberg algebra generated by the $\alpha_{-m}(\gamma)$'s with $v_\emptyset = 1 \in H^0(\mathrm{Hilb}^0 X)$ being the highest weight vector.

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A linear basis for $\mathbb{H}$ is given by

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Nakajima basis is given by cohomology weighted partitions:

$$\vec{\lambda} = \{(m_1, \gamma_1), \ldots, (m_k, \gamma_k)\} \leftrightarrow \frac{1}{Z(\vec{\lambda})} \prod_{i=1}^{k} \alpha_{-m_i}(\gamma_i)$$

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Cohomology degree of this element is $2(|\lambda| - \ell(\lambda)) + \sum \deg \gamma_i^{PD}$.

Example

Nakajima basis for $H_T^*(\mathrm{Hilb}^2\mathbb{C}^2, \mathbb{Q})$ over $\mathbb{Q}[t_1, t_2]$:

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Lehn: The Chern character of the tautological bundle $ch_v(\mathcal{O}^{[n]})$ acts on $\mathbb{Q}[t_1,t_2][q_{-1},q_{-2},\dots]$ by

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$H^*_T(\mathrm{Hilb}^m\mathbb{C}^2, \mathbb{Q})_{t_1t_2}$ is generated as $\mathbb{Q}[t_1, t_2]_{t_1t_2}$-algebra by $ch_v(\mathcal{O}^{[n]})$'s and the relations between these generators are those of the restriction of the given differential operators on the degree m part.

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In the same way, the maps between coherent sheaf and exact sequences of coherent sheaf correspond (after restriction to U_i) to homomorphism of R_i-modules and the short exact sequences of R_i-modules...

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ch defines a ring isomorphism

$$K(X)_{\mathbb{Q}} \xrightarrow{ch} H^{2*}(X, \mathbb{Q}).$$

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Suppose $\mathcal{F} \in \mathrm{coh}(X)$ given by $\mathcal{F}|_{U_i} = \widetilde{F_i}$ as before. Global sections of $\mathcal{F}$ are obtained by gluing local sections i.e. elements of $\Gamma(U_i, \mathcal{F}) = F_i$'s.

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$\det \mathcal{F}$ is a line bundle and by definition

$$c_1(\mathcal{F}) = c_1(\det \mathcal{F}).$$

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From this, the moduli space of rank 1 torsion free sheaves on X (with fixed Chern classes) is isomorphic to $\text{Hilb}^m X$ for some m which we have studied so far.

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We denote by $N_X^H(2, c_1, c_2)$ the moduli space of rank 2 stable vector bundles on X with fixed first and second Chern classes c_1, c_2.

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Let $\sigma : T \times X \to X$ be the action and $p : T \times X \to X$ be the projection. A coherent sheaf $\mathcal{F}$ is called T-equivariant if there is an isomorphism $\phi : \sigma^* \mathcal{F} \cong p^* F$ that satisfies the cocycle condition i.e.

$$\begin{array}{ccc} T \times T \times X & \xrightarrow{1\times\sigma} & T \times X \\ \downarrow pr & \searrow^{\mu\times 1} & \\ T \times X & & T \times X \end{array}$$

$$(\mu \times 1)^* \phi = pr^* \phi \circ (1 \times \sigma)^* \phi.$$

Klyachko's result

The category of T-equivariant vector bundles on $\mathbb{P}^2$ is equivalent with the category of 2-dimensional $\mathbb{C}$-vector spaces E endowed with a triple of filtrations $(E^1(\ell), E^2(\ell), E^3(\ell))$ $\ell \in \mathbb{Z}$.

$$\cdots \subset E^j(\ell - 1) \subset E^j(\ell) \subset E^j(\ell + 1) \subset \dots$$

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For any T-equivariant vector bundle V on $\mathbb{P}^2$, we have the T-weight decomposition

$$\Gamma(U_j, V) = \oplus_{m \in X(T)} \Gamma(U_j, V)_m$$

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Under the equivalence above for any $m = (\ell_1, \ell_2)$ we have

$$\Gamma(U_1, V)_m = E^1(\ell_1) \cap E^2(\ell_2),$$

$$\Gamma(U_2, V)_m = E^2(\ell_1) \cap E^3(\ell_2),$$

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where

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To give a stable triple of filtrations we require to specify three distinct 1-dimensional subspaces of E, they can be fixed by the action of $Sl(2, \mathbb{C})$. So this proves $N_{\mathbb{P}^2}(c_1, c_2)^T$ consists of only isolated points.

