July 8: Uhlenbeck's gauge fixing theorem and compactness theorem

<u>Lemma</u>: Let P be the trivial $SU(2)$-bundle over S^4, let E be the associated $\mathbb{C}^2$-bundle, let A_0 be the trivial connection on E. Then there exists constants $\varepsilon > 0$, $C > 0$ such that the following holds:

Suppose $A(t)$ is a continuous 1-parameter family of L_1^2-connections with $\| F_{A(t)} \|_{L^2} < \varepsilon$ and $A(0) = A_0$.

Then for each t, there $\exists$ $g(t) \in \mathcal{G}_2^2$, s.t.

(1) $d^*\big(g(t)\, A(t)\big) = 0$.

(2) $\| g(t)\, A(t) \|_{L_1^2} \le C \cdot \| F_{A(t)} \|_{L^2}$

Remark: The constants ε and C depend on the metric on S^4.

<u>Proof</u>. Let S be the set of t s.t. the desired result holds. We show that S is both open and closed. (Since $0 \in S$, the desired lemma would follow.)

(i) S is closed. If
$$\begin{cases} t_i \to t, \\ d^*\big(g(t_i)\, A(t_i)\big) = 0 \\ \| g(t_i)\, A(t_i) \|_{L_1^2} \le C \| F_{A(t_i)} \|_{L^2}. \end{cases}$$

we show that $g(t_i)$ has a weak limit in $\mathcal{G}_2^2$.

Let $B_i := g(t_i) A(t_i)$, $A_i := A(t_i)$, $g(t_i) := g_i$.

then $B_i = A_i - (d_{A_i} g_i) g_i^{-1}$

$= A_i - [A_i, g_i] g_i^{-1} - (dg_i) g_i^{-1}$

$\Rightarrow -B_i g_i + g_i A_i = (dg_i)$

Since $\|g_i\|_{L^\infty} \leq C \Rightarrow \|-B_i g_i + g_i A_i\|_{L^4} \leq C$

$\Rightarrow \|g_i\|_{L_1^4} \leq C$ L^∞-bounded L_1^2-bounded L^6-bounded

$\|B_i g_i\|_{L_1^2} \lesssim \|\nabla B_i \cdot g_i\|_{L^2} + \|B_i \cdot \nabla g_i\|_{L^2}$

$\leq C$.

Similarly, $\|g_i A_i\|_{L_1^2} \lesssim C$

$\Rightarrow \|g_i\|_{L_2^2} \lesssim C$.

So $\{g_i\}$ has a subsequence that is weakly convergent in L_2^2 (and hence strongly convergent in L_1^2).

By the assumptions, $A_i \to A(t)$ in L_1^2

after taking a subsequence, $B_i \rightharpoonup B$ in L_1^2

$g_i \to g$ in L_1^2.

$\Rightarrow B_i g_i \to Bg$ in L^p (for $\forall$ $p<2$).

$g_i A_i \to gA$ in L^p

$dg_i \to dg$ in L^2

$$\Rightarrow \quad -Bg + gA = dB$$

We also have $d^*B = 0$, $\|B\|_{L_1^2} \leq C\|F_B\|$ by similar arguments.

(ii) S is open. Suppose $t \in S$. WLOG, assume $\begin{cases} g(t) = id \\ d^*A(t) = 0 \end{cases}$

Consider the map

$$\Phi : L_2^2(\mathrm{ad}\, P) \times L_1^2(T^*S^4 \otimes \mathrm{ad}\, P) \to L^2(\mathrm{ad}\, P)$$

defined by $\quad \Phi(\xi, a) := d^*\left(\exp(\xi)\,(A + a)\right)$

Then $\quad d\Phi\big|_{(0,0)}(\xi, a) = d^*\left([\xi, A] + a - d\xi\right)$.

Notice that $\quad s \in \ker d^*d \iff d^*ds = 0$

$$\Rightarrow \langle d^*ds, s\rangle = 0$$

$$\Rightarrow \langle ds, ds\rangle = 0$$

$$\Rightarrow ds = 0.$$

So $\ker d^*d = \ker d$.

"$\Rightarrow$" $\mathrm{Im}\,(d^*d) = \mathrm{Im}\, d^* \subseteq L^2(\mathrm{ad}\, P)$.

Therefore $\mathrm{Im}\,\left(\xi \mapsto (d^*[\xi, A] - d^*d\xi)\right)$

$= \mathrm{Im}\, d^*$ when $\|A\|_{L_1^2}$ is sufficiently small.

So statement (1) is an open condition.

To show that (2) is an open condition, notice that if $d^* A = 0$, then

$$F_A = (d + d^*) A + A \wedge A$$

Since $\ker (d + d^*) = 0$ on S^4, we have

$$\| A \|_{L^2_1} \le C_1 \cdot \| F_A \|_{L^2} + C_1 \cdot \| A \wedge A \|_{L^2}$$

$$\le C_1 \cdot \| F_A \|_{L^2} + C_2 \| A \|^2_{L^2_1}$$

Therefore, $\| A \|_{L^2_1} < \frac{1}{2 C_2} \Rightarrow \| A \|_{L^2_1} \le 2 C_1 \| F_A \|_{L^2}$

and the first inequality is an open condition. $\square$.

Cor. If A is a L^2_1 connection on $B^4(1)$ s.t.

