Lecture 1 : Connections and curvatures on principal bundles.

Def 1. Suppose M is a manifold. A real vector bundle with rank k $(k \in \mathbb{Z}^+)$ consists of the following data:

(1) A smooth mfld E and a smooth map $\pi : E \to M$

(2) For each $x \in M$, a linear structure on $\pi^{-1}(x)$ s.t. $\pi^{-1}(x)$ is an $\mathbb{R}$-linear space with rank k.

(3) $\forall x \in M$, an open nbhd U_x of x in M and a diffeomorphism $\phi_x : \pi^{-1}(U_x) \to U_x \times \mathbb{R}^k$

s.t. $\left(\pi^{-1}(U_x) \xrightarrow{\phi_x} U_x \times \mathbb{R}^k \xrightarrow{\text{proj}} U_x \right) = \pi \big|_{\pi^{-1}(U_x)}$

and $\left(\pi^{-1}(U_x) \xrightarrow{\phi_x} U_x \times \mathbb{R}^k \to \mathbb{R}^k \right)$ is a linear isomorphism on each fiber.

Examples. Trivial bundles. tangent bundles. cotangent bundles.

Remark: If $U_x \cap U_y \neq \emptyset$ in definition 1, then

$\exists\ g_{xy} : U_x \cap U_y \to GL_k(\mathbb{R})$ s.t.

$$\phi_y \circ \phi_x^{-1} (p, v) = (p, g_{xy}(p) v)$$

for all $p \in U_x \cap U_y$, $v \in \mathbb{R}^k$.

Def 2.
$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowright"] & E' \arrow[dl, "\pi'"] \\ M & \end{tikzcd}$$
is a vector bundle isomorphism iff f is a diffeomorphism that restricts to linear isomorphisms ~~on all fibers~~ on all fibers.

Alternatively, vector bundles can be characterized by the above property.

<u>Def 1'</u>. A real vector bundle of rank k over M consists of the following data:

(1) A smooth mfld E, a smooth map $\pi : E \to M$.

(2) $\forall x \in M$, $\exists$ an open nbhd $\mathcal{U}_x$ of x and a diffeomorphism $\phi_x : \pi^{-1}(\mathcal{U}_x) \xrightarrow{\phi_x} \mathcal{U}_x \times \mathbb{R}^k$

$$\begin{tikzcd} \pi^{-1}(\mathcal{U}_x) \arrow[rr, "\phi_x"] \arrow[dr, "\pi"'] & & \mathcal{U}_x \times \mathbb{R}^k \arrow[dl, "\text{projection}"] \\ & \mathcal{U}_x & \end{tikzcd}$$

Such that:

(3) if $\mathcal{U}_x \cap \mathcal{U}_y \neq \emptyset$, $\exists$ $g_{xy} : \mathcal{U}_x \cap \mathcal{U}_y \to GL_k(\mathbb{R})$

s.t. $\phi_y \circ \phi_x^{-1}(p, v) = (p, g_{xy}(p)\, v)$

<u>Rmk</u>. The linear ~~str~~ space structures can be recovered from Def 1' by requiring $\left(\pi^{-1}(\mathcal{U}_x) \xrightarrow{\phi_x} \mathcal{U}_x \times \mathbb{R}^k \to \mathbb{R}^k\right)$ are linear isomorphisms on fibers. By condition (3), the linear structures are well-defined.

Another point of view: the vector bundle can be specified (up to isomorphisms) by the "transition maps (functions)" $g_{xy} : \mathcal{U}_x \cap \mathcal{U}_y \to GL_k(\mathbb{R})$

<u>Def 3</u>. Suppose M is a mfld and G is a Lie group. A principal G-bundle over M consists of the following data:

(1) $P \xrightarrow{\pi} M$ smooth map

(2) A <u>right</u> G-action on each fiber $\pi^{-1}(x)$

~~such that~~:

(3) $\forall x \in M$, $\exists\ \mathcal{U}_x$ open nbhd of x, and a diffeomorphism $\phi_x : \pi^{-1}(\mathcal{U}_x) \longrightarrow \mathcal{U}_x \times G$

s.t. (i)
$$\begin{tikzcd} \pi^{-1}(\mathcal{U}_x) \arrow[r, "\phi_x"] \arrow[dr, "\pi"'] & \mathcal{U}_x \times G \arrow[d, "\text{projection}"] \\ & \mathcal{U}_x \end{tikzcd}$$

(ii) ϕ_x preserves the G-actions.

<u>Def 4</u>.
$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\pi"'] & P' \arrow[dl, "\pi'"] \\ M & \end{tikzcd}$$
is a principal bundle isomorphism iff f is a diffeomorphism and preserves the G-action.