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Twisting $V \mapsto V \otimes \mathcal{O}(k)$ induces the isomorphism

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Note: $\Delta(V) = 4c_2(V) - c_1^2(V)$ remains unchanged after twisting by a line bundle.

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From this we see that $N_{\mathbb{P}^2}(c_1, c_2)$ only depends on the discriminant $-\Delta = c_1^2 - 4c_2$, and so we simply denote it by $N_{\mathbb{P}^2}(\Delta)$.

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$$e(N_{\mathbb{P}^2}(\Delta)) = \begin{cases} 3H(\Delta) & \Delta \equiv -1 \mod 4 \\ 3H(\Delta) - 3/2d(\Delta/4) & \Delta \equiv 0 \mod 4. \end{cases}$$

Here $H(\Delta)$ is the Hurwitz function which gives the number of classes of integral binary quadratic forms Q of discriminant $-\Delta$ taken with weight $2/|\mathrm{Aut} Q|$.

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Idea of proof

By the formula above for c_1, c_2, we have

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which is of discriminant $-\Delta$. Then the inequalities $v_1 \leq v_2 \leq v_3$ is equivalent to Gaussian condition

$$C > A; -A < B \leq A \quad or \quad C = A; 0 \leq B \leq A$$

with the extra condition $B > 0$. The one can check all the multiplicities match up (!) the formulas in the theorem are obtained.

A few words about $H(\Delta)$

Two quadratic binary froms $F(X,Y)$ and $G(X,Y)$ with integer coefficients are called equivalent

$F(X,Y) = G(aX+bY, cX+dY)$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2,\mathbb{Z})$.

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Gauss observed that it is possible to find a unique reduced binary quadratic form in each equivalence class, that is with

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A few words about $H(\Delta)$

Two quadratic binary froms $F(X,Y)$ and $G(X,Y)$ with integer coefficients are called equivalent

$F(X,Y) = G(aX+bY, cX+dY)$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2,\mathbb{Z})$.

In this case F, G have the same discriminant.

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collection of vector spaces $\{F(k_1,\ldots,k_n)\}_{(k_1,\ldots,k_n)\in\mathbb{Z}^n}$ and linear maps

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$$\ldots$$

$$\chi_n(k_1,\ldots,k_n) : F(k_1,\ldots,k_n) \to F(k_1,\ldots,k_n+1),$$

such that $\chi_i \circ \chi_j = \chi_j \circ \chi_i$ for all $i, j, (k_1,\ldots,k_n)$.

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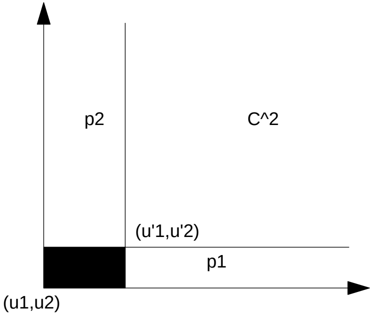
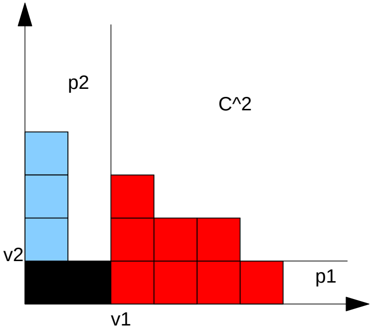


Toric rank 2 tf sheaf on $\mathbb{C}^2$

The same picture as for vector bundles except that we need to cut out two Young diagrams from the positions $(v_1, 0)$ and $(0, v_2)$:



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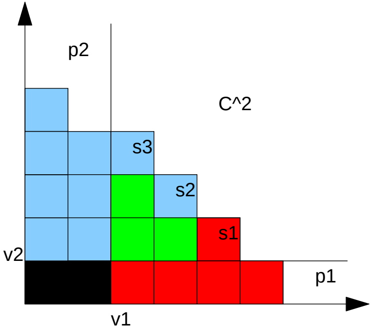
Two partition can intersect which may cause some of the squares to get extra labeling $s_1, s_2, \cdots \in \mathbb{P}^1$.



Here green boxes are in the intersection of two partitions blue and red.

