TS 2: Spectral Density and Distribution Function

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1 Wiener-Khinchine Theorem

Wiener–Khintchine theorem (1930, 1934) states that the autocovariance of a weakly stationary random process has a spectral representation. The theorem is also known as Wiener–Khinchin–Einstein theorem or the Khinchin–Kolmogorov theorem. Einstein explained the idea in a 2-page letter in 1914.

Without loss of generality, assume $\{X_t, t = 0, \pm 1, ...\}$, is real weakly stationary with mean 0 and variance 1:

$$R(v) = Cov(X_t, X_{t+v}) = EX_t X_{t+v}, \quad R(0) = EX_t^2 = 1$$

Definition: $\sigma(v), v = 0, \pm 1, ...$, is called positive semidefinite if for all real numbers $a_1, a_2, ..., a_n$, $n = 1, 2, ...$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \sigma(i - j) \geq 0$$

Theorem: Suppose $R(v)$ is the covariance function of X_t as above. Then:
a. $R(v)$ is symmetric and positive semidefinite.
b. There is a symmetric probability distribution function F ($F(\omega) = 1 - F(-\omega)$) on $[-\pi, \pi]$ such that

$$R(v) = \int_{-\pi}^{\pi} \cos(v\omega) dF(\omega), \; v = 0, \pm 1, ...$$

Proof:
a)

$$0 \leq E|\sum_{i=1}^{n} a_i X_{t+i}|^2 = \sum_{i=1}^{n} \sum_{i=j}^{n} a_i a_j EX_{t+i}X_{t+j} = \sum_{i=1}^{n} \sum_{i=j}^{n} a_i a_j R(i - j)$$

b) Now choose specific a_i, and let $-\pi \le \omega \le \pi$.
Let $a_i = \cos(i\omega)$. Then from a) we have,

$$0 \le \sum_{i=1}^{n}\sum_{j=1}^{n} \cos(i\omega)\cos(j\omega)R(i-j)$$

Let $a_i = \sin(i\omega)$. Then from a) we have

$$0 \le \sum_{i=1}^{n}\sum_{j=1}^{n} \sin(i\omega)\sin(j\omega)R(i-j)$$

Summing:

$$\begin{aligned} 0 &\le \frac{1}{2\pi n}\sum_{i=1}^{n}\sum_{j=1}^{n} R(i-j)[\cos(i\omega)\cos(j\omega)+\sin(i\omega)\sin(j\omega)] \\ &= \frac{1}{2\pi n}\sum_{i=1}^{n}\sum_{j=1}^{n} R(i-j)\cos((i-j)\omega) \\ &= \frac{1}{2\pi n}\sum_{v=-(n-1)}^{n-1} (n-|v|)R(v)\cos(v\omega) \equiv f_n(\omega), \quad \omega \in [-\pi,\pi] \end{aligned}$$

Now, $f_n(\omega) \ge 0$ and,

$$\int_{-\pi}^{\pi} f_n(\omega)d\omega = \frac{1}{2\pi n}\sum_{v=-(n-1)}^{n-1} (n-|v|)R(v)\int_{-\pi}^{\pi}\cos(v\omega)dw$$

But $\int_{-\pi}^{\pi}\cos(v\omega)dw = 0$ unless $v = 0$. Therefore,

$$\int_{-\pi}^{\pi} f_n(\omega)d\omega = \frac{1}{2\pi n} \times n \times 2\pi \times R(0) = R(0) = 1$$

It follows that f_n is a *symmetric pdf* on $[-\pi,\pi]$.

Let $F_n(\omega)$ be the CDF corresponding to $f_n(\omega)$:

$$F_n(\omega) = \int_{-\pi}^{\omega} f_n(\lambda)d\lambda$$

Now,

$$\int_{-\pi}^{\pi}\cos(u\omega)dF_n(\omega) = \int_{-\pi}^{\pi}\cos(u\omega)f_n(\omega)d\omega$$

and observe that

$$\int_{-\pi}^{\pi}\cos(u\omega)\cos(v\omega)d\omega = \begin{cases} 0, & \text{if } v \ne u; \\ 2\pi, & \text{if } v = u. \end{cases}$$

Therefore,

$$\int_{-\pi}^{\pi} \cos(u\omega) dF_n(\omega) = \left(1 - \frac{|u|}{n}\right) R(u)$$

By the Helly-Bray lemma there is a distribution function F and a subsequence $\{F_{n_k}\}$ such that for every continuous bounded function h,

