July 9: Taubes' gluing

• The BPST instanton

Let $S^7 \subseteq \mathbb{R}^8 = \mathbb{H}^2$ be the unit sphere

identify $SU(2)$ with $\{ x^0 + x^1 I + x^2 J + x^3 K \in \mathbb{H} \mid \Sigma (x^i)^2 = 1 \}$.

Then $SU(2)$ acts on S^7 (on both left and right)

The quotient space of the right action is given by

$$\{ (x, y) \in \mathbb{H}^2 \mid |x|^2 + |y|^2 = 1 \} \Big/ (xq, yq) \sim (x, y)$$

$$=: \mathbb{HP}^1 \cong S^4.$$

Therefore S^7 can be regarded as a principal $SU(2)$-bundle over S^4. The associated $\mathbb{C}^2$-bundle is given by

$$\gamma := \left\{ ([x, y], s, t) \in \mathbb{HP}^1 \times \mathbb{H} \times \mathbb{H} \;\middle|\; \begin{array}{l} \exists q \text{ s.t.} \\ s = xq \\ t = yq \end{array} \right\}.$$

Define $\nabla_A := \Pi \circ d$, where d is the exterior differential for sections of the trivial $\mathbb{H}^2$-bundle, and Π is the orthogonal projection onto γ.

Local chart of $\mathbb{HP}^1$: $\{ [1, x] \mid x \in \mathbb{H} \}$

local basis: $$s := \begin{pmatrix} \frac{1}{\sqrt{1+|x|^2}} \\ \frac{x}{\sqrt{1+|x|^2}} \end{pmatrix} \in \mathbb{H}^2.$$

$$\nabla_A s = \overline{\Pi} \cdot d \begin{pmatrix} \dfrac{1}{\sqrt{1+|x|^2}} \\ \dfrac{x}{\sqrt{1+|x|^2}} \end{pmatrix}$$

$$= s \cdot s^* \cdot d \begin{pmatrix} \dfrac{1}{\sqrt{1+|x|^2}} \\ \dfrac{x}{\sqrt{1+|x|^2}} \end{pmatrix}$$

$$= s \cdot \left(\frac{1}{\sqrt{1+|x|^2}} , \frac{\bar{x}}{\sqrt{1+|x|^2}} \right) \begin{pmatrix} -\frac{1}{2}\left(1+|x|^2\right)^{-\frac{3}{2}} d|x|^2 \\ -\frac{1}{2}\left(1+|x|^2\right)^{-\frac{3}{2}} d|x|^2 \cdot x \\ + dx \cdot \left(1+|x|^2\right)^{-\frac{1}{2}} \end{pmatrix}$$

$$= s \cdot \Big(-\frac{1}{2(1+|x|^2)^2} d|x|^2 - \frac{|x|^2}{2(1+|x|^2)^2} d|x|^2 + \frac{\bar{x}\, dx}{1+|x|^2} \Big)$$

$$= s \cdot \Big(-\frac{d|x|^2}{2(1+|x|^2)} + \frac{\bar{x}\, dx}{1+|x|^2} \Big)$$

$$= s \cdot \frac{\frac{1}{2}\left(\bar{x}(dx) - (d\bar{x}) \cdot x\right)}{1+|x|^2}$$

$$= s \cdot \frac{\operatorname{Im}(\bar{x}\, dx)}{1+|x|^2} .$$

So the connection matrix of $A = \dfrac{\operatorname{Im}(\bar{x}\, dx)}{1+|x|^2}$

If we write $x = x^0 + x^1 I + x^2 J + x^3 K$

where $I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.

Then $A(x^0, x^1, x^2, x^3)$

$$= \frac{1}{1+\sum (x^i)^2}\Big((x^0 dx^1 - x^1 dx^0 - x^2 dx^3 + x^3 dx^2) I$$

$$+ (x^0 dx^2 - x^2 dx^0 - x^3 dx^1 + x^1 dx^3) J$$

$$+ (x^0 dx^3 - x^3 dx^0 - x^1 dx^2 + x^2 dx^3) k) \Big)$$

Notice that if $b = b^0 + b^1 I + b^2 J + b^3 k \in \Omega^1 \otimes \mathbb{H}$,

then $\operatorname{Im}(b \wedge b) = 2(b^2 \wedge b^3 I + b^3 \wedge b^1 J + b^1 \wedge b^2 k)$

$$= \operatorname{Im}(b) \wedge \operatorname{Im}(b)$$

Therefore $F_A = dA + A \wedge A$

$$= \frac{\operatorname{Im}(d\bar{x} \wedge dx)}{1+|x|^2} - \frac{\operatorname{Im}(d|x|^2 \wedge (\bar{x}\, dx))}{(1+|x|^2)^2}$$