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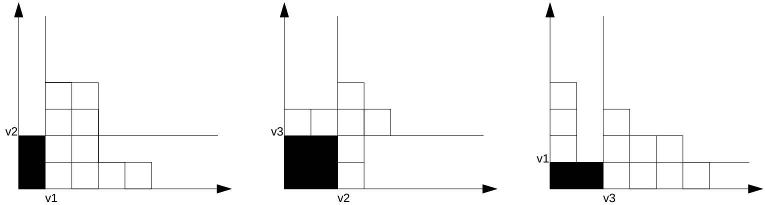

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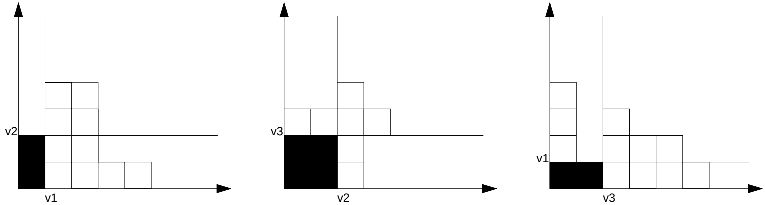

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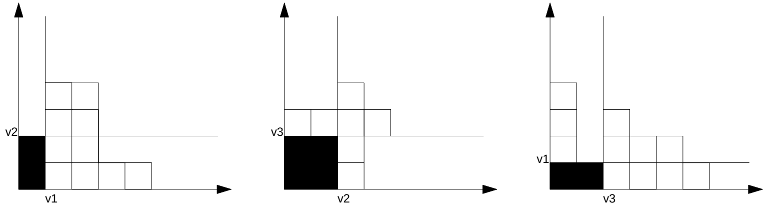

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The coarse moduli space X of $\mathfrak{X}$ is obtained by gluing these open subschemes. X is then a toric variety (possibly singular) in the usual sense.

Toric DM stacks

E.g. Weighted projective plane $\mathbb{P}(a,b,c)$ for the integers $a,b,c \geq 1$ can be covered by the open substacks

$$\mathfrak{U}_1 \cong [\mathbb{C}^2/\mu_a], \quad \mathfrak{U}_2 \cong [\mathbb{C}^2/\mu_b], \quad \mathfrak{U}_3 \cong [\mathbb{C}^2/\mu_c]$$

where say μ_a acts by $(\omega^b, \omega^c), \quad \mu_a = < \omega >$.
If $n = 0$, the resulting quotient stack is denoted by BG.
Using "groupoid notation" the quotient stack $[\mathbb{C}^n/G]$ is denoted by

$$\mathbb{C}^n \times G \rightrightarrows \mathbb{C}^n$$

where the upper arrow denoted by σ is the action and the lower arrow denoted by p is the projection.
The coarse moduli space of the quotient stack is the usual quotient scheme which can be singular.
The coarse moduli space X of $\mathfrak{X}$ is obtained by gluing these open subschemes. X is then a toric variety (possibly singular) in the usual sense.

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	$\mathbb{C}^{*2}$-weights on $\mathbb{C}^2$
$[\mathbb{C}^2/\mu_a]$	$(b,0),(0,c)$
$[\mathbb{C}^2/\mu_b]$	$(-a,0),(-c,c)$
$[\mathbb{C}^2/\mu_c]$	$(0,-a),(b,-b)$

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More precisely, $\mathrm{Hilb}^c[\mathbb{C}^2/\mu_n]$ where $c \in K_0([\mathbb{C}^2/\mu_n]) \cong \mathrm{Rep}(\mu_n)$ is the regular representation.

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The latter is equivalent to the category of finitely generated $\mathbb{C}[x_1, \ldots, x_n]$-modules with an $X(T)$-grading and an $X(G)$-fine grading.

Torus action

Let T act linearly on $\mathbb{C}^n$ with characters $\chi(m_1), \ldots, \chi(m_n)$ i.e. $t \cdot x_i = \chi(m_i)(t) x_i$ for all i and $t \in T$.

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Stacky S-family

Let $T = \mathbb{C}^{*d}$ and G be a finite abelian group acting on $\mathbb{C}^d$ by

$$t \cdot x_i = \chi(m_i)(t) x_i, \qquad g \cdot x_i = \chi(n_i)(g) x_i.$$

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Theorem

The category of T-equivariant coherent sheaves on $[\mathbb{C}^d/G]$ is equivalent to the category of stacky S-families.

