July 12. Donaldson's diagonalization theorem.

Theorem. Suppose X is a simply-connected, closed, oriented, smooth 4-mfld. If the intersection form of X is negative-definite, then the intersection form is diagonalizable.

Compare. Freedman's theorem: for any unimodular form Q over $\mathbb{Z}$, there is a (topological) simply-connected, closed, oriented 4-mfld whose intersection form is isomorphic to Q.

Number of negative-definite unimodular forms up to isomorphism:

rank ~~dim~~ = 24 : about 10^2

rank ~~dim~~ = 26 : about 10^3

rank ~~dim~~ = 28 : $\sim 10^5$

$\vdots$

rank = 32 : $> 10^{16}$

By Donaldson's theorem, most of those unimodular forms cannot be realized by smooth structures.

Corollary / Theorem: There exists a smooth structure on $\mathbb{R}^4$ that is not diffeomorphic to the standard smooth structure.

<u>Proof</u>: We need the following two results of Casson:

<u>Theorem 1</u> $\exists$ a collared topological embedding of $X = 3(S^2 \times S^2) - D^4$ in $K3$ such that $H_2(X)$ generate the $3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ - component in the intersection form of $K3$.

<u>Theorem 2</u>. Any non compact 4-mfld V without boundary s.t. $\pi_1(V) = 0$, $H_2(V; \mathbb{Z}) = 0$ and V has a single end homeomorphic to $S^3 \times [0, +\infty)$ is homeomorphic to $\mathbb{R}^4$.

The collar in Theorem 1 is a topological $S^3 \times [0,1]$, but it contains no smooth, separating S^3. Therefore $3(S^2 \times S^2) - \overline{X}$ is a smooth manifold that is homeomorphic but not diffeomorphic to $\mathbb{R}^4$.

The proof of the theorem relies on the study of the moduli space of ASD connections.

Recall: $\mathcal{G}$: gauge group ($\mathcal{G}_k^p$)

$\mathcal{A}$: affine space of connections ($\mathcal{A}_k^p$)

Suppose $A \in \mathcal{A}$. let $\mathrm{Stab}(A) \subseteq \mathcal{G}$ be the subgroup of $\mathcal{G}$ given by $\{ g \in \mathcal{G} \mid g(A) = A \}$

Then $g(A) = A \iff A - (d_A g) g^{-1} = A$

$\iff d_A g = 0$

$\iff D_A g = 0$

Obviously, $id \in \mathcal{G}$ is always an element of $\mathrm{Stab}(A)$.

Since $-id \in \mathcal{G}$ is in the center of $\mathcal{G}$, it is straightforward to verify that $-id \in \mathrm{Stab}(A)$ for all A.

Now suppose $g \in \mathrm{Stab}(A)$ and $g \neq \pm id$.

On each $p \in X$, $g(p) \in SU(E|_p)$ is diagonalizable. Let $\{ \lambda_1(p), \lambda_2(p) \}$ be the set of eigenvalues of $g(p)$ (counted with multiplicity), then $\lambda_1(p) = \lambda_2^{-1}(p) \in U(1)$.

$$d(\operatorname{Tr} g) = \operatorname{Tr}(d_A g) = 0.$$

$\Rightarrow$ $\lambda_1(p) + \lambda_2(p)$ is independent of p.

$\Rightarrow$ $\{\lambda_1(p), \lambda_2(p)\}$ is independent of p.

Let $g(p) = \lambda_1 \cdot \pi_1(p) + \lambda_2 \cdot \pi_2(p)$ be the diagonalization of $g(p)$, where π_1, π_2 are the orthogonal projections onto the respective eigenspaces, then $\nabla_A \pi_1(p) = 0$, $\nabla_A \pi_2(p) = 0$.

Let $E_1 = \operatorname{Im} \pi_1$, $E_2 = \operatorname{Im} \pi_2$, then $E = E_1 \oplus E_2$. Then A decomposes as a direct sum of a connection A_1 on E_1, and a connection A_2 on E_2. And $\operatorname{Stab}(A) = \{\lambda \pi_1 + \lambda^{-1} \pi_2 \mid \lambda \in U(1)\}$, so $\operatorname{Stab}(A) \cong U(1)$.

<u>Def</u>. We say that A is <u>irreducible</u> if $\operatorname{Stab}(A) \cong \{\pm 1\}$. (i.e. A does not decompose as a direct sum). We say that A is reducible if $\operatorname{Stab}(A) \cong U(1)$.

It is possible to show that $\mathcal{A}_k^p / \mathcal{G}_{k+1}^p$ is a Banach manifold near irreducible points.

Example: $U(1) \curvearrowright \mathbb{C}^2$.

$$\mathrm{Stab}(x) = \begin{cases} \{id\} & \text{if } x \neq (0,0) \quad (\text{irreducible}) \\ U(1) & \text{otherwise} \quad (\text{reducible}) \end{cases}$$

Then ~~$\mathbb{CP}^2$~~ $\mathbb{C}^2 / U(1)$ is smooth near irreducible points.

Let $(\mathcal{A}_k^p)^* \subseteq \mathcal{A}_k^p$ be the set of irreducible connections, then $(\mathcal{A}_k^p)^*$ is an open subset of $\mathcal{A}_k^p$.

Let $(\mathcal{M}_k^p)^*$ be the moduli space of <u>irreducible</u> ASD connections.