$\int_{B^4(1)} \| F_A \|^2 < \varepsilon$. then $\exists\, g \in G^2_2 (B^4(1-\delta))$

s.t. (1) $d^* (g(A)) = 0$ on $B^4(1-\delta)$

(2) $\| g(A) \|_{L^2_1 (B^4(1-\delta))} \le C \cdot \| F_A \|_{L^2_1(B^4)}$

Pf. Define $A(t) := f(t)^* A$, where

$f(t) : S^4 \to B^4$ is a smooth family of maps, s.t.

$f(0)$ = the zero map, and $f(1)$ is an isometry from the ~~northen~~ northern hemisphere to $B^4(1-\delta)$

Proposition. If A is an L^2_3-connection on $B^4(1)$ s.t. $F_A^+ = 0$, then A is gauge equivalent to a C^∞-connection on $B^4(1-\delta)$.

Pf. After gauge transformation, assume $d^*A = 0$ on $B^4(1-\frac{\delta}{2})$. Then

$$(d^* + d^+) A + (A \wedge A)^+ = 0.$$

$$A \in L^2_3(B^4(1-\tfrac{\delta}{2})) \Rightarrow (A \wedge A)^+ \in L^2_3(B^4(1-\tfrac{2}{3}\delta))$$

$$\Rightarrow A \in L^2_4(B^4(1-\tfrac{3}{4}\delta))$$

(induction) $\Rightarrow A \in L^2_k(B^4(1-\delta))$

for all k.

$$\Rightarrow A \in C^\infty(B^4(1-\delta))$$

Remark. If $A \in \mathcal{A}^p_k$ for $(1+k)p > 4$, the same argument shows that $A \in C^\infty$ on $B^4(1-\delta)$.

Proposition. If $A \in \mathcal{A}^2_3$ and $F_A^+ = 0$, then $\exists\ g \in \mathcal{G}^2_4$ s.t. $g(A)$ is smooth on X.

Proof. $\exists$ an open covering $\{\mathcal{U}_i\}$ of X s.t. on each $\mathcal{U}_i$, $\exists\ g_i$ w/ $g_i(A_i)$ smooth.

We need to "patch" $\{g_i\}$ together.

Suppose $X = U_1 \cup U_2 \cup \cdots \cup U_n$

Let $V_m := U_1 \cup \cdots \cup U_m$.

We use induction to find a desired gauge transformation on V_m.

Suppose g_{V_m} is defined on V_m, $g_{U_{m+1}}$ is defined on U_{m+1}. s.t.

$$g_{V_m}(A) \in C^\infty(V_m)$$

$$g_{U_{m+1}}(A) \in C^\infty(U_{m+1})$$

We may assume that $g_{V_m}, g_{U_{m+1}}$ are close to id in C^0 by replacing g_{V_m} with $\tilde{g}_{V_m}^{-1} g_{V_m}$, where $\tilde{g}_{V_m}$ is a C^∞ approximation of g_{V_m}. (and Similar for $g_{U_{m+1}}$).

Then $g := g_{V_m} \cdot g_{U_{n+1}}^{-1} \in C^\infty(V_m \cap U_{m+1})$

(In fact, let $B_1 = g_{V_m}(A)$, $B_2 = g_{U_{m+1}}(A)$, then $dg = B_1 g - g B_2$.) and g is close to id in C^0. So g extends to U_{m+1} as a C^∞-gauge transformation (after shrinking U_i if necessary)

□.

Lemma. Suppose A is an ASD connection on $B^4(1)$. and

$$\begin{cases} \|F_A\|_{L^2(B(1))} \le \varepsilon \\ d^* A = 0 \quad \text{on } B(1-\delta) \\ \|A\|_{L^2_1(B(1-\delta))} \le C \cdot \|F_A\|_{L^2(B(1))} \end{cases}$$

Then for $\forall k$, $\exists$ a polynomial f with zero constant term, s.t.

$$\|A\|_{L^2_k(B(1-2\delta))} \le f\left(\|F_A\|_{L^2(B(1))}\right)$$

Pf. Let η be a cut-off function.

$$(d^+ + d^*)\eta A = ((d^+ + d^*)\eta) \wedge A - \eta (A \wedge A)^+$$

$$\cancel{d^+ + d^*} \Rightarrow \|\eta A\|_{L^2_{k+1}(B(1-2\delta))} \le C \cdot \|A\|_{L^2_k(B(1-\delta))}$$

$$+ C \cdot \|A\|^2_{L^2_k(B(1-\delta))}$$

when $k \ge 3$.

So we only need to bound L^2_2 and L^2_3 norms of A.

For that, we have

$$\|\eta A\|_{L^2_k(B(\cancel{1-\delta}\,1))} \le C \cdot \|A\|_{L^2_{k-1}(B(1-\delta))}$$

$$+ C \cdot \|A\|_{L^2_1(B(1-\delta))} \cdot \|\eta A\|_{L^2_k(B(1))}$$

If ε is sufficiently small, we obtain L^2_2 and L^2_3 estimates by rearrangement.

Theorem (Uhlenbeck). If $\{A_i\}$ is a sequence of ASD connections with instanton number k, then there exists a subsequence that is convergent on $X - \{\text{finite set}\}$ in C^∞ after ~~to~~ gauge transformations.