$\mathbb{P}(a, b, c)$

The weighted projective plane $\mathbb{P}(a, b, c)$ is by definition the quotient stack $[\mathbb{C}^3 \backslash \{0\} / \mathbb{C}^*]$, where $\mathbb{C}^*$ acts on $\mathbb{C}^3$ by

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If $d_{12} = d_{23} = d_{13} = 1$, then the structure map from the stack to the coarse moduli space is an isomorphism away from the points among

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where $\mu_a \times \mu_b$ acts by $(\omega_b \omega_a^{-1}, \omega_a^c)$. Note that

$$[\mathbb{C}^2/\mu_a] \times_{\mathbb{P}} [\mathbb{C}^2/\mu_b] \times_{\mathbb{P}} [\mathbb{C}^2/\mu_c] \cong [\mathbb{C}^{*2}/\mu_a \times \mu_b \times \mu_c] \cong \mathfrak{T}.$$

$\mathbb{P}(a, b, c)$

In the case $d = 1$, $\mathbb{P}(a, b, c)$ is an orbifold meaning that the structure map is an isomorphism away from the lines

$$(0 : Y : Z), (X : 0 : Z), (X : Y : 0) \subset \mathbf{P}(a, b, c).$$

In general, $\mathbb{P}(a, b, c)$ is a $B\mu_d$-gerbe over the orbifold $\mathbb{P}(a/d, b/d, c/d)$.
We glue toric coherent sheaves we constructed on $\mathfrak{U}_i$'s along $\mathfrak{U}_i \times_{\mathbb{P}} \mathfrak{U}_j$. It turns out that

$$[\mathbb{C}^2/\mu_a] \times_{\mathbb{P}} [\mathbb{C}^2/\mu_b] \cong [\mathbb{C}^* \times \mathbb{C}/\mu_a \times \mu_b]$$

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Theorem

The category of T-equivariant sheaves on $\mathbb{P}(a, b, c)$ is equivalent to the category of triples $\{\hat{F}_i\}_{i=1,2,3}$ of stacky S-families on $\mathfrak{U}_i$'s satisfying certain delicate gluing conditions at the intersections.

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$$K_0(\mathbb{P}(a,b,c)) \cong \mathbb{Z}[g, g^{-1}]/(1-g^a)(1-g^b)(1-g^c),$$

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E.G. The classes of the structure sheaves of the fixed points of the T-action are

$$[\mathcal{O}_{P_i}] = (1 - g^a)(1 - g^b)(1 - g^c)/(1 - g^{\hat{i}}),$$

where

$$\hat{\ } : \{1, 2, 3\} \to \{a, b, c\}$$

sends 1 to a, 2 to b, and 3 to c.

Inertia stack $I\mathbb{P}(a, b, c)$

Define

$$D := \{l/d\}_{l=0,\dots,d-1},\ D_{ij} := \{l/d_{ij}\}_{l=0,\dots,d_{ij}-1} \setminus D,$$

$$D_i := \{l/\hat{i}\}_{l=0,\dots,\hat{i}-1} \setminus (D \cup D_{ij} \cup D_{ik}),$$

$$F = D \sqcup \coprod_{i,j} D_{ij} \sqcup \coprod_{i} D_i \quad \forall \{i, j, k\} = \{1, 2, 3\}.$$

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The 2-dimensional components of the inertia stack

$$I\mathbb{P}(a, b, c) := \mathbb{P}(a, b, c) \times_{\mathbb{P}\times\mathbb{P}} \mathbb{P}(a, b, c)$$

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There is a natural map $\pi : I\mathbb{P}(a,b,c) \to \mathbb{P}(a,b,c)$ (local immersion).

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There is a natural map $\pi : I\mathbb{P}(a,b,c) \to \mathbb{P}(a,b,c)$ (local immersion). The eigenvalue of $\pi^*\mathcal{O}(1)$ when restricted to the component corresponding to $f \in F$ is $e^{2\pi\sqrt{-1}f}$.

Chern character

For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}(a, b, c)$, define

$$\widetilde{\mathrm{ch}} : K_0(\mathbb{P})_{\mu_\infty} \to A^*(I\mathbb{P})_{\mu_\infty}, \; \widetilde{\mathrm{ch}}(\mathcal{F}) := \sum_{f \in F} \sum_i \omega_{f,i} \cdot \mathrm{ch}(\mathcal{F}_{f,i}),$$

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The reason for this notational convention is that we are dealing with Chern characters of sheaves on components of *different dimension* of the inertia stack $I\mathbb{P}$ so it is more natural to keep track of dimension than codimension.