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} h(\omega) dF_{n_k}(\omega) = \int_{-\pi}^{\pi} h(\omega) dF(\omega)$$

In particular, take $h(\omega) = \cos(u\omega)$, and we have:

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} \cos(u\omega) dF_{n_k}(\omega) = \int_{-\pi}^{\pi} \cos(u\omega) dF(\omega) = \lim_{k\to\infty} \left(1 - \frac{|u|}{n_k}\right) R(u)$$

Hence, finally, we obtain the celebrated spectral representation of the covariance function,

$$R(u) = \int_{-\pi}^{\pi} \cos(u\omega) dF(\omega), \quad u = 0, \pm 1, ...$$

Note:

1. F is right-continuous *spectral distribution function.*
2. If F is absolutely continuous then its derivative f is the *spectral density* of X_t.
3. Can get a more general theory if we define for some α's and β's

$$f_n(\omega) \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} R(i-j)[\alpha_i \alpha_j + \beta_i \beta_j]$$

4. So, what is $R(v)$??? What can you say about $R(0)$???
5. Decomposition

$$F = F_{ac} + F_{sf} + F_{singular}$$

   We will assume $F = F_{ac} + F_{sf}$ where F_{ac} has a spectral density, and F_{sf} has jumps p_k (spectral function).
6. Thus, if F is absolutely continuous, then there is a spectral density

$$f(\omega) = \frac{d}{d\omega} F(\omega), \; \omega \in [-\pi, \pi]$$

   and

$$R(v) = \int_{-\pi}^{\pi} \cos(v\omega) f(\omega) d\omega = \int_{-\pi}^{\pi} \exp(iv\omega) f(\omega) d\omega, \; v = 0, \pm 1, ...$$

7. Fact: F is differentiable with derivative f iff $\sum_{v=-\infty}^{\infty} |R(v)| < \infty$, in which case,

$$f(\omega) = \frac{1}{2\pi} \sum_{v=-\infty}^{\infty} \cos(v\omega) R(v) = \frac{1}{2\pi} \sum_{v=-\infty}^{\infty} e^{-iv\omega} R(v)$$

8. So, under a conditions, the information in $R(v)$ is equivalent to the information in $f(\omega)$. However, $R(v)$ is a time domain function while $f(\omega)$ is a frequency domain function. Under some conditions there is a 3rd domain that we shall explore.

9. $\frac{2\pi}{\omega_0}$ is the period associated with frequency ω_0 measured in radians per unit time.

1.1 Examples of Weekly Stationary Processes

Example WN: X_t, $t = 0, \pm 1, ...$ are uncorrelated with $EX_t = 0$ and variance $Var(X_t) = \sigma^2$. Then

$$R(v) = \begin{cases} \sigma^2, & \text{if } v = 0; \\ 0, & \text{if } v = \pm 1, \pm 2, ... \end{cases}$$

Clearly $\sum_{v=-\infty}^{\infty} |R(v)| < \infty$. Therefore

$$f(\omega) = \frac{1}{2\pi} \sum_{v=-\infty}^{\infty} \cos(v\omega) R(v) = \frac{\sigma^2}{2\pi}, \quad \omega \in [-\pi, \pi]$$

which is a constant, whence the term WN.

$$F(\lambda) = \int_{-\pi}^{\lambda} f(\omega) d\omega = \frac{\sigma^2}{2\pi}(\lambda + \pi)$$

Example MA(1):
a) $X_t = u_t + u_{t-1}$, where u_t is WN with mean 0 and variance 1. Then $EX_t = 0$ and,

$$R(k) = EX_t X_{t+k} = \begin{cases} 2, & \text{if } k = 0; \\ 1, & \text{if } k = 1 \\ 0, & \text{if } k \geq 2. \end{cases}$$

which is absolutely summable, hence

$$f(\lambda) = \frac{1}{\pi}(1 + \cos\lambda), \quad \lambda \in [-\pi, \pi]$$

We see that *LOW* frequencies are emphasized. See Fig 1.

b) $X_t = u_t - u_{t-1}$, where u_t is WN with mean 0 and variance 1. Then $EX_t = 0$ and,

$$R(k) = EX_t X_{t+k} = \begin{cases} 2, & \text{if } k = 0; \\ -1, & \text{if } k = 1 \\ 0, & \text{if } k \geq 2. \end{cases}$$

which is absolutely summable, hence

$$f(\lambda) = \frac{1}{\pi}(1 - \cos\lambda), \quad \lambda \in [-\pi, \pi]$$

We see that *HIGH* frequencies are emphasized. See Fig 2.