<u>Def 3'</u>: Alternatively, a principal G-bundle over M consists of the following:

(1) $P \xrightarrow{\pi} M$ smooth map

(2) $\forall x \in M$, $\phi_x : \pi^{-1}(\mathcal{U}_x) \longrightarrow \mathcal{U}_x \times G$
$$\begin{tikzcd} \pi^{-1}(\mathcal{U}_x) \arrow[r] \arrow[dr, "\pi"'] & \mathcal{U}_x \times G \arrow[d, "\text{projection}"] \\ & \mathcal{U}_x \end{tikzcd}$$

s.t. (3) $\forall x, y$ with $\mathcal{U}_x \cap \mathcal{U}_y \neq \emptyset$, $\exists\ g_{xy} : \mathcal{U}_x \cap \mathcal{U}_y \to G$

s.t. $\phi_y \circ \phi_x^{-1}(p, g) = (p, g_{xy}(p) \cdot g)$

Since left multiplications on G preserve the right G-action, we have well-defined right G-actions on the fibers of π from (2), (3) of Def 3'.

Another point of view: the principal G-bundle is specified (up to isomorphisms) by the transition maps (functions) $g_{xy} : \mathcal{U}_x \cap \mathcal{U}_y \to G$.

Remark: Real bundles of rank k $\Longleftrightarrow$ $GL_k(\mathbb{R})$-bundles.

Proof 1: Both structures ~~have transition functions~~ are described by transitions functions $\mathcal{U}_x \cap \mathcal{U}_y \longrightarrow GL_k(\mathbb{R})$.

Proof 2: (i) If $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ rank-k, real vector bundle,

define $P := \left\{ (x, v_1 \dots v_k) \;\middle|\; \begin{matrix} x \in M \\ (v_1, \dots, v_k) \text{ is a linear} \\ \text{basis of } \pi^{-1}(x) \end{matrix} \right\}$.

then P is a principal $GL_k(\mathbb{R})$ bundle.

(ii) If $\begin{matrix} P \\ \downarrow \\ M \end{matrix}$ is a principal $GL_k(\mathbb{R})$-bundle,

define $E := P \times_{GL_k(\mathbb{R})} \mathbb{R}^k = P \times \mathbb{R}^k \big/ (pg, v) \sim (p, gv)$

for all $g \in GL_k(\mathbb{R})$

More generally, if $\begin{matrix} P \\ \downarrow \\ M \end{matrix}$ is a principal G-bundle and $\rho: G \to GL_k(\mathbb{R})$ is a linear representation of G, then we can define a vector bundle

$$P \times_\rho \mathbb{R}^k := P \times \mathbb{R}^k \Big/ (pg, v) \sim (p, \rho(g)v)$$

(for all $g \in G$)

Example: $G = SO(k)$, ~~then $P \times_\rho$~~ let ρ be the standard representation of G in $\mathbb{R}^k$.

Then $P \times_\rho \mathbb{R}^k$ is a vector bundle endowed with an inner product, because the standard Euclidean structure on $\mathbb{R}^k$ is preserved by the G-action, and hence we can define $\| [p, v] \| := |v|_{\mathbb{R}^k}$.

This establishes a one-to-one correspondence between (1) $SO(k)$-bundles and (2) rank-k real vector bundles with inner products and orientations.

Remark: similar examples work for $U(k)$, $Sp(k)$ ...

Let $\mathfrak{g}$ be the Lie algebra of G, define

$$\mathrm{ad}\, P := P \times_{\mathrm{ad}} \mathfrak{g},$$ where

$\mathrm{ad}: G \to GL(\mathfrak{g})$ is the adjoint representation.

<u>Lemma</u>

If $\rho : G \to GL(V)$ is a linear representation.

then its tangent map induces a Lie algebra homomorphism

$$f : \mathfrak{g} \longrightarrow \mathrm{End}(V).$$

This action can be lifted to principal bundles:

$$f : \mathrm{ad}\, P \longrightarrow \mathrm{End}(P \times_\rho V).$$

<u>Proof</u>: Let $\xi \in \mathfrak{g}$, $v \in V$.

~~Then~~ $\cancel{\rho(g)(f(\xi)(g^{-1}v)) = f((\mathrm{ad}\, g)(\xi))(v)}$.

Define $[p, \xi]([p, v]) := [p, \xi \cdot v]$

then $[pg, g^{-1}\xi g]([pg, g^{-1}v])$

$= [pg, (g^{-1}\xi g) \cdot g^{-1} v]$

$= [pg, g^{-1}\xi v] \sim [p, \xi v]$

So the lifting is well-defined. $\square$

Connections on vector bundles:

<u>Def</u>. If E is a vector bundle, then a <u>connection</u> ∇ on E is an operator $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$

s.t. (1) $\nabla(s_1 + s_2) = \nabla s_1 + \nabla s_2$

(2) $\nabla(fs) = df \otimes s + f \cdot \nabla s$.

Example: If E is the trivial $\mathbb{R}$-bundle, then

$$d : \Gamma(E) = C^\infty(M) \longrightarrow \Gamma(T^*M \otimes E) = C^\infty(T^*M)$$

is a connection.