Rank 1 torsion free sheaves

Recall

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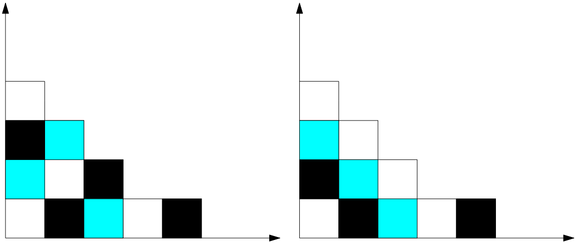

$\mu_3 \curvearrowright \mathbb{C}^2$ by (ω, ω^2) in the left picture and by (ω, ω) in the right picture.

Relations

Introduce the variables

$$p_0, \ldots, p_{a-1}, \quad q_0, \ldots, q_{b-1}, \quad r_0, \ldots, r_{c-1},$$

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one for each color. They satisfy certain relations imposed by the geometry of $\mathbb{P}(a, b, c)$. In fact the relation in the Grothendieck group forces these relations among the variables:

$$p_0 p_d \cdots p_{a-d} = q_0 q_d \cdots q_{b-d} = r_0 r_d \cdots r_{c-d},$$
$$p_1 p_{d+1} \cdots p_{a-d+1} = q_1 q_{d+1} \cdots q_{b-d+1} = r_1 r_{d+1} \cdots r_{c-d+1},$$
$$\ldots$$
$$p_{d-1} p_{2d-1} \cdots p_{a-1} = q_{d-1} q_{2d-1} \cdots q_{b-1} = r_{d-1} r_{2d-1} \cdots r_{c-1},$$
$$p_0 p_{d_{12}} \cdots p_{a-d_{12}} = q_0 q_{d_{12}} \cdots q_{b-d_{12}}, \ldots$$
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Rank 1 torsion free sheaves on $\mathbb{P}(a, b, c)$

For a fixed $\beta \in A^1(I\mathbb{P})$ let $G_\beta(q) = \sum_c e(M_\beta(c))q^c$ where $c \in K_0(\mathbb{P})_\mathbb{Q}$ runs over all classes of rank 1 torsion free sheaves with $c_1(c) = \beta$.

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G-Jiang-Kool (2012)

The generating function of the Euler characteristics of the moduli space of rank 1 torsion free sheaves on $\mathbb{P}(a, b, c)$ with trivial determinant ("Hilbert scheme of points") is given by

$$G_0(q) = \left(\sum_{\lambda \in \Pi_1} \prod_{l=0}^{a-1} p_l^{\#_l \lambda} \right) \left(\sum_{\lambda \in \Pi_2} \prod_{l=0}^{b-1} q_l^{\#_l \lambda} \right) \left(\sum_{\lambda \in \Pi_3} \prod_{l=0}^{c-1} r_l^{\#_l \lambda} \right),$$

where the p_l, q_l, r_l satisfy relations above.

Colored Partition

When the action of μ_k on $\mathbb{C}^2$ is *balanced*, i.e. is of the form $\omega \cdot (x, y) = (\omega x, \omega^{-1} y)$, there is an elegant formula appearing in the physics literature (Dijkgraaf).

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$$\sum_{\text{colored partitions } \lambda} q_0^{\#_0 \lambda} \cdots q_{k-1}^{\#_{k-1}\lambda} =$$

$$\frac{1}{\prod_{j>0}(1 - (q_0 \cdots q_{k-1})^j)^k}$$

$$\sum_{n_1, \ldots, n_{k-1} \in \mathbb{Z}} (q_0 \cdots q_{k-1})^{\sum_i n_i^2 - n_i n_{i+1}} \prod_{r=1}^{k-1} q_{k-r}^{r^2/2 + n_1 r - r/2}.$$

One can count colored partitions keeping track of the number of boxes with color 0 only by setting $q_0 = q$ and $q_1 = \cdots = q_{k-1} = 1$.

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One can count colored partitions keeping track of the number of boxes with color 0 only by setting $q_0 = q$ and $q_1 = \cdots = q_{k-1} = 1$. Then formula above is related to the character formula of the affine Lie algebra $\widehat{su}(k)$

$$\sum_{\text{colored partitions } \lambda} q^{\#_0 \lambda} = \frac{q^{k/24}}{\eta(q)} \chi^{\widehat{su}(k)}(0).$$

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Setting $q_1 = r_1 = r_2 = 1$ and $p_0 = q_0 = r_0 = q$ we get

$$G_0(q) = \frac{q^{1/4}}{\eta(q)^6} \chi^{\widehat{su}(2)}(0) \chi^{\widehat{su}(3)}(0)$$
$$= \frac{q^{1/4}}{\eta(q)^6} \theta_3(q)(\theta_3(q)\theta_3(q^3) + \theta_2(q)\theta_2(q^3)),$$

where $\theta_2(q)$, $\theta_3(q)$ are Jacobi theta functions.

