July 7: The moduli space of connections and the ASD equation

Let X be an oriented, closed 4-mfld

Let P be a principal $SU(2)$ bundle over X. let E be the associated $\mathbb{C}^2$-bundle.

Lemma: $c_1(E) = 0$

Proof 1: Since P is an $SU(2)$-bundle, $\det(E)$ is trivial, therefore $c_1(E) = c_1(\det(E)) = 0$.

Proof 2: Let A be an $SU(2)$ connection, then F_A is a section of $\Lambda^2 T^* X \otimes \underline{su(2)}$, therefore $tr(F_A) = 0$. $\Rightarrow$ $c_1(E) = 0 \in H^2(X; \mathbb{C})$.

Proposition. Let P_1, P_2 be two principal $SU(2)$ bundles, let E_1, E_2 be the associated $\mathbb{C}^2$-bundle.

Then $P_1 \cong P_2 \iff c_2(P_1) = c_2(P_2)$

Proof. Since $\pi_1(SU(2)) = \pi_2(SU(2)) = 0$

$\pi_3(SU(2)) \cong \mathbb{Z}$.

$\Rightarrow$ $\pi_i(BSU(2)) = 0$ for $i = 0, 1, 2, 3$

$\pi_4(BSU(2)) \cong \mathbb{Z}$.

$\Rightarrow \quad [X, BSU(2)] \cong \mathbb{Z}$, and the homotopy class of $f: X \to BSU(2)$ is determined by the map

$$f^*: H^4(BSU(2)) \to H^4(X).$$

It can be shown that $f^*(\underset{\substack{\uparrow \\ \text{generator}}}{1}) = c_2(E)$ $\square$.

Since F_A is a section of $\Lambda^2 T^*X \otimes \underline{su(2)}$, we have

$$\det F_A = -\frac{1}{2}\operatorname{Tr}(F_A \wedge F_A)$$

therefore

$$\det\left(1 + \frac{i}{2\pi} F_A\right) = 1 + \frac{1}{8\pi^2}\operatorname{Tr}(F_A \wedge F_A).$$

and we have:

Lemma: $\langle c_2(E), [X] \rangle = +\frac{1}{8\pi^2}\int \operatorname{Tr}(F_A \wedge F_A)$ $\square$.

Recall that $\operatorname{ad} P = P \times_{ad} \underline{su(2)} \subseteq \operatorname{End}(E)$.

Introduce a metric $\langle , \rangle$ on $\underline{su(2)}$ by taking

$$\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\}$$

as an orthonormal basis, then $\langle , \rangle$ is invariant under the adjoint action. (In fact, $\langle \xi_1, \xi_2 \rangle = -\operatorname{Tr}(\xi_1 \xi_2)$

$= -\frac{1}{2}\operatorname{Tr}([\xi_1, [\xi_2, \cdot]])$)

Then the metric lifts to an inner product structure on ad P, and F_A is a section of $\Omega^2(X) \otimes \mathrm{ad}(P)$.

We have

$$\mathrm{Tr}\,(F_A \wedge F_A) = - \langle F_A \wedge F_A \rangle$$

Recall: $\quad * : \Omega^2(X) \to \Omega^2(X) \quad$ s.t. $*^2 = 1$.

decompose

$$\Omega^2(X) = \Omega^2_+(X) \oplus \Omega^2_-(X)$$

where $\quad * = \mathrm{id} \quad$ on $\Omega^2_+(X)$

$\quad\quad\quad = -\mathrm{id} \quad$ on $\Omega^2_-(X)$, $\quad$ ~~and $\Omega^2_+ \wedge \Omega^2_- = 0$~~ and $\alpha \wedge \beta = 0$ if $\alpha \in \Omega^2_+$, $\beta \in \Omega^2_-$.

$\Rightarrow \quad F_A = F_A^+ + F_A^-$, where

F_A^+ is a section of $\Omega^2_+(X) \otimes \mathrm{ad}\,P$

F_A^- is a section of $\Omega^2_-(X) \otimes \mathrm{ad}\,P$.

As a consequence,

$$F_A \wedge F_A = (F_A^+ \wedge F_A^+ + F_A^- \wedge F_A^-) \cdot d\mathrm{vol}.$$

So $\quad \mathrm{Tr}\,(F_A \wedge F_A) = - (|F_A^+|^2 - |F_A^-|^2)\, d\mathrm{vol}.$

Corollary:

$$\int |F_A^+|^2 - |F_A^-|^2 \, d\mathrm{vol}$$

$$= -8\pi^2 \langle c_2(E), [X] \rangle$$

$$\int |F_A|^2 = +8\pi^2 \langle c_2(E), [X] \rangle + 2 \int |F_A^+|^2$$

$$\geq +8\pi^2 \langle c_2(E), [X] \rangle \qquad (*)$$

Def. $+\langle c_2(E), [X] \rangle =: k$. "instanton number"

then $\int |F_A|^2 \geq 8\pi^2 k$.

Def. A connection is called "anti-self-dual" (ASD) if $F_A^+ = 0$. A connection is ASD iff $(*)$ achieves equality.

Locally, $F_A^+ = 0 \iff d^+ A + (A \wedge A)^+ = 0$.

Object of study: the space of solutions to the ASD equation.

~~What structures~~

- Topology?
- other structures?

It turns out to be more convenient to study the ASD equation in Sobolev spaces.

Def. We say that a connection A is in L^p_k, if locally its matrix representations are in L^p_k.

