Problem Session 2

1. Suppose M is a closed oriented Riemannian n–manifold, let G be a Lie group endowed with a bi-invariant metric, let $\mathfrak{g}$ be the Lie algebra of G.

   Let P be a principal G–bundle over M, let $\operatorname{ad} P := P \times_{\operatorname{ad}} \mathfrak{g}$ be the vector bundle associated with the adjoint representation. The ad-invariant inner product on $\mathfrak{g}$ induces an inner product structure on $\operatorname{ad} P$. Let $\mathcal{A}$ be the affine space of connections on P. Consider the following functional defined over $\mathcal{A}$:
$$\mathcal{F}(A) := \frac{1}{2}\int_M |F_A|^2.$$

   (a) Let a be an $\operatorname{ad} P$–valued 1-form on M. Show that
$$\frac{d}{dt}\Big|_{t=0}\mathcal{F}(A+ta) = \int_M \langle *a, d_A(*F_A)\rangle.$$

   (b) We say that A is a critical point of $\mathcal{F}$ if and only if
$$\frac{d}{dt}\Big|_{t=0}\mathcal{F}(A+ta) = 0$$
   for all a. Show that A is a critical point of F if and only if $d_A^* F_A = 0$.

   (c) For $A \in \mathcal{A}$, define
$$\operatorname{grad}\mathcal{F}(A) := d_A^* F_A.$$
   Let a be an $\operatorname{ad} P$–valued 1-form. Verify that
$$\left\langle \frac{d}{dt}\Big|_{t=0} \operatorname{grad}\mathcal{F}(A+ta), b\right\rangle = \langle d_A a, d_A b\rangle + \langle F_A, [a\wedge b]\rangle$$
   for all $b \in \Gamma(T^*M \otimes \operatorname{ad} P)$, and compute
$$\frac{d}{dt}\Big|_{t=0} \operatorname{grad}\mathcal{F}(A+ta).$$

   (d)* Suppose A is a fixed smooth connection that is a critical point of $\mathcal{F}$. Consider the following second-order differential operator:
$$\operatorname{Hess}_A : a \mapsto \frac{d}{dt}\Big|_{t=0} \operatorname{grad}\mathcal{F}(A+ta).$$
   Show that $\operatorname{Im} d_A \subset \ker \operatorname{Hess}_A$.

2. Show that the quotient space of $\mathfrak{sl}_2(\mathbb{C})$ by the adjoint action of $\mathrm{SL}_2(\mathbb{C})$ is not Hausdorff.

3. Suppose G is a compact Lie group acting smoothly on a compact manifold M. Show that the quotient topological space M/G is Hausdorff.