Example: $\mathbb{P}(1, c, c)$ with $c \geq 2$

Relations above give $p_0 = q_0 \cdots q_{c-1}$ and $q_i = r_i$.

$$G = \frac{1}{\prod_{k>0}(1-(r_0 \cdots r_{c-1})^k)}$$
$$\frac{1}{(\prod_{k>0}\prod_{i=0}^{c-2}(1 - r_0 \cdots r_i(r_0 \cdots r_{c-1})^{k-1}))^2}.$$

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In our case, we fix the standard polarization $\mathcal{O}_{\mathbf{P}}(1)$ on $\mathbf{P}$ and we choose a generating sheaf $\mathcal{E} := \bigoplus_{n=0}^{E-1} \mathcal{O}_{\mathbb{P}}(-n)$, where E is any positive integer such that the least common multiple m of a, b, c divides E.

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Fix $\alpha \in A^0(I\mathbb{P})_{\mu_\infty}$ and $\beta \in A^1(I\mathbb{P})_{\mu_\infty}$. Define the generating functions

$$H_{\alpha,\beta}(q) := \sum_{\substack{\widetilde{\mathrm{ch}}^2(\mathrm{c}) = \alpha \\ \widetilde{\mathrm{ch}}^1(\mathrm{c}) = \beta}} e(M_{\mathcal{E}}(\mathrm{c}))q^{\mathrm{c}}, \quad H^{vb}_{\alpha,\beta}(q) := \sum_{\substack{\widetilde{\mathrm{ch}}^2(\mathrm{c}) = \alpha \\ \widetilde{\mathrm{ch}}^1(\mathrm{c}) = \beta}} e(N_{\mathcal{E}}(\mathrm{c}))q^{\mathrm{c}}.$$

So in terms of $\widetilde{\mathrm{ch}}$, these generating functions sum over all 0-dimensional (i.e. codegree 0) parts $(\widetilde{\mathrm{ch}}_f)^0$.

Rank 2 torsion free sheaves

Technical point: To define a correct notion of stability for higher rank sheaves on DM stacks besides polarization one needs also a π-very ample sheaf called the generating sheaf.
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Proposition

$H_{\alpha,\beta}(q) = H^{vb}_{\alpha,\beta}(q) \prod_{i=1}^3 G_{\mathfrak{U}_i}(q)^2.$

Rank 2 vector bundles

We classify T-equivariant rank 2 vector bundles on $\mathbb{P}(a, b, c)$ into three types I, II and III, according to the number of nonzero components of the box elements in the stacky S-families attached.

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The stacky S-families of a stable rank 2 vector bundle $\mathcal{F}$ of type I are entirely determined by integers u_1, u_2, u_3 and $v_1, v_2, v_3 > 0$ satisfying

$$b \mid v_1, \;\; c \mid v_2, \;\; a \mid v_3,$$

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$$\Big(1 + g^{v_1+v_2+v_3} - (1-g^{v_1})(1-g^{v_2}) - (1-g^{v_2})(1-g^{v_3}) \\ - (1-g^{v_3})(1-g^{v_1})\Big) g^{u_1+u_2+u_3}.$$

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Recall: If $\mathcal{F}$ is a rank 2 vector bundle on $\mathbb{P} = \mathbb{P}(a, b, c)$ then its generalized Chern character $\widetilde{\mathrm{ch}}(\mathcal{F})$ takes values in $A^*(I\mathbb{P})_{\mu_\infty}$.