In general, if $s_1 \dots s_k$ is a local basis of V, then

$$\nabla\left(\sum_{i=1}^k f_i s_i\right) = \sum_{i=1}^k df_i \otimes s_i + \sum_{i=1}^k f_i \cdot \nabla s_i$$

Suppose $\nabla s_i = \sum_{j=1}^k a_{ji} \otimes s_j$, where $a_{ij} \in \Gamma(T^*M)$

then $\nabla\left(\sum_{i=1}^k f_i s_i\right) = \sum_{i=1}^k df_i \otimes s_i + \sum_{i=1}^k \sum_{j=1}^k f_i \, a_{ji} \otimes s_j$

The matrix $A = (a_{ji})_{1 \le j, i \le k}$ is called the <u>connection matrix</u>, and can be regarded as the local coordinate representation of ∇.

Intrinsically, $A \in \Gamma(T^*M \otimes \mathrm{End}(E))$.

The intrinsic description of A can be seen as follows:

If ∇, ∇' are two connections, then

$$\begin{cases} (\nabla-\nabla')(fs) = f(\nabla-\nabla')s \\ (\nabla-\nabla')(s_1+s_2) = (\nabla-\nabla')(s_1) + (\nabla-\nabla')(s_2) \end{cases}$$

So $(\nabla-\nabla')$ is a pointwise linear map from E to $T^*M \otimes E$. i.e. $\nabla-\nabla' \in \Gamma(T^*M, \mathrm{End}(E))$.

If $s_1 \cdots s_k$ is a local basis, let ∇' be the respectively trivialized connection (i.e. $\nabla' s_i = 0$ for $i=1,\cdots,k$) then (a_{ji}) is the matrix representation of $\nabla-\nabla'$.

Connections on ~~...~~ principal bundles:

<u>Def</u>. A connection on $\begin{matrix} P \\ \downarrow \\ M \end{matrix}$ is a splitting

$$TP = T^V P \oplus T^H P \quad \text{s.t.}$$

(1) $T^V P$ = the ~~tang~~ tangent spaces of the fibers

(2) $T^H P$ is equivariant under the G-action.

<u>Rmk</u>. If $G = GL_k(\mathbb{R})$, this is equivalent to affine connections on vector bundles.

If E is a vector bundle, ∇ is an affine connection on E, let $P = \{ (x, v_1, \ldots, v_k) \mid \begin{smallmatrix} x \in M \\ (v_1, \ldots, v_k) \end{smallmatrix} \text{ basis of } \pi^{-1}(x) \}$ be the corresponding principal bundle, define $T^H P$ such that: $(x(t), v_1(t), \ldots, v_k(t))$ is tangent to $T^H P$ at $t=0$ $\Leftrightarrow$ $\nabla_{\dot{x}(t)} v_i(t) = 0$ at $t=0$ for all $i = 1 \ldots k$.

If $G = SO(k)$, then a G-connection is equivalent to a connection on $P \times_G \mathbb{R}^k$ s.t. parallel translations preserve orthogonality (i.e. $\langle \nabla s_1, s_2 \rangle + \langle s_1, \nabla s_2 \rangle = d\langle s_1, s_2 \rangle$)

Similar properties hold for $SU(k)$, $Sp(k)$ ...

Curvatures on vector bundles.

If $\nabla_A : \Gamma(E) \to \Gamma(T^*M \otimes E)$ is a connection, define

$$d_A : \Gamma(\Lambda^k T^*M \otimes E) \to \Gamma(\Lambda^{k+1} T^*M \otimes E)$$

by the property $d_A(\omega \otimes s) = d\omega \otimes s + (-1)^{|\omega|} \omega \otimes \nabla_A s$.

Consider $d_A \circ d_A : \Gamma(E) \to \Gamma(\Lambda^2 T^* M \otimes E)$

Then $d_A \circ d_A (fs) = d_A (df \otimes s + f d_A s)$

$$= d^2 f \otimes s - df \otimes d_A s$$
$$+ df \otimes d_A s + f d_A^2 s$$

$$= f d_A^2 \otimes s$$

$\Rightarrow$ d_A^2 is a pointwise linear map from $\cancel{\Gamma(E)}$ E to $\Lambda^2 T^* M \otimes E$.

Let $F_A \in \Gamma(\Lambda^2 T^* M \otimes \mathrm{End}(E))$ be the section given by d_A^2.

Under local coordinates, if $d_A = d + A$,

then
$$d_A^2 s = (d + A)(d + A) s$$
$$= d(As) + A(ds) + A \wedge A s$$
$$= (dA + A \wedge A) s.$$

(Since A is a matrix of 1-forms, we have $d(As) = (dA) s - A\, ds$)

So F_A is given by $dA + A \wedge A$.

Chern - Weil theory: If E is a complex vector bundle,

define $\tilde{c}_i(E) \in \Gamma(\Lambda^{2i} T^* M)$ by

$$\det\left(1 + \frac{i}{2\pi} F_A\right) =: 1 + \tilde{c}_1(E) + \cdots + \tilde{c}_k(E)$$

Then $d\,\tilde{c}_i(E) = 0$ and

$$c_i(E) := [\tilde{c}_i(E)] \in H^{2i}_{deR}(M; \mathbb{C})$$

is an invariant of E (i.e. independent of A).

By definition, $c_i(E)$ is the i^{th} Chern class of E.

Remark: It is also possible to define Chern classes in $H^*(M; \mathbb{Z})$.