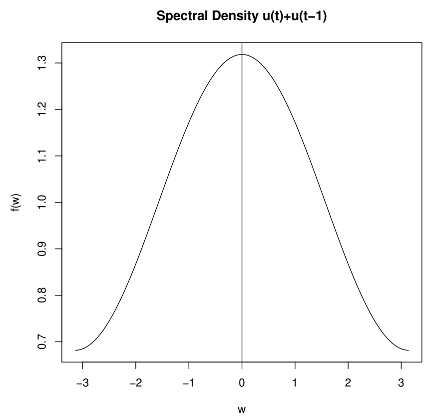

Figure 1: Spectral density of $u_t + u_{t-1}$, $u_t \sim WN$

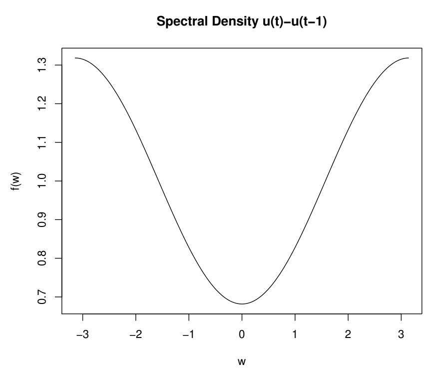

Figure 2: Spectral density of $u_t - u_{t-1}$, $u_t \sim WN$

Simulation of MA(1): Go back to latex source

```
#u(t)+U(t+1)
par(mfrow=c(2,2))
x1 <- arima.sim(list(order=c(0,0,1), ma=1), n=100)
ts.plot(x1, main="ARIMA(0,0,1): ma=1; r1 > 0")
acf1 <- acf(x1) ### acf normalized R(k)

#u(t)-u(t-1)
x2 <- arima.sim(list(order=c(0,0,1), ma=-1), n=100)
ts.plot(x2, main="ARIMA(0,0,1): ma=-1; r1 < 0")
acf2 <- acf(x2) ### acf normalized R(k)
```

Example: Random Telegraph Signal with Parameter λ

$$R(v) = e^{-2\lambda|v|}$$

Get the spectral density by a simulation.

Simulation of Random Telegraph Signal

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```
par(oma = c(0, 0, 4, 0)) ##4 lines of outer margin at the top
par(mfrow=c(1,1))


###Generate a Poisson process by generating the interarrival times
which are independent Exp(lam).

f <- function(a,x){a*exp(-a*x)} #Exponential(a) pdf

#Histogram and pdf
lam <- 0.1
x <- rexp(100,lam) ##Generated Exp(lam)
hist(x,probability=T,cex=1.2,main="Exp(0.1)")
q <- seq(0.001,60,length=500)
lines(q,f(lam,q), type="l")

mtext("Exponential Distribution",
side = 3, outer = T, cex = 1.5, line=1) #OVERALL TITLE


###Sum: T1, T1+T2, T1+T2+T3,.....  (same as cumsum(x)!!!)

SUM <- 0
for(i in 1:length(x)){
SUM[i] <- sum(x[1:i])}


###Creating the indicator I[a < t <= b]
NOTE:
w <- function(x){(2 < x < 5)} does not work!
w <- function(x){(2 < x ) & (x < 5)} ok!


w1 <- function(a,b,t){ ##Indicator function
1*((a < t) & (t <= b))}

#Sum of two indicators with different signs
X<- 0
```

```
tpoints <- seq(0,SUM[50],length=5000)
X <- w1(0,SUM[1],tpoints)-w1(SUM[1],SUM[2],tpoints)

###Sampling of Random Telegraph signal: Delta=1
#> SUM[50]
#[1] 508.1134 <--- Get a ts of length > 500

tpoints <- seq(0,SUM[50],length=SUM[50])
X <- 0

#Summing all the indicators with alternating signs
  to get random telegraph square signal:

for(i in 2:50){
X <- X + ((-1)^(i+1))*w1(SUM[i-1],SUM[i],tpoints)}
X <- X+w1(0,SUM[1],tpoints)

ts.plot(X,xlab="t", main="Random Telegraph: Lambda=0.1 Sampling Delta = 1")
abline(0,0)


#acf of Rndom Telegraph, Lambda=0.1, Dela t = 1.