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G-Jiang-Kool (2014)

For any $\alpha \in A^0(I\mathbb{P})_{\mu_\infty}$ and $\beta \in A^1(I\mathbb{P})_{\mu_\infty}$

$$H^{vb}_{\alpha,\beta} = \sum_{(u,v_1,v_2,v_3)\in C_{\alpha,\beta}} \prod_{f\in D} p_f^{\widetilde{\mathrm{ch}}^0(u,v_1,v_2,v_3)_f} \prod_{\substack{i<j \\ f\in D_{ij}}} q_{ij,f}^{\widetilde{\mathrm{ch}}^0(u,v_1,v_2,v_3)_f} \prod_{i,f\in D_i} r_{i,f}^{\widetilde{\mathrm{ch}}^0(u,v_1,v_2,v_3)_f},$$

where

$$C_{\alpha,\beta} := \Big\{(u,v_1,v_2,v_3) \in \mathbb{Z}\times\mathbb{Z}^3_{>0} \ : \ b \mid v_1, \ c \mid v_2, \ a \mid v_3, \\ \widetilde{\mathrm{ch}}^2(u,v_1,v_2,v_3) = \alpha, \widetilde{\mathrm{ch}}^1(u,v_1,v_2,v_3) = \beta, \\ v_i < v_j + v_k \ \forall\{i,j,k\} = \{1,2,3\}\Big\}.$$

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We define

$$H_{\mathbb{P}}^{vb}(q) = \sum_{\Delta \geq 0} e(M_{\mathbb{P}}(\Delta)) q^{\Delta}.$$

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$$e(M(\Delta)) = \begin{cases} 2H(\Delta) & \Delta = 8k-1 \\ H(\Delta) + 2H(\Delta/4) - (1/2)d(\Delta/4) - d(\Delta/16) & \Delta \equiv_{16} 0 \\ H(\Delta) + 2H(\Delta/4) - (1/2)d(\Delta/4) & \Delta \not\equiv_{16} 0 \& \Delta \equiv_4 0 \\ 0 & \text{otherwise.} \end{cases}$$

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It can be seen that this a holomorphic part of a modular form of weight $3/2$.

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Cheah: $\displaystyle\sum_{n=0}^{\infty} e(\mathrm{Hilb}_{0,n}(X))q^n = M(q)^{e(X)}$.

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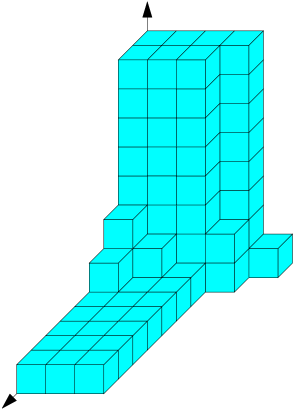
$$|\pi| := \#\{\pi \cap ([0,1,\ldots,N]^3)\} - (N+1)\sum_{i=1}^{3} |\lambda_i|$$

for $N \gg 0$.

3d partition with legs



- $\lambda_1 = 1^3$
- $\lambda_2 = 2^3 1$
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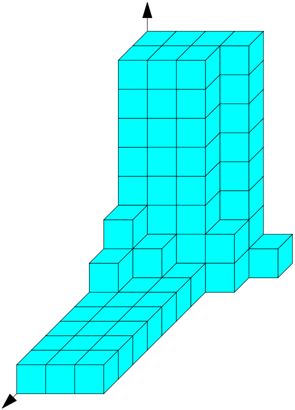
3d partition with legs



- $\lambda_1 = 1^3$
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- $|\pi| = 1$ (with $N = 4$, $51 - 5 \cdot (3 + 7 + 0)$)

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$$Z(\lambda_3) q^{-\binom{\lambda_1}{2} - \binom{\lambda_2^t}{2} - |\lambda_1|/2 - |\lambda_2|/2} \sum_{\eta} s_{\lambda_1^t/\eta}(q^{-\lambda_3 - \rho}) s_{\lambda_2/\eta}(q^{-\lambda_3^t - \rho})$$

where

$$Z(\nu) = \frac{q^{-\binom{\nu}{2} - |\nu|/2} s_{\nu^t}(q^{-\rho})}{\prod_{k>0}(1 - q^k)^k}.$$

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virtual class is crucial for defining GW and DT invariants etc. in general (giving deformation invariance of the invariants!).

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Take the quotient...

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The deformation theory of the moduli problems often gives us infinitesimal version of Y, E, s on a moduli space M, with Cok becomes the obstruction sheaf.

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Recall: If $[B_0 \to B_1]$ is 2-term complex of vector bundles, its top Chern class is defined by $c_{top}(B_0)/c_{top}(B_1)$ provided $c_{top}(B_1)$ is invertible.

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Virtual localization formula (Graber-Pandharipande):

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Thomas in his PhD thesis took this idea to construct a natural perfect obstruction theory over $\mathcal{M}$.