<u>Proposition</u>. ~~For a~~ Suppose $c_2(E) \neq 0$. Then for a generic ~~pos~~ metric on X, $(\mathcal{M}_k^p)^*$ is a smooth, orientable manifold with ~~dimension~~

$$\text{dimension} = 8 \langle c_2(E), [X] \rangle - 3(1 - b_1(X) + b_2^+(X))$$

4

Sketch of the proof of Donaldson's theorem assuming the proposition.

Choose E so that $(C_2(E), [X]) = 1$

By the assumptions. $b_1(X) = 0$, $b_2^+(X) = 0$.

So $\dim((\mathcal{M}_k^P)^*) = 5$. for a generic metric

$(\mathcal{M}_k^P)^*$ is not a compact manifold.

If $\{[A_n]\} \subseteq (\mathcal{M}_k^P)^*$ is a sequence of orbits of ASD connections. then at least one of the following holds:

(1) ~~$[A_n]$~~ A subsequence of $[A_n]$ converges to an element in $(\mathcal{M}_k^P)^*$

(2) A subsequence of $[A_n]$ converges to a reducible point in $\mathcal{M}_k^P$

(3) A subsequence of $[A_n]$ converges in $X - \{p_1, \cdots, p_m\}$. where

$$\|F_{A_n}\|^2 \to \sum e_i \cdot \delta_{p_i} + (\text{a non-negative function}).$$

with $e_i > 0$.

In case (3), Uhlenbeck's theorem also states that $\frac{e_i}{8\pi^2} \in \mathbb{Z}$.

Since $\int \| F_{A_m} \|^2 = 8\pi^2 \cdot (c_2(E), [X])$
$$= 8\pi^2,$$
we have $m = 1$, and $e_1 = 8\pi^2$, and

$$\| F_{A_m} \|^2 \to 8\pi^2 \delta_{p_1}.$$

So $(M_k^P)^*$ has two types of ends: one given by reducible solutions and the other given by Uhlenbeck's compactness.

Near a reducible solution, one can show that (generically) $(M_k^P) \cong$ cone over $\mathbb{C}P^2$ (or $\overline{\mathbb{C}P^2}$).

On the second end, we have a map

$$\phi : (M_k^P)^* \supseteq \text{end} \longrightarrow X \times (0, \varepsilon).$$

where $\phi(A) = (p, r)$.

p is the "center of mass" of $\| F_{A_m} \|^2$

r is the "standard deviation" of $\| F_A \|^2$.

By a generalized version of Taubes' gluing, ϕ is a diffeomorphism on the end

Therefore, $(\mathcal{M}^P_k)^*$ gives a cobordism from X to $\#_r \mathbb{CP}^2 \#_s \overline{\mathbb{CP}^2}$, where $(r+s)$ equals the number of reducible points on $\mathcal{M}^P_k$.

The number of reducible points

$$= \# \{x \mid Q(x,x) = -1\}/_2$$

where Q is the intersection form.

So $r+s \leq b_2(X) = b_2^-(X)$

and equality holds $\Leftrightarrow$ Q is diagonalizable.

~~By the property of~~

Since cobordisms ~~preserve~~ preserve the signature,

$$-b_2^-(X) = \sigma(X) = r - s$$

$$\Rightarrow \left\{ \begin{array}{l} r = 0 \\ s = b_2^-(X) \\ r+s = b_2(X) \end{array} \right.$$

therefore Q is diagonalizable.

Proof of transversality

Let $\mathcal{M}et$ be the set of conformal classes of metrics

Consider

$$\begin{tikzcd} \mathcal{M}et & \mathcal{M}et \times \mathcal{A} \arrow[l, "\text{projection}"'] \arrow[d, "F^+"] \\ & \Gamma(\Omega^+_2) \end{tikzcd}$$

Let $\widetilde{\mathcal{M}} = (F^+)^{-1}(0)$, we show that

$\widetilde{\mathcal{M}} \subseteq \mathcal{M}et \times \mathcal{A}$ is a Banach mfld.

Let $(g, A) \in \widetilde{\mathcal{M}}$. Then $T_g \mathcal{M}et$

can be represented by $\Gamma(\mathrm{Hom}(\Omega_g^-, \Omega_g^+))$.

And we can show that

$$d(F^+)\Big|_{(g,A)} (f, a) = f(F_A^-) + d_A^+ a,$$

where $f \in \Gamma(\mathrm{Hom}(\Omega_g^-, \Omega_g^+))$

$a \in \Gamma(\Omega^1(X) \otimes \mathrm{ad}\, P)$

Suppose $u \perp \operatorname{Im} d(\mathcal{F}^+)|_{(g,A)}$ (in L^2), then

$$\begin{cases} (d_A^+)^* u = 0. & (1) \\ u = 0 \quad \text{whenever } F_A^- \neq 0 & (2) \end{cases}$$

(1) $\Rightarrow$ u satisfies the unique continuity property

(2) $\Rightarrow$ $u = 0$ on an open subset

Therefore $u = 0$ (since we assume $\int \|F^-\|^2 > 0$),

and hence $d(\mathcal{F}^+)$ is surjective.

We can then prove that the projection of $\widehat{\mathcal{M}}$ to $\mathcal{M}et$ is Fredholm, and invoke the Sard-Smale theorem. $\square$.