MATH 734, SPRING 2024: MIDTERM EXAM

The exam contains 6 problems. You may use notes and textbooks during the exam. Collaborations, internet resources, and electronic devices are not allowed.

Theorems from Hatcher: *Algebraic Topology*, Chapters 0–2 and the appendix, as well as results we proved in class, can be cited without proof.

In the following, all homology and cohomology groups are defined with $\mathbb{Z}$ coefficients. We use the word "surfaces" to denote 2-dimensional manifolds.

Problem 1. (20 pts) Let $D^2 = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$ denote the open disk in $\mathbb{R}^2$. Assume $K \subset S^3$ is a subset of S^3 homeomorphic to S^1. Also assume that there is an open neighborhood U of K homeomorphic to $D^2 \times S^1$, such that the homeomorphism $\varphi : U \to D^2 \times S^1$ sends K to $\{(0,0)\} \times S^1$. Show that

$$H_1(S^3 - K) \cong \mathbb{Z}.$$

Hint: $H_1(S^3 - K)$ can be computed using the Mayer–Vietoris sequence.

Note: you are not allowed to cite Corollary 3.45 (the Alexander duality theorem) for this problem.

Problem 2. (30 pts)

(1) Let X be a topological space. Show that $H^1(X)$ does not contain torsion elements.
(2) Let M be a compact oriented manifold with dimension n, show that $H_{n-1}(M)$ does not contain torsion elements.
(3) Assume M is a manifold such that $H_1(M) = 0$, show that M is orientable.

Hint: you may need to cite the Hurewicz isomorphism on the fundamental group (see Theorem 2A.1) for part (3).

Problem 3. (30 pts)

(1) Show that $\mathbb{C}P^2$ is not homotopy equivalent to $S^2 \vee S^4$.
(2) Show that there exists a continuous map from S^3 to S^2 that is *not* homotopic to the constant map.

Hint: for part (1), consider the ring structures on cohomology.

Problem 4. (20 pts) Let $f : \mathbb{C}P^2 \to \mathbb{C}P^1$ be a continuous map. Show that the induced map

$$f_* : H_2(\mathbb{C}P^2) \to H_2(\mathbb{C}P^1)$$

is zero.

Hint: consider the ring structures on cohomology.

Problem 5. (30 pts) Let Σ be a compact oriented surface with genus g.

(1) Compute the cohomology groups of Σ.
(2) Compute the cohomology ring of Σ.

Hint: For Part 1, you may use the celluar structure of Σ shown on page 5 of the textbook; see also Example 2.36 for the computation of the homology group. A strategy of the computation of the ring structure is sketched in Section 3.2, Problem 1 of the textbook. It is also possible to compute the ring structure using Poincaré duality and the geometric interpretation of intersection numbers.

Problem 6. (20 pts) Assume M, M' are two oriented connected compact manifolds with dimension n, and $f : M \to M'$ is a continuous map. Recall that the *degree* of f is defined to be the integer that characterizes the map

$$f_* : H_n(M) \to H_n(M'),$$

where both $H_n(M)$ and $H_n(M')$ are canonically isomorphic to $\mathbb{Z}$ via the orientations.

Let Σ_g, Σ_h be compact oriented surfaces with genera g and h respectively. Show that there exists a map $f : \Sigma_g \to \Sigma_h$ with a non-zero degree if and only if $g \geq h$.

Hint: consider the ring structures on cohomology.