par(oma = c(0, 0, 4, 0))
par(mfrow=c(2,1))

acf(X)
lines(tpoints,exp(-(2*lam*tpoints)),type="l")
mtext("Random Telegraph: Lambda=0.1",
        side = 3, outer = T, cex = 1.5, line=1)
mtext("Estimated vs Theoretical ACF",
        side = 3, outer = T, cex = 1.5, line=-4)

#Spec. density of Rndom Telegraph on linear scale evaluated at
frequencies in radians per unit time. Lambda=0.1, Dela t = 1.

par(oma = c(0, 0, 4, 0))
par(mfrow=c(2,1))

ESPa <- spectrum(X,spans=c(3,5,7)) #First in Hz on [0,0.5]

#Now in radians/unit time
SpecDensity <- 10^(ESPa$spec/10)
plot(2*pi*ESPa$freq, SpecDensity, type="l",
main="Linear Scale in [0,pi]", cex=1.3,ylab="")
```

```
mtext("Random Telegraph: Lambda=0.1, Delta t =1
Daniel(3,5,7): 10*Log_10 in [0,0.5] vs Linear Scale in [0,pi]",
side = 3, outer = T, cex = 1.5, line=1) #OVERALL TITLE
```

Example: Sum of Sinusoids with Random Amplitudes
Discrete spectrum: No spectral density.

$$Y_t = \sum_{j=0}^{m} [A_j \cos(\omega_j t) + B_j \sin(\omega_j t)]$$

$EA_j = EB_j = 0$, $EA_j^2 = EB_j^2 = \sigma_j^2$, $EA_j B_k = 0$ for all $j, k = 0, 1, ..., m$.
$0 \leq \omega_0 \leq \omega_1, ..., \leq \omega_m \leq \pi$
Note: The frequencies are all non-negative, but it is convenient to define F on $[-\pi, \pi]$.

$$R(k) = EY_t Y_{t+k} = \sum_{j=0}^{m} \sigma_j^2 \cos(\omega_j k)$$

Therefore, from this we get the relative contribution of ω_j to the variance:

$$R(0) = \sum_{j=0}^{m} \sigma_j^2$$

Since

$$R(k) = \int_{-\pi}^{\pi} \cos(\omega k) dF(\omega)$$

we can get F by uniqueness: F is a step function with jumps $\frac{1}{2}\sigma_j^2$ at $\pm\omega_j$, and jump σ_0^2 at $\omega_0 = 0$.

We can define the *spectral function* as $p(\omega_j) \equiv dF(\omega_j)$.

Observe: $R(k)$ is NOT absolutely summable, and $R(0) = F(\pi) = \sum_{j=0}^{m} \sigma_j^2$.

Observe the acf,

$$\rho(k) = \frac{R(k)}{R(0)} = \frac{\sum_{j=0}^{m} \sigma_j^2 \cos(\omega_j k)}{\sum_{j=0}^{m} \sigma_j^2}$$

Example: Harmonic Process with K Frequencies
Discrete spectrum: No spectral density.

$$X_t = \sum_{i=1}^{K} A_i \cos(\omega_i t + \phi_i), \quad \phi_i \sim Unif(-\pi, \pi),\ independent$$

A_i are fixed, not random.

$$E\cos\phi_i = E\sin\phi_i = 0, \quad E\cos\phi_i \sin\phi_j = 0\ for\ all\ i, j.$$

$$E\cos^2\phi_i = E\sin^2\phi_i = \frac{1}{2}$$

Also here the amplitudes are random:

$$X_t = \sum_{i=1}^{K} A_i[\cos(\omega_i t)\cos\phi_i - \sin(\omega_i t)\sin\phi_i]$$

$$R(k) = \sum_{i=1}^{K} (\frac{1}{2}A_i^2)\cos(k\omega_i)$$

$$R(0) = \sum_{i=1}^{K} (\frac{1}{2}A_i^2)$$

$$\rho_k = \frac{\sum_{i=1}^{K} \cos(k\omega_i)}{\sum_{i=1}^{K} A_i^2}$$

$$dF(-\omega_i) = dF(\omega_i) = \frac{1}{4}A_i^2$$

```
par(mfrow=c(3,1))
#Sum of 3 sinusoids
t <- 1:200
ts.plot(2*(cos(0.7*t + runif(1,-pi,pi)))+ 0.5*(cos(1.2*t +
runif(1,-pi,pi)))-2.8*(cos(2.3*t + runif(1,-pi,pi))),ylab="")
title("Sum of 3 Sinusoids")

