Problem Session 3.

1. Let E be an Hermitian vector bundle over a Riemannian manifold M. Let A, B be two Hermitian connections on E. Show that $d_A^*(A-B)=0$ if and only if $d_B^*(A-B)=0$.

2. Suppose V, W are two Banach spaces. Consider a map

$$\begin{aligned} f : V &\to W \\ x &\mapsto L(x)+Q(x,x), \end{aligned}$$

where L is a linear isomorphism, Q is a quadratic map, and there exists a constant $C>0$ such that $\|Q(x_1,x_2)\| \le C\cdot\|x_1\|\cdot\|x_2\|$ for all $x_1, x_2 \in V$.

   (a) Let $r := \dfrac{1}{4C\cdot\|L^{-1}\|}$, show that f is injective on the open ball $B(r)$ centered at zero with radius r.

   (b) Show that $f^{-1} : f(B(r)) \to B(r)$ is continuous.

   (c)* Show that $f^{-1} : f(B(r)) \to B(r)$ is a smooth map. (Hint: By the inverse function theorem, we only need to show that df are isomorphisms.)

3. (a) Suppose P is an SU(2) bundle on a closed oriented 3-manifold M. Consider the associated $\mathrm{SL}_2(\mathbb{C})$ bundle

$$\widetilde{P} := P \times_{\mathrm{SU}(2)} \mathrm{SL}_2(\mathbb{C})$$

   and the associated Lie algebra bundle

$$\operatorname{ad} P := P \times_{\mathrm{SU}(2)} \mathfrak{su}(2).$$

   Suppose A is a connection on P and ϕ is a section of $T^*M \otimes \operatorname{ad} P$, then $A + i\phi$ defines a connection on $\widetilde{P}$ by adding the T^*M–valued matrices in local coordinates.

   Show that every connection $\hat{A}$ of $\widetilde{P}$ decomposes uniquely as

$$\hat{A} = A + i\phi$$

   as described above.

   (b) Let $\hat{A} = A + i\phi$ be an $\mathrm{SL}_2(\mathbb{C})$–connection. Show that $F_{\hat{A}} = 0$ if and only if

$$\begin{aligned} F_A &= \phi\wedge\phi, \\ d_A\phi &= 0. \end{aligned}$$

(c)* Let (A_n, ϕ_n) be a sequence of solutions to the following system of equations:

$$\begin{aligned} F_A &= \phi \wedge \phi, \\ d_A \phi &= 0, \\ d_A^* \phi &= 0. \end{aligned}$$

Suppose U is a small open ball on M, and suppose that after choosing a local trivialization and applying suitable gauge transformations, we have

$$\begin{aligned} d^* A_n &= 0, \\ \|A_n\|_{L^2_3(U)} &\leq C, \\ \|\phi_n\|_{L^2_3(U)} &\leq C, \end{aligned}$$

for all n. Show that (A_n, ϕ_n) has a convergent subsequence in C^∞ on compact subsets of U.