Problem Session 1.

1. If E is a smooth vector bundle over a smooth manifold M, show that affine connections on E always exist.

2. Suppose M is a smooth manifold, let E_1, E_2 be two vector bundles over M, let A_1, A_2 be affine connections on E_1 and E_2 respectively. Let F_1 and F_2 be the curvatures of A_1 and A_2.

   Let A be the connection on $E_1 \oplus E_2$ given by the direct sum of A_1 and A_2, let F be the curvature of A. Show that F is given by the direct sum of F_1 and F_2.

3. Suppose E is a vector bundle over M, let $f : N \to M$ be a smooth map. Let A be a connection on E, let f^*A be the pull-back of A to f^*E. Show that F_{f^*A} is the pull-back of F_A.

4. Let A be a connection on a vector bundle, F_A be its curvature. Prove that $d_A F_A = 0$.

5. Let A be a connection on a vector bundle. Show that A can be locally trivialized if and only if $F_A = 0$.

6.* Let E be a complex line bundle over $\mathbb{C}P^1$ defined as follows:

$$E := \{(p, z, w) \in \mathbb{C}P^1 \times \mathbb{C}^2 \,|\, p = [x, y] \in \mathbb{C}P^1, yz = xw\}.$$

By definition, the line bundle E is a sub-bundle of the trivial $\mathbb{C}^2$–bundle. Let ∇^0 be the trivial connection on the trivial $\mathbb{C}^2$–bundle, and let π_E be the orthogonal projection of the trivial $\mathbb{C}^2$–bundle onto E.

(a) Define a connection ∇_A on E by $\nabla_A(s) = \pi_E \nabla^0(s)$. Verify that A is a connection.

(b) Compute the curvature F_A.

(c) Compute $c_1(E)$ using F_A.