#Sum of 3 sinusoids + noise
t <- 1:200
u <- rnorm(200)
ts.plot(2*(cos(0.7*t + runif(1,-pi,pi)))+ 0.5*(cos(1.2*t +
runif(1,-pi,pi)))-2.8*(cos(2.3*t + runif(1,-pi,pi))
+u[t]),ylab="")
title("Sum of 3 Sinusoids Plus Gaussian Noise e(t)")

#More noise
t <- 1:200
```

```
u <- rnorm(200)
ts.plot(2*(cos(0.7*t + runif(1,-pi,pi)))+ 0.5*(cos(1.2*t +
runif(1,-pi,pi)))-2.8*(cos(2.3*t + runif(1,-pi,pi))
+2*u[t]),ylab="")
title("Sum of 3 Sinusoids Plus Gaussian Noise 2*e(t)")
```

Example: Harmonic Process K=1.

$$X_t = A\cos(\omega_0 t + \phi), \quad \phi \sim Unif(-\pi, \pi)$$

$$\rho(k) = \cos(\omega_0 k)$$

Hence, $\rho(k)$ is also a sinusoid and it never decays!

Since

$$R(k) = \frac{1}{2}A^2\cos(\omega_0 k) = \int_{-\pi}^{\pi} \cos(\omega k) dF(\omega)$$

then by uniqueness

$$dF(\omega) = \begin{cases} \frac{A^2}{4}, & \text{at } \pm\omega_0; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$F(\omega) = \begin{cases} 0, & \text{if } \omega < -\omega_0 \\ \frac{A^2}{4}, & \text{if } -\omega_0 \le \omega < \omega_0 \\ \frac{A^2}{2}, & \text{if } \omega_0 \le \omega. \end{cases}$$

and

$$R(0) = F(\pi) = \frac{A^2}{2}$$

Example: Sinusoid Plus Noise

$$X_t = A\cos(\omega_0 t) + B\sin(\omega_0 t) + \epsilon_t \equiv Y_t + \epsilon_t$$

$A, B \sim N(0, \sigma^2)$, independent; $\epsilon_t \sim N(0, \sigma_\epsilon^2)$, independent of A, B.

$$R(k) = R_y(k) + R_\epsilon(k) = \sigma^2 \cos(\omega_0 k) + \delta_k \sigma_\epsilon^2$$

We define

$$dF_y(\omega) = \begin{cases} \frac{\sigma^2}{2}, & \text{if } \omega = \pm\omega_0 \\ 0, & \text{otherwise.} \end{cases}$$

$$dF_\epsilon(\omega) = \frac{\sigma_\epsilon^2}{2\pi} d\omega$$

Then we can see,

$$R(k) = \int_{-\pi}^{\pi} e^{ik\lambda} dF_x(\lambda) = \int_{-\pi}^{\pi} e^{ik\lambda}[dF_y(\lambda) + dF_\epsilon(\lambda)]$$

Fourier Pair

Under absolute summability of the covariance function we have.

Discrete case:

$$R(v) = \int_{-\pi}^{\pi} e^{iv\omega} f(\omega) d\omega, \quad v = 0, \pm 1, ..., , \lambda > 0.$$

$$f(\omega) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} e^{-iwv} R(v), \quad -\pi \le \omega \le \pi$$

Example: Discrete time. Consider the cov. fun. $R(v)$ such that $R(0) = 1$, $v = 0, \pm 1, ...$

$$R(v) = \rho(v) = \lambda^{|v|}, \ |\lambda| < 1.$$

$$\begin{aligned} f(\omega) &= \frac{1}{2\pi}\{1 + \sum_{v=1}^{\infty} \lambda^v e^{i\omega v} + \sum_{v=1}^{\infty} \lambda^v e^{-i\omega v}\} \\ &= \frac{1-\lambda^2}{2\pi(1-\lambda e^{i\omega})(1-\lambda e^{-i\omega})} = \frac{1-\lambda^2}{2\pi(1-2\lambda\cos\omega + \lambda^2)} \end{aligned}$$

Continuous case:

$$R(v) = \int_{-\infty}^{\infty} e^{iv\omega} f(\omega) d\omega, \quad -\infty < v < \infty$$

$$f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iwv} R(v), \quad -\infty < \omega < \infty$$

Example: Continuous time. $R(v) = e^{-\lambda|v|}$, $\lambda > 0$, $R(0) = 1$.

$$f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda|v|} e^{-i\omega v} dv = \frac{\lambda}{\pi(\lambda^2 + \omega^2)}$$