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If $Ext^3(\mathcal{F}, \mathcal{F})$ doesn't change dimension over $\mathcal{M}$, then we can define a truncation of $E^\bullet$ that is concentrated in degree $[-1, 0]$ (we use the same symbol).
Thomas proved that there exists a perfect obstruction theory $\phi : E^\bullet \to L^\bullet\mathcal{M}$ and hence a virtual cycle $[\mathcal{M}]^{vir} \in A_d(\mathcal{M})$ where $d = ext^1(\mathcal{F}, \mathcal{F}) - ext^2(\mathcal{F}, \mathcal{F})$.

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There is a universal curve $\pi : \mathcal{C} \to M_g(X, \beta)$ and a universal morphism $f : \mathcal{C} \to X$.
There is a natural morphism of stacks $M_g(X, \beta) \to \mathfrak{M}_g$.
The morphism $(R\pi_* f^* TX)^\vee \to L_\tau$ gives rise to a perfect obstruction theory for $M_g(X, \beta)$.

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$$h^0 - h^1 + 3g - 3 = \int_C \mathrm{ch}(f^*TX) \cdot \mathrm{td}(C) + 3g - 3$$

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Generating functions:

$$Z_{GW}(X; q, v) = \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{g \geq 0} N_{g,\beta} u^{2g-2} v^\beta.$$

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We have seen $\mathrm{Hilb}_{\beta,n}(\overline{X})^T$ correspond to the tuples of the generalized 3d partitions. Note that by our choice of β, the partitions have no legs along the edges corresponding to lines outside of S.

Localization

Let $\mathcal{I} \in \mathrm{Hilb}_{\beta,n}(\overline{X})^T$ correspond to $\{\pi_\alpha, \lambda_{\alpha\alpha'}\}$. Then

$$\beta = \sum_{\alpha,\alpha'} |\lambda_{\alpha\alpha'}|[C_{\alpha\alpha'}], \qquad n = \sum_{\alpha} |\pi_\alpha| + \sum_{\alpha,\beta} f_{m_{\alpha,\beta},m'_{\alpha,\beta}}(\lambda_{\alpha,\beta})$$

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Let $I_\alpha = \mathcal{I}|_{U_\alpha}$ then the vertex α contribution to $Ext^1(\mathcal{I},\mathcal{I}) - Ext^2(\mathcal{I},\mathcal{I})$ is

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$$V_\alpha := F_\alpha + \sum_{i=1}^{3} \frac{F_{\alpha\beta_i}(t_{i'}, t_{i''})}{1 - t_i},$$

where $\alpha\beta_1, \alpha\beta_2, \alpha\beta_3$ are the three edges passing the vertex α and $\{t_i, t_{i'}, t_{i''}\} = \{t_1, t_2, t_3\}$, and similarly

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<u>Theorem:</u> The T-character of $Ext^1(\mathcal{I}, \mathcal{I}) - Ext^2(\mathcal{I}, \mathcal{I})$ is $\sum_\alpha V_\alpha + \sum_{\alpha\beta} E_{\alpha\beta}$.

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$$w(\pi_\alpha)(s_1, s_2, s_3) = \prod_{k \in \mathbb{Z}^3} (s, k)^{-v_k}$$

where $s = (s_1, s_2, s_3)$ and v_k is the coefficient of t^k in V_α.

Calabi-Yau Torus

Applying localization formula

$$Z'(X;q)_\beta = \frac{\sum_n q^n \sum_{\mathcal{I}\in \mathrm{Hilb}_{\beta,n}(\overline{X})^T} e(Ext^2\mathcal{I},\mathcal{I})/e(Ext^1\mathcal{I},\mathcal{I})}{\sum_n q^n \sum_{\mathcal{I}\in \mathrm{Hilb}_{0,n}(\overline{X})^T} e(Ext^2\mathcal{I},\mathcal{I})/e(Ext^1\mathcal{I},\mathcal{I})}.$$

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Note: In U_α with coordinates (x_1,x_2,x_3), the subtorus T_0 is given by $t_1t_2t_3=1$.

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Let $E_{\alpha\beta} = E^+_{\alpha\beta} + E^-_{\alpha\beta}$ where $\overline{E}^+_{\alpha\beta}|_{t_1 t_2 t_3 = 1} = -E^-_{\alpha\beta}|_{t_1 t_2 t_3 = 1}$ using

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Next year we will talk about the proof of MNOP conjecture for general toric threefolds. Thank you!