Sobolev multiplication theorem: if $\frac{k}{n} > \frac{1}{p}$, then $f \in L_k^p,\ g \in L_k^p \Rightarrow f \cdot g \in L_k^p$. (In ~~our~~ our case, $n=4$)

$\Rightarrow$ If A is in L_k^p, with $k \geq 1$, $\frac{k}{n} > \frac{1}{p}$, then F_A is in L_{k-1}^p.

<u>Notation</u>. $\mathcal{A}_k^p$: ~~affine~~ space of L_k^p connections.

$\mathcal{A}_k^p$ is an affine Banach space on $L_k^p(T^*M \otimes \operatorname{ad} P)$

Now consider the automorphism group of P. Locally, these are given by maps from local charts to $SU(2)$.

<u>Notation</u>: $\mathcal{G}_k^p$ space of L_k^p-automorphisms of P.

~~If g is a smooth automorphism, A is a smooth connection,~~

By the Sobolev multiplication thm, $\mathcal{G}_k^p$ is a group when $\frac{k}{n} > \frac{1}{p}$. It turns out that $\mathcal{G}_k^p$ is a Banach Lie group in this case.

(For our case, $n=4$)

<u>Sketch of proof</u>: $\mathcal{G}_k^p \subseteq \mathrm{End}_k^p(E)$

and is given by the zero locus of a regular map. $\square$

If g is a smooth gauge transformation, A a smooth connection, then

$$g_*(A) = A - (\nabla_A g)\cdot g^{-1}$$

By the Sobolev multiplication theorem again, if $\frac{k}{n} > \frac{1}{p}$, then $\mathcal{G}_{k+1}^p$ acts (smoothly) on $\mathcal{A}_k^p$.

The ASD equation is invariant under the $\mathcal{G}_{k+1}^p$ action.

Define $\mathcal{C}_k^p := \mathcal{A}_k^p / \mathcal{G}_{k+1}^p$

$$\mathcal{M}_k^p := \{A \in \mathcal{A}_k^p \mid F_A^+ = 0\} / \mathcal{G}_{k+1}^p.$$

Then $\mathcal{M}_k^p \subseteq \mathcal{C}_k^p$.

Lemma: $\mathcal{B}_k^p$ is Hausdorff.

Pf. Suppose $A_i \to A$ in $\mathcal{A}_k^p$
$B_i \to B$

and $\exists\, g_i \in \mathcal{G}_{k+1}^p$ s.t. $g_i(A_i) = B_i$

need to show that $\exists\, g \in \mathcal{G}_{k+1}^p$ s.t. $g(A) = B$.

By the assumption:

$$A_i - (d_{A_i} g_i)\, g_i^{-1} = B_i$$

$$d_{A_i} g_i = (B_i - A_i)\, g_i$$

locally:

$$dg_i + [A_i, g_i] = (B_i - A_i)\, g_i$$

Since $SU(2)$ is compact, $\|g_i\|_{L^\infty} \le C$.

$$\Rightarrow \| -[A_i, g_i] + (B_i - A_i)\, g_i \|_{L^p} \le C$$

$$\Rightarrow \| dg_i \|_{L^p} \le C$$

$$\Rightarrow \| g_i \|_{L_1^p} \le C.$$

Sobolev multiplication

$$\Rightarrow \| -[A_i, g_i] + (B_i - A_i)\, g_i \|_{L_1^p} \le C$$

$$\Rightarrow \| g_i \|_{L_2^p} \le C$$

(Bootstrap)

$$\Rightarrow \| g_i \|_{L_{k+1}^p} \le C.$$

$\exists$ a weakly convergent subsequence in L_{k+1}^p

$$g_i \rightharpoonup g$$

s.t. $\quad g_i \to g_\infty \quad$ in L^p_k.

$\Rightarrow \quad g \in \mathcal{G}^p_{k+1}$. and $g A = B$.

If $\quad A \in \mathcal{G}^p_k. \quad g \in \mathcal{G}^p_{k+1}$. Then

$$g(A) = A \iff \nabla_A g = 0.$$

$$\overset{\text{(locally)}}{\iff} \quad dg + [A, g] = 0.$$

~~If g~~

If $\quad g = -id$ pointwise. then this condition always holds.

If $\quad g \neq \pm id$ and $\nabla_A g = 0$. Then

$g \neq \pm id$ everywhere, and the eigenspace decomposition by g decomposes E into $E_1 \oplus E_2$. with eigenvalues λ_1, λ_2. $\quad$ ~~$(\lambda_1 * \lambda_2 = 0)$~~ $\quad (\lambda_1 \cdot \lambda_2 = 1)$

and $\quad g = \lambda_1 \cdot \pi_{E_1} + \lambda_2 \cdot \pi_{E_2}$

Since $\quad d(\operatorname{Tr} g) = 0$
$\quad d(\operatorname{Tr} g^2) = 0 \quad \Rightarrow \quad \lambda_1, \lambda_2$ are constants

$\Rightarrow \quad \nabla \pi_{E_1} = 0, \quad \nabla \pi_{E_2} = 0$

$\Rightarrow \quad A$ decomposes as the direct sum of connections on E_1 and E_2

Def. A is called <u>irreducible</u> if $\mathrm{Stab}(A) = \{\pm 1\}$

is called <u>reducible</u> if $\mathrm{Stab}(A) \cong U(1)$

Let $(\mathcal{A}_k^p)^* \subsetneq \mathcal{A}_k^p$ be the set of irreducibles,

then $\mathcal{A}_k^p - (\mathcal{A}_k^p)^*$ has codimension ∞.

and $(\mathcal{A}_k^p)^* / \mathcal{G}_{k+1}^p$ is a Banach mfld.