$$+ \frac{\operatorname{Im}(\bar{x}\, dx\ \bar{x}\, dx)}{(1+|x|^2)^2}$$

$$= \frac{\operatorname{Im}(d\bar{x} \wedge dx)}{1+|x|^2} - \frac{\operatorname{Im}((d\bar{x})\, x \cdot \bar{x}\, dx)}{(1+|x|^2)^2}$$

$$= \frac{\operatorname{Im}(d\bar{x} \wedge dx)}{1+|x|^2} - \frac{|x|^2}{(1+|x|^2)^2} \operatorname{Im}(d\bar{x} \wedge dx)$$

$$= \frac{\operatorname{Im}(d\bar{x} \wedge dx)}{(1+|x|^2)^2}$$

$$dx = dx^0 + dx^1 I + dx^2 J + dx^3 k$$

$$d\bar{x} = dx^0 - dx^1 I - dx^2 J - dx^3 K$$

$$\Rightarrow \quad d\bar{x} \wedge dx = 2\left(dx^0 \wedge dx^1 - dx^2 \wedge dx^3\right) I$$
$$+ 2\left(dx^0 \wedge dx^2 - dx^3 \wedge dx^1\right) J$$
$$+ 2\left(dx^0 \wedge dx^3 - dx^1 \wedge dx^2\right) k \in \Omega^2_-.$$

Therefore A is an ASD connection.

By definition, the BPST instanton is invariant under ~~that~~ the action of $Sp(2) \circlearrowright \mathbb{H}^2$, which reduces to the action of $SO(5) \circlearrowright S^4$.

Example of bubbling: $A_\lambda := f_\lambda^*(A)$, where $f_\lambda(x) = \lambda x$. Then $\lim_{\lambda \to +\infty} \|F_{A_\lambda}\|^2 \to 8\pi^2 \delta$ as distributions.

- Functional analysis.

$L: V \to W$ linear surjection between Banach spaces

$P: W \to V$ s.t. P is ~~bounded~~ bounded, linear and $L \circ P = id_W$.

$Q: V \times V \to W$ bounded bilinear map.

$$\|Q(x,y)\| \le \|Q\| \cdot \|x\| \cdot \|y\|$$

Claim 1: if $\|L' - L\| < \frac{1}{2\|P\|}$, then L' is surjective, with a right inverse P', and we can choose P' s.t.

$$\|P'\| \le 2\|P\|.$$

Proof. Let $\delta = L' - L$.

Then $P' := P \sum_{i=0}^{\infty} (-1)^i (\delta P)^i$ is a right inverse of L'. □

Claim 2 Consider the map

$$f: V \to W$$
$$x \mapsto L(x) + Q(x,x)$$

Then $Im(f) \supseteq B_W\left(\frac{1}{2\|P\|^2\|Q\|}\right)$

Proof. $f^{-1} : B_W\left(\frac{1}{2\|P\|^2\|Q\|}\right) \to V$

where $f^{-1}(x) = P\left(\sum_{i=0}^{\infty}(-1)^i (Q \circ P)^i\right)$

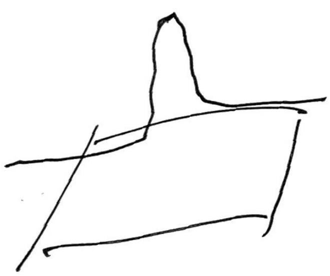
defines an inverse of f. $\square$

· Taubes' gluing.

X closed 4-mfld. A_X : ASD connection on X

For simplicity, assume X is locally flat near p.

After rescaling, $A_X \approx 0$ under a local trivialization.



Let A_{S^4} be the rescaled BPST instanton on S^4 with curvature concentrated near a point

Let $A := \chi A_X + (1-\chi) A_{S^4}$

with $\|D\chi\|_{L^4} \ll 1$.

Then $\|F_A^+\|_{L^2} \ll 1$

Need to find "a" s.t. $F^+_{A+a} = 0$.

$$F^+_{A+a} = F^+_A + d^+_A a + (a \wedge a)^+ .$$

If $\exists$ a right inverse P of d^+_A s.t. $\|P\| \leq C$, then the desired result follows from Claim 2 above.

By (a variation of) Claim 1, we can show that such a map P exists if

(1) $d^+_{A_X}$, $d^+_{A_{S^4}}$ are both surjective

(2) we use the conformal-invariant Sobolev norm $\cancel{\|\cdot\|_{L^4} +}$ $\|\cdot\|_{L^4} + \|\nabla_A \cdot\|_{L^2}$ on $\Omega^1 \otimes \mathrm{ad}\, P$.

Proposition. $d^+_{A_{S^4}}$ is surjective for the BPST instanton.

Observation: d^+_A is surjective if A is the trivial connection and $b_2^+